# NINE-PARAMETER PATH SYNTHESIS OF PLANAR FOUR-BAR MECHANISM WITH THE AID OF THE GENERAL EQUATION OF COUPLER CURVE

by

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*The design of a four-bar mechanism to generate a prescribed path with minimal error is possible by using the maximum number of parameters that are effective in the path synthesis of the mechanism. In this study, the design of four-bar mechanisms, which intersect the given path curve at nine points, was dealt with in two stages. In the first step, the kinematic equations of the mechanism were used to implement the preliminary design based on the five parameters and closed-form solving. Thus, all the possible solution values have been reached with five parameters. In the second stage, which is the final design, the general algebraic form of coupler curve, which is dependent on the nine dimensions of the mechanism and of the sixth order, was obtained. An objective function derived from the obtained equation is subjected to an optimization process with nine-parameters by using the dimensions obtained from the preliminary design as an initial value, and the error between the actual and the desired path is minimized. The efficiency of the method is shown by numerical example made by choosing difficult paths to produce four-bar mechanisms.*

Key words: *nine-parameter, path synthesis, planar four bar mechanism, general equation, coupler curve*

## Introduction

Equations expressing kinematic synthesis and analysis of mechanism systems consist of non-linear algebraic and/or higher order equations. The vast majority of the problems of kinematic synthesis and analysis require the solution of such equations to be found as a consequence of the designers' demands. Non-linear motion equations of mechanics include trigonometric functions, or their multiplications, which depend on the type of kinematic links of the mechanical system. Graphical, sequential repetitive numerical techniques and closed-form solution methods are widely used in the solution of such equations. Optimization methods are the most widely used methods among those requiring numerical techniques. For example, in the study of a constrained optimization method in the design of a four-bar mechanism that generates a path, the inequality and equality constraints have been added with the help of the Lagrange multiplier [1]. In another study [2], the direct search method Hooke and Jeeves [3] was used to find the smallest value of a penalty function defined in selected sensitivity regions. Watnabe [4] minimized the perpendicular distance between the given path and the generated path points. The author transformed the coupler curve of the four-bar mecha-

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nisms into a single valued and open function by taking the arc length as a parameter. In view of the fact that guarantee of the mentioned methods and that a single solution value is finally reached, appropriate initial values must be selected or found. Since methods based on closed-form solutions, which are used in the solution of kinematic synthesis problems, require very large symbolic operations, proper results cannot be obtained on the parameters of unknown much parameters. However, in cases where the number of unknown parameters is not more than five, the solution can easily be reduced to a single unknown polynomial root so that all possible solutions can be obtained, either real or imaginary. Recently, methods based on closed-form solutions have been used in kinematic path and position analysis of multi-link mechanisms. Using sequential elimination techniques for displacement analysis of various 8, 9, and 10 link mechanisms, the solution was reduced to a high-degree polynomial roots [5-8]. Similarly, the closed-form equation of the coupler curve of some multi-link mechanisms has been obtained by Dhingra *et al*. [9]. However, it is not so common to apply closed-form solution methods to solve kinematic synthesis problems. In cases where the effectiveness of the methods used in [10-15] is tested in studies involving similar methods, the selected curves are generally chosen to suit the nature of the four bar mechanism. In this study, in order to demonstrate the effectiveness of the method used, it is preferred to use the general equation of the ellipsoid to select a difficult path which is not suitable for the nature of the four bar mechanisms. In this study, the design of the four bar mechanism, which generates the desired path with the minimum error rate considering prescribed nine path points, is performed in two stages. In the first stage, called the preliminary design, the solution of the path synthesis problem using kinematic analysis equations of the mechanism has been reduced to polynomial roots at $14^{th}$ and $12^{th}$, using successive eliminating techniques called intersection function method [16] for the given five path points.

This successive elimination method has been based on the selection of an intersection function method which requires the organization of the appropriate polynomial form of the synthesis equations [16].

In this way, the dimensions of all four bar mechanisms that intersect the desired path at five points were determined. In the second stage, which is called the final design, the closed-form of the $6^{th}$ order of algebraic expression of the coupler curve is obtained by the kinematic equations of the mechanism. An objective function was then created, taking into account the condition of the nine path points given in this equation. This function has rendered the dimensions of the mechanism that minimizes of the generated curve objective function. The Newton-Raphson method was used for the purpose of minimizing the objective function. As the first value to initiate the optimization process, it is guaranteed to approximate a solution value since the dimensions of the four bar mechanisms are used. As the initial guess value is used dimensions of the preliminary design obtained in the first stage of four bar mechanism, convergence to a solution, is guaranteed in solution method. In addition, since the preliminary design is based on closed-form solution, there can be many preliminary design mechanisms that have practical meaning, as well as the final design mechanisms with many different dimensions following the given path. This will give alternative options to the designer.

## Closed-form solution and formulation of preliminary design

When the four bar mechanism in which the kinematic dimensions are shown in fig. 1 is in any position, the following expression can be written, assuming that the *x*- and *y*-co-

-ordinates of a P point on the coupler link are treated as vectors according to the fixed O$xy$ reference system:

$$x = x_7 + x_1\cos\psi + d_{45}\cos(\delta+\gamma) \quad (1)$$

$$y = x_8 + x_1\sin\psi + d_{45}\sin(\delta+\gamma) \quad (2)$$

Similarly, from the MABQ loop:

$$\begin{aligned} x_1\cos\psi + x_2\cos\delta &= x_6 + x_3\cos\alpha \\ x_1\sin\psi + x_2\sin\delta &= x_9 + x_3\sin\alpha \end{aligned} \quad (3)$$

expression can be obtained.

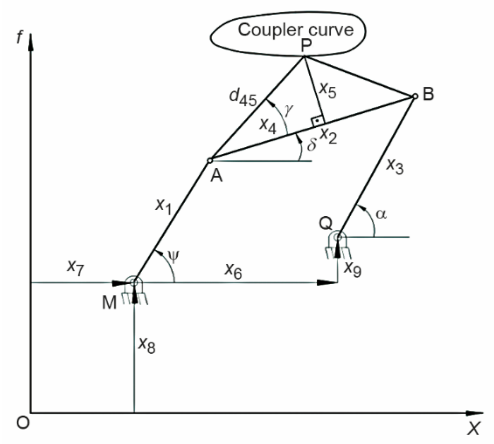


**Figure 1. Design parameters of four bar mechanism for path generation**

In order to obtain a design expression, $\cos(\delta + \gamma)$ and $\sin(\delta + \gamma)$ terms are left alone on one side of the equation and then squared and summed on both sides to eliminate the $\delta + \gamma$ in the eqs. (1) and (2), the following relation can be written:

$$x_1^2 + x_7^2 + x_8^2 - d_{45}^2 + x^2 + y_2^2 - 2xx_7 - 2yx_8 + \cos\psi(2x_1x_7 - 2xx_1) + \sin\psi(2x_1x_8 - 2yx_1) = 0 \quad (4)$$

If similar operations are repeated in eliminating the angle $\alpha$ from eq. (3), the following equation is obtained:

$$x_1^2 + x_2^2 + x_6^2 + x_9^2 - x_3^2 - 2x_2x_6\cos\delta + 2x_1x_2\cos(\delta-\psi) -$$
$$-2x_1x_6\cos\psi - 2x_2x_9\sin\delta - 2x_1x_9\sin\psi = 0 \quad (5)$$

The unknown angle in the first position of the input arm of the four bar mechanism in eq. (4), which changes depending on the coordinates of a given path ($x$, $y$), is called $\psi_0$. The arbitrary angle added to this angle to show subsequent positions is called $\psi'$. If the resultant ($\psi = \psi_0 + \psi'$) transformation is written and substituted in eq. (4), the following expressions are found:

$$F_i(\psi_0, x_1, x_7, x_8, d_{45}; x_i, y_i, \psi_i') = a_i + x_7(-b_i + d_{ci}x_1\cos\psi_0 - d_{si}x_1\sin\psi_0) +$$
$$+x_8(-c_i + d_{ci}x_1\sin\psi_0 - d_{si}x_1\cos\psi_0) + x_1(-v_{ci}\cos\psi_0 - v_{si}\sin\psi_0) + f_iP = 0, \quad i = 1,..,5 \quad (6)$$

$$P = x_1^2 + x_7^2 + x_8^2 - d_{45}^2 \quad (7)$$

Equation (6) of five unknown $\psi_0, x_1, x_7, x_8, d_{45}$ parameters of the four-bar mechanism shown in fig. 1 can be expressed in five different ($x_i$, $y_i$, $i$ = 1, 2, 3, 4, 5) points and arbitrarily selected $(\psi_i', i = 1,2,3,4,5)$ represents the position where the mechanism should be located. Accordingly, coefficients ($a_i$, $b_i$, $c_i$, $f_i$, $d_{ci}$, $d_{si}$, $v_{ci}$, $v_{si}$, $i$ = 1, 2, 3, 4, 5) in eq. (6) can be calculated:

$$\begin{aligned} &a_i = x_i^2 + y_i^2, \quad b_i = 2x_i, \quad c_i = 2y_i, \quad d_{ci} = 2\cos\psi_i', \quad d_{si} = 2\sin\psi_i' \\ &v_{ci} = 2(x_i\cos\psi_i' + y_i\sin\psi_i'), \quad v_{si} = 2(x_i\sin\psi_i' + y_i\cos\psi_i'), \quad f_i = 1 \end{aligned} \quad i = 1,2,3,4,5 \quad (8)$$

Equation (6) is also a set of five non-linear equations in five unknowns. If two equations of the equation ($P$, $x_7$, $x_8$) are eliminated first and then a half-angle tangent formula is used for $\psi_0$, then two equations are obtained with the third order polynomials connected to ($t$, $x_1$) and $x_1$.

$$\begin{gathered}(a_{pk}t^4 + 2a_{pk}t^2 + a_{pk}) + x_1(-b_{pk}t^4 + 2c_{pk}t^3 + 2c_{pk}t + b_{pk}) + \\ + x_1^2(d_{pk} + f_{pk})t^4 - 4e_{pk}t^3 + (2d_{pk} - 6f_{pk})t^2 + 4e_{pk}t + (d_{pk} + f_{pk})] + \qquad k = 1,2 \\ + x_1^3(-g_{pk}t^4 + 2h_{pk}t^3 + 2h_{pk}t + g_{pk}) = 0\end{gathered} \tag{9}$$

$$t = \tan\frac{\psi_0}{2} \tag{10}$$

If $x_1$ is eliminated from the eq. (9), the following 14th degree polynomial is obtained.

$$\begin{gathered}w_0(1-t^{14}) + w_1(1+t^{12})t + w_2(1-t^{10})t^2 + w_3(1+t^8)t^3 + w_4(1-t^6)t^4 + \\ + w_5(1+t^4)t^5 + w_6(1-t^2)t^6 + w_7t^7 = 0\end{gathered} \tag{11}$$

Equation (11) with coefficients ($w_i$, $i$ = 0, .., 7) which can be calculated with the help of coefficients ($a_{pk}$, $b_{pk}$, $c_{pk}$, $d_{pk}$, $e_{pk}$, $f_{pk}$, $g_{pk}$, $h_{pk}$; $k$ = 1, 2) are not included here because they are too long symbolic representations. Furthermore, it can be seen that only the eight coefficients are sufficient for eq. (11) polynomial to be 14th order.

Following the solution of eq. (11) for $t_i$, 14 different both real and complex roots can be found for $\psi_0$:

$$\psi_{0i} = 2\tan^{-1}t_i, \quad i = 1,2,...,14 \tag{12}$$

For the calculation of the remaining four ($d_{45}$, $x_7$, $x_8$, $x_1$,) parameters of the following four-bar mechanism, the second order polynomial is obtained from $x_1$ if $x_1^3$ is eliminated first from eq. (9).

$$(aa_2dd_1 - aa_1dd_2) + (bb_2dd_1 - bb_1dd_2)x_1 + (cc_2dd_1 - cc_1dd_2)x_1^2 = 0 \tag{13}$$

where

$$\begin{gathered}aa_k = a_{pk}(t_i^4 + 2t_i^2 + 1) \\ bb_k = -b_{pk}t_i^4 + 2c_{pk}t_i^3 + 2c_{pk}t_i + b_{pk} \qquad i = 1,2,\ldots,14 \\ cc_k = (d_{pk} + f_{pk})t_i^4 - 4e_{pk}t_i^3 + (2d_{pk} - 6f_{pk})t_i^2 + 4e_{pk}t_i + (d_{pk} + f_{pk}) \qquad k = 1,2 \\ dd_k = -g_{pk}t_i^4 + 2h_{pk}t_i^3 + 2h_{pk}t_i + g_{pk}\end{gathered} \tag{14}$$

In eq. (14), there are two solutions $x_{1\pm}$ for each solution ($t_i$, $i$ = 1, 2,...,14). Thus, revealed a total of 28 sets of solutions, are involved including imaginary or real. The linear equation obtained by substituting $(\psi_{0i}, x_{1\pm i}, \quad i = 1,2,...,14)$ into eq. (6) for these 28 sets of solutions is easily solved for any three of the sets $(x_{7i\pm}, x_{8i\pm}, P_{i\pm} \quad i = 1,2,...,14)$. Finally, the length $d_{45}$ is calculated:

$$d_{45i\pm} = \sqrt{x_{1i\pm}^2 + x_{7i\pm}^2 + x_{8i\pm}^2 - P} \quad i = 1,2,...,14 \tag{15}$$

The number of practical solutions can be found as a result of looking at the condition of achieving the eq. (6) of the real solutions in the solution set ($x_{1j}$, $x_{7j}$, $x_{8j}$, $d_{45j}$, $\psi_{0j}$ $j$ = 1, 2, ..., 28). Furthermore, in order to detect the branching problem during the successive movement of the mechanism, the obtained solution set must be evaluated in eqs. (1) and (2). The required ($\delta + \gamma$) angle is calculated as follows for the five exact path points given:

$$(\delta+\gamma)_{i\pm} = \frac{-B_i \pm \sqrt{B_i^2 - 4A_iC_i}}{2A_i}, \quad i=1,2,3,4,5 \tag{16}$$

where

$$\begin{gathered} A_i = (d_{45} - x_7 + x_i)^2 - (x_1 + x_8 - y_i)(x_1 - x_8 + y_i) \\ B_i = 4d_{45}(x_8 - y_i) \qquad\qquad i=1,2,3,4,5 \\ C_i = (d_{45} + x_7 - x_i)^2 - (x_1 + x_8 - y_i)(x_1 - x_8 + y_i) \end{gathered} \tag{17}$$

Only one of the $(\delta+\gamma)_{\pm}$ directions in the 16 equation is compatible with the given path point. As a result of the mentioned design method, the input link (crank) of the four-bar mechanisms shown in fig. 1, which is effective in path synthesis, has a horizontal distance $x_7$ with respect to the stationary O$xy$ reference system, a vertical distance $x_8$ with the entrance length $x_1$ and a trailing P the distance $d_{45}$ of the connecting point to the link joint hinge is determined. The connection length $x_2$, the outlet length $x_3$, the $x_4$ horizontal distance $x_6$, which determines the position of the output shaft body joint with respect to the input shaft body joint, and the $x$ angle between the connecting shaft length $x_2$ and the $d_{45}$ length, to determine a total of five parameters (3) to (4). For this, first of all the angle $\alpha$, is eliminated from these expressions. Then, because the expressions of unknown parameters do not take the angle of $\gamma$ directly, if the terms of $\delta$ are written in ($\delta + \gamma - \gamma$) and are opened and substitute according to the sum of sinus and cosine expressions ($\delta + \gamma$) + ($-\gamma$) are corrected, the following five sets of non-linear equations in five unknowns are obtained.

$$\begin{aligned} G_i(x_2,x_3,x_6,x_9,\gamma;x_i,y_i,\psi_i') = {} & pa_iZ_1 + Z_3(pb_iZ_2\sin\gamma - pe_iZ_2\cos\gamma - pb_i\cos\gamma - pe_i\sin\gamma) + \\ & + Z_4(cp_i\cos\gamma + pg_i\sin\gamma) - pf_iZ_2 - dp_i = 0, \quad i=1,..,5 \end{aligned} \tag{18}$$

where

$$Z_1 = \frac{x_1^2 + x_2^2 + x_6^2 + x_9^2 - x_3^2}{2x_1x_6}, \quad Z_2 = \frac{x_9}{x_6}, \quad Z_3 = \frac{x_2}{x_1}, \quad Z_4 = \frac{x_2}{x_6} \tag{19}$$

Obtained.

Furthermore, the coefficients of (18) ($pa_i$, $pb_i$, $pc_i$, $pd_i$, $pe_i$, $pf_i$, $pg_i$, $i$ = 1, 2, 3, 4, 5) can be calculated:

$$\begin{gathered} pa_i = 1, \quad pb_i = \cos[(\delta+\gamma)_i], \quad pc_i = \cos[(\delta+\gamma)_i - (\psi_0 + \psi_i')] \\ pd_i = \cos[\psi_0 + \psi_i'], \quad pe_i = \sin[(\delta+\gamma)_i], \quad pf_i = \sin[\psi_0 + \psi_i'] \qquad i=1,2,3,4,5 \\ pg_i = \sin[(\delta+\gamma)_i - (\psi_0 + \psi_i')] \end{gathered} \tag{20}$$

Although the angle $\gamma$ is not known in eq. (20), it can easily be calculated from the eq. (16) for the given [($\delta + \gamma$)$_i$, $i$ = 1, 2, 3, 4, 5] path points and arbitrarily selected $(\psi_i', i=1,2,3,4,5)$ angles. In order to solve unknown parameters from the set of equations

(18), the following two unknown equations are obtained when the parameters of ($Z_1$, $Z_2$, $Z_4$) are eliminated first by this expression and the coefficients are dependent on the angle $\gamma$ and are second order polynomials according to $Z_2$:

$$\begin{gathered}(pa_{pk}\cos\gamma + pb_{pk}\sin\gamma) + Z_3(pe_{pk} + pf_{pk}\sin2\gamma + pg_{pk}\cos2\gamma) \\ Z_3^2(pc_{pk}\cos\gamma + pd_{pk}\sin\gamma) = 0\end{gathered} \qquad k = 1, 2 \tag{21}$$

If $Z_1$ is eliminated removed from eq. (21), the polynomial is obtained in the following 12$^{\text{th}}$ order:

$$\begin{gathered}q_0(v^{12}+1) + q_1(v^{10}-1)v + q_2(v^8+1)v^2 + q_3(v^6-1)v^3 + \\ +q_4(v^4+1)v^4 + q_5(v^2-1)v^5 + p_6v^6 = 0\end{gathered} \tag{22}$$

where

$$v = \tan\frac{\gamma}{2} \tag{23}$$

where ($pa_{pk}$, $pb_{pk}$, $pc_{pk}$, $pd_{pk}$, $pe_{pk}$, $pf_{pk}$, $pg_{pk}$, $k$ = 1, 2) coefficients found in the connection of the polynomial (21) and calculated with the help of the coefficients in (20) are not given here since the coefficients ($q_i$, $i$ = 0, .., 6) of the polynomial are very long symbolic representations. Similarly, although the polynomial (22) in this case has a order of 12, only 7 coefficients are sufficient. As a result of finding all the roots of polynomial (22) depend to $v$, 12 solution clusters are calculated as follows, real or imaginary:

$$\gamma_i = 2\tan^{-1}v_i, \quad i = 1, 2 \ldots, 12 \tag{24}$$

with the aid of eq. (24), ($Z_{2i}$ ,$i$ = 1, 2, .., 12) is solved from the linear equation which emerges as the result of the $Z2^2$ termination from eq. (21). Then, ($Z_{1i}$, $Z_{3i}$, $Z_{4i}$, $i$ = 1, .. 12) magnitudes solved from any three of the five sets of linear equations formed by replacing the obtained ($\gamma_i$, $Z_{2i}$, $i$ = 1, .., 12) solution set in eq. (18), and ($x_{2i}$, $x_{3i}$, $x_{6i}$, $x_{9i}$, $i$ = 1, .., 12) lengths of the mechanism are calculated by substituting in eq. (19).

The actual solutions found in the ($x_{2j}$, $x_{3j}$, $x_{6j}$, $x_{9j}$, $\gamma_j$, $j$ = 1, 2, ..., 12) solution set can be found in eq. (18) set and there can be a number of solutions with practical meaning. Furthermore, it is possible to detect the branching problem during the sequential motion of the mechanism by evaluating the obtained solution set (3), (4). For this, first of all, the angles $\alpha$ from eqs. (3) and (4) are eliminated and the angle of $\delta$ the link horizontally is solved as follows for each given path point:

$$\delta_{i\pm} = \frac{-B_i' \pm \sqrt{B_i'^2 - 4A_i'C_i'}}{2A_i'}, \quad i = 1, 2, 3, 4, 5 \tag{25}$$

where

$$\begin{gathered}A_i = x_1^2 - x_3^2 + x_9^2 + (x_2 - x_6)^2 + 2x_1[(x_2 - x_6)\cos(\psi_0 + \psi_i') - x_9\sin(\psi_0 + \psi_i')] \\ B_i' = -4x_2[x_9 - x_1\sin(\psi_0 + \psi_i')] \qquad i = 1,.,5 \\ C_i' = x_1^2 - x_3^2 + x_9^2 + (x_2 + x_6)^2 - 2x_1[(x_2 + x_6)\cos(\psi_0 + \psi_i') + x_9\sin(\psi_0 + \psi_i')]\end{gathered} \tag{26}$$

The $\delta_{i+}$ angular dimensions obtained from eq. (25) are only compatible with the given trajectory points given in eqs. (1) and (2).

## Results and discussion

### *Formulation of final design*

The maximum number of parameters of the preliminary four-bar mechanism that accurately passing through the given five path points are nine. This allows the designer to have maximum control when generated a given coupler generated. Where preliminary design results are not sufficient for applications that require too much precision, it is necessary that the maximum design is used. In addition to the previously described eqs. (1)-(4) of the four-bar mechanisms shown in fig. 1, the following expressions can be written from the OQBP open vector loop.

$$x = x_6 + x_7 + x_3\cos\alpha + (x_4 - x_2)\cos\delta - x_5\sin\delta \tag{27}$$

$$y = x_8 + x_9 + x_3\sin\alpha + (x_4 - x_2)\sin\delta + x_5\cos\delta \tag{28}$$

As previously similarly applied, $\psi$ angle is eliminated from eqs. (1), (2) and $\alpha$ angle from eqs. (27), (28). Then, $x_{42} = (x_{4} - x_2)$ expression abbreviation is performed and the half angle tangent formula is used, the following two equations are achieved.

$$\begin{gathered} x^2 + y^2 - x_1^2 + x_4^2 + x_5^2 + x_7^2 + x_8^2 - 2x(x_4 + x_7) - \\ -2y(x_5 + x_8) + 2(x_4x_7 + x_5x_8) + 4\xi(xx_5 - yx_4 - x_5x_7 + x_4x_8) + \\ +\xi^2[x^2 + y^2 - x_1^2 + x_4^2 + x_5^2 + x_7^2 + x_8^2 + 2x(x_4 - x_7) + 2y(x_5 - x_8) - 2(x_4x_7 + x_5x_8)] = 0 \end{gathered} \tag{29}$$

$$\begin{gathered} x^2 + y^2 - x_3^2 + x_{42}^2 + x_5^2 + x_6^2 + x_7^2 + x_8^2 + x_9^2 - 2x(x_{42} + x_6 + x_7) - \\ -2y(x_5 + x_8 + x_9) + 2(x_{42}x_6 + x_{42}x_7 + x_6x_7 + x_5x_8 + x_5x_9 + x_8x_9)] + \\ +4\xi\left[xx_5 - yx_{42} - x_5x_6 - x_5x_7 + x_{42}x_8 + x_{42}x_9\right] + \\ + \xi^2[x^2 + y^2 - x_3^2 + x_{42}^2 + x_5^2 + x_6^2 + x_7^2 + x_8^2 + x_9^2 + 2x(x_{42} - x_6 - x_7) + \\ +2y(x_5 - x_8 - x_9) - 2(x_{42}x_6 + x_{42}x_7 - x_6x_7 + x_5x_8 + x_5x_9 + x_8x_9)] = 0 \end{gathered} \tag{30}$$

where

$$\xi = \tan\frac{\delta}{2}$$

If $\xi$ is removed from eqs. (29) and (30), the closed-form of the sixth-order gradual equation, which expresses coupler curve of the four-bar mechanism and contains the dimensions of the ($\sigma_i$, $i$ = 0,1, ..., 15) coefficients, becomes as follows according to a fixed reference system:

$$\begin{gathered} (x,y) = \sigma_0 + \sigma_1 y + \sigma_2 y^2 + \sigma_3 y^3 + \sigma_4(y^4 + x^2y^2) + \sigma_5(x^4y + y^5 + 2x^2y^3) + \\ + \sigma_6(x^6 + 3x^2y^4 + 3x^4y^2 + y^6) + \sigma_7(x^5 + xy^4 + 2x^3y^2) + \sigma_8(x^4 + x^2y^2) + \sigma_9 x^3 + \\ +\sigma_{10}(x^3y + xy^3) + \sigma_{11}x^2 + \sigma_{12}x^2y + \sigma_{13}x + \sigma_{14}xy + \sigma_{15}xy^2 = 0 \end{gathered} \tag{31}$$

In the general equation of the coupler curve, when the preliminary design result is written instead of nine points selected from the dimensions of the mechanism and the given trajectory, it can be said that five of these points used in the preliminary design will achieve this equality but not the other four points. Using eq. (31), the following nine unknown systems of nine non-linear equations can be written as follows to ensure that the preliminary design mechanism passes through the given nine path points:

$$H_i\left(x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9; x_i, y_i\right) = 0, \quad i = 1, 2, \ldots, 9 \tag{32}$$

The solution of eq. (32) set can be accomplished by minimizing an objective function defined as follows and normalized to coefficients:

$$E(x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9) = \frac{\sum_{i=0}^{9} H_i^2(x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9; x_i, y_i)}{\sum_{i=0}^{15} \sqrt{\sigma_i^2}} = \min! \tag{33}$$

The dimension of the mechanism that provides the equation of the coupler curve given in eq. (32) makes the objective function in eq. (33) the minimum. Conversely, the dimensions of the mechanism that minimizes eq. (33) may be a solution to the system of the eq. (32).

An advantage of using the eq. (33) is that if the number of path points given is more than 9, then the design becomes the problem of generating the path with the least squared total error.

The minimum value of the expression of the objective function in eq. (33) requires that the partial derivative of this function is equal to zero as many given as the unknown parameters. This can be expressed as follows using matrix notation:

$$\underline{f}(\underline{X}) = \frac{\partial E(\underline{X})}{\partial \underline{X}} = 2\sum_{i=1}^{9} H_i \frac{\partial H_i}{\partial \underline{X}} = \underline{0} \tag{34}$$

where

$$\underline{f}(\underline{X}) = \left[2\sum_{i=1}^{9} H_i \frac{\partial H_i}{\partial x_1}, 2\sum_{i=1}^{9} H_i \frac{\partial H_i}{\partial x_2}, \ldots, 2\sum_{i=1}^{9} H_i \frac{\partial H_i}{\partial x_9}\right]^T = (0, 0, \ldots., 0)^T \tag{35}$$

$$\underline{X} = (x_1, x_2, \ldots., x_9)^T \tag{36}$$

The solution of the system of nine unknowns of nine non-linear equations obtained from eq. (35) and composed of the first derivatives of the coupler curve can be the solution of the set of eq. (32) at the same time while making the function of the objective function in the eq. (33) minimum. Accordingly, for the solution of the set of non-linear equations in the eq. (35) according to $\underline{X}$, the following consecutive repetition of the Newton-Raphson method can be used:

$$\underline{X}_{k+1} = \underline{X}_k - \lambda \underline{J}(\underline{X}_k)^{-1} \underline{f}(\underline{X}_k), \quad k = 0, 1, 2 \ldots\ldots \tag{37}$$

where $\underline{X}_{k+1}$ is a next successive parameter obtained again illustrates the given square matrix $\underline{J}(\underline{X}_k)^{-1}$ matrix in the first order derivatives of (35) set of equation and comprising or second order derivatives of the general expression of the coupler curve inverse matrix of the Jacob matrix.

$$\underline{J}[\underline{X}_k]=\begin{bmatrix} \sum_{i=1}^{9}\left[\left(\frac{\partial H_i}{\partial x_1}\right)^2+H_i\frac{\partial^2 H_i}{\partial x_1^2}\right] & \ldots\ldots & \sum_{i=1}^{9}\frac{\partial H_i}{\partial x_9}\frac{\partial H_i}{\partial x_1}+H_i\frac{\partial^2 H_i}{\partial x_9\partial x_1}\} \\ \ldots\ldots\ldots\ldots & \ldots\ldots & \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots & \ldots\ldots & \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \sum_{i=1}^{9}\left(\frac{\partial H_i}{\partial x_1}\frac{\partial H_i}{\partial x_9}+H_i\frac{\partial^2 H_i}{\partial x_1\partial x_9}\right) & \ldots\ldots & \sum_{i=1}^{9}\left[\left(\frac{\partial H_i}{\partial x_9}\right)^2+H_i\frac{\partial^2 H_i}{\partial x_9^2}\right] \end{bmatrix} \tag{38}$$

Also, in the case where there is no convergence in the previous expression, the convergence factor $\lambda$ relaxation factor is used. When $\lambda = 1$ is initially selected, eq. (37) is the Newton-Raphson method. If the consecutive repetition has a tendency to diverge or branch off from a particular solution value, convergence is achieved by systematically changing $\lambda$ (for example, by changing $\lambda = -\lambda/n$ to a multiple number, such as one greater than n).

Accordingly, the following steps are taken to achieve a solution value from eq. (37).

First, a maximum consecutive repetition limit, $N$, and a small number $x_1$ for the successive repetition termination criterion are selected.

Assignment of $j = 1$ is made to determine the number of consecutive repetitions made.

The dimensions of the four-bar mechanisms obtained from the preliminary design are taken as initial values of $\underline{X}_{j-1}$ to be successive revolutions and these values are calculated from eq. (33) $E(\underline{X}_{j-1})$, $(x_3)$ with $\underline{f}(\underline{X}_{j-1})$ and $(x_3)$ with the help of eq. (38).

The $\lambda = 1$ is done, $\underline{X}_j$ is calculated by substituting in the calculated quantities (37) in step III.

Equation (33) is calculated as $E(\underline{X}_j)$.

The $E(\underline{X}_j)$ and $E(\underline{X}_{j-1})$ are compared. If it is $E(\underline{X}_j) > E(\underline{X}_{j-1})$, it is set $\lambda = -\lambda/n$ ($n > 1$) operations are repeated from the V. step.

If

$$\left|\frac{E(\underline{x}_{j-1})-E(\underline{x}_j)}{E(\underline{x}_{j-1})}\right| < \varepsilon_s \text{ or } j > N,$$

the design process is completed. Otherwise j is incremented by one ($j = j + 1$) and the process is repeated.

The analytical ideas described above are developed into computer software using the PASCAL language and two example results are shown in the following section.

### *Numerical example*

In this section, some arbitrary paths have been examined and the effectiveness of the method has been tested.

*Example:* In this example, it is intended to generate the complete elliptical path shown in fig. 2. The five trajectory points ($P$) necessary to obtain the preliminary design mechanism, which is the first stage of design, are selected as follows according to the reference system shown in fig. 2.

$$P_1 = (0.469131, 0.099115),\ P_2 = (0.137733, 0.275435),\ P_4 = (-0.452019, 0.122247)$$

$$P_6 = (-0.358365, -0.199805),\ P_8 = (0.263175, -0.243619)$$

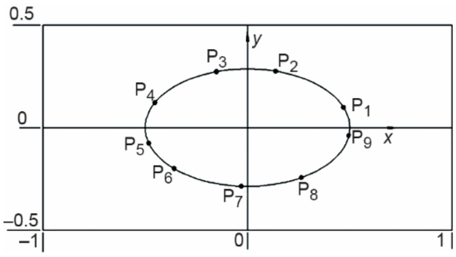


**Figure 2. The points for closed elliptical path in the first example**

In addition, the amounts of change for the mentioned five path points of the input angle of the mechanism are selected as follows in a sequential manner so that the input arm of the mechanism moves sequentially.

$$(\psi_i', i = 1,2,..,5) = (0°, –85°, –185°, –220°, –320°)$$

The kinematic dimensions of the two selected from the preliminary design mechanisms obtained considering these data are below.

*First solution of preliminary design four-bar mechanism:*

$$x_1 = 0.377131,\ x_2 = –0.816084,\ x_3 = 1.612492,\ x_4 = –1.533799,\ x_5 = –0.701133$$
$$x_6 = –1.947780,\ x_7 = 0.158814,\ x_8 = 1.5848965,\ x_9 = –0.372924,\ \psi_0 = 171.450740$$

*Second solution of preliminary design four-bar mechanism:*

$$x_1 = 0.377131,\ x_2 = –0.806335,\ x_3 = 1.254800,\ x_4 = –1.644748,\ x_5 = 0.372738$$
$$x_6 = 1.302556,\ x_7 = 0.158814,\ x_8 = 1.5848965,\ x_9 = –0.976681,\ \psi_0 = 171.450740$$

The first and second preliminary design mechanisms have been subjected to the optimization process for a total of nine points selected as the result of adding the four points ($P$) of the elliptical path given previously to the above five path points.

$$P_3 = (–0.358365, –0.199805),\ P_5 = (–0.029535, –0.286019)$$
$$P_7 = (0.263175, –0.243619),\ P_9 = (0.495391, –0.038812)$$

The final design mechanism, which is the result of optimization of the preliminary design mechanisms for the nine path points, is as follows.

– First final design four-bar mechanism

$$x_1 = 0.323046,\ x_2 = –1.130587,\ x_3 = 1.144712,\ x_4 = –0.490894,\ x_5 = –0.923637$$
$$x_6 = 0.244027,\ x_7 = –0.234085,\ x_8 = 1.035137,\ x_9 = –0.414735$$

– Second final design four-bar mechanism

$$x_1 = 0.297858,\ x_2 = –3.874090,\ x_3 = 3.886851,\ x_4 = –2.159555,\ x_5 = 3.500216$$
$$x_6 = –0.852722,\ x_7 = 0.846282,\ x_8 = 4.024049,\ x_9 = –1.361500$$

The obtained first and second preliminary design and kinematic diagrams of the final design mechanisms and the coupler curves generated are shown in figs. 3 and 4, respectively. When the shapes are examined, it can be seen that the elliptical path given to the preliminary design mechanisms is quite different at the other path points intersect at five points. The path generated by the final design mechanisms that emerged at the end of the optimization process can be seen in the same way that they are very close to the elliptical path. The first solution producing the trajectory is the kinematic dimensions of the four-bar mechanisms shown in fig. 5, and the second solution is as shown in fig. 6.

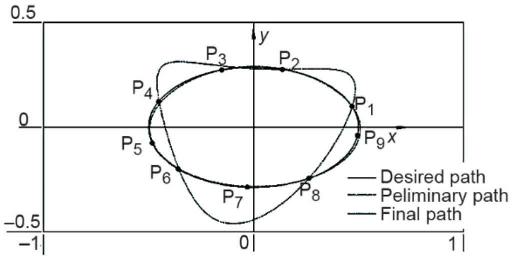


**Figure 3. The first solver preliminary design and final design path for the elliptical path in *Example***

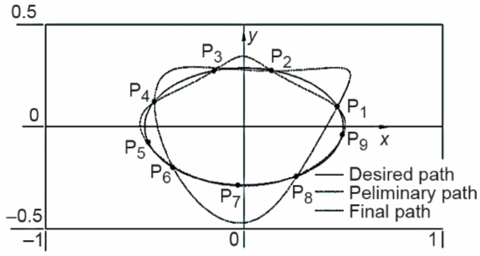


**Figure 4. The second solver preliminary design and final design path for the elliptical path in *Example***

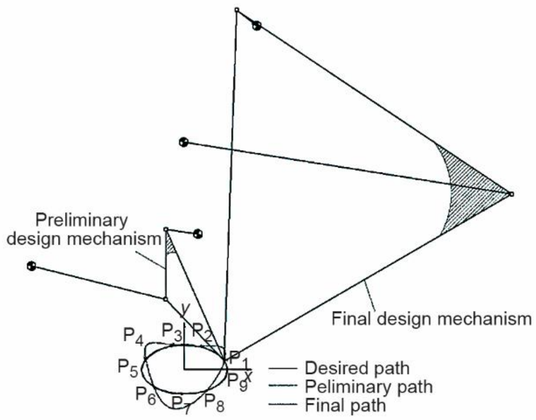


**Figure 5. The four-bar mechanism geometry belong to the first solving preliminary design and the final design, which generates the elliptical path in *Example***

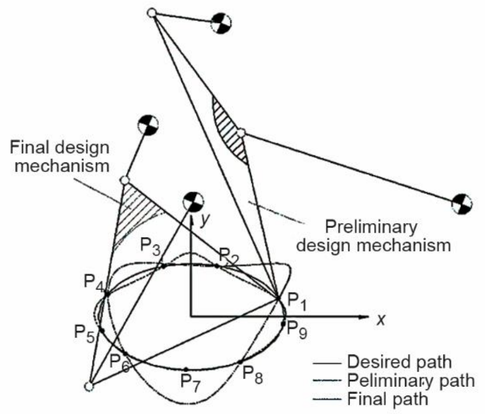


**Figure 6. The second solver preliminary design and the final design that generate the elliptical path in *Example* are the four-bar mechanism geometry**

## Conclusion

The kinematic synthesis problem of non-linear expressions or equation sets are known to be used intensively. One of the most important problems encountered in the algorithms of computer programs developed in this scope are the calculation singularities that arise in the mathematical expressions used and are unknown beforehand when they are encountered and convergence provider appropriate selection of the first guess value. The singularity condition is usually caused by points where many functions or expressions in the used equations are discontinuous or undefined. This leads to the presence of a large number of control lines within the software algorithm and that cause to the increase of the software codes. Furthermore, if general-purpose software is desired to be developed, these problems may make the software difficult to use by others or make impossible to use the software. In this study, two-stages approach in the problem of generating a path in four-bar mechanism were performed considering aforementioned matter to minimize. When the two-stages approach used are examined, closed solver-based orbital synthesis of four bar mechanisms is performed for the given five orbital points in the first stage, called preliminary design. In this approach, which does not require initial guessing values, the solution is reduced to the finding of polynomial roots in the $14^{th}$ and $12^{th}$ time, ensuring that all possible real and complex roots are found. In the second step, a goal function with the unknown number of parameters nine obtained from the general equation of the coupler curve of the four-bar mechanism is minimized by an iterative approach. As the goal function determined in the minimization process can be derivation continuously and at desired degree in all points, computational impossibilities are not encountered in the developed computer software. Since the initial predictor values that will initiate the iterative process use the solution values of the four-bar mechanism in the previous stage, the designer does not need to find the initial predictor values. The solutions obtained in the first stage, referred to as the preliminary design, have not encountered the convergence problems in the second stage iterative solution which is called the final design due to the close of the optimum solution. Since the number of unknown parameters used in the second step is nine, it can be seen from the examples given that the solution obtained at this stage is always closer to the given path. in addition to these, all possible solutions can be obtained in the preliminary design, so example shows that there can be more than one number of different mechanisms to produce the best given path. The approach shown in this study can be easily adopt to different mechanisms other than the four-bar mechanism.

## References

[1] Bakthavachalam, N., Kimbrell, J., Optimum Synthesis of Path-Generating Four-Bar Mechanisms, *Journal of Engineering for Industry*, *97* (1975), 1, pp. 314-321
[2] Sandor, G., Selective Precision Synthesis-a General Method of Optimization for Planar Mechanisms, *Journal of Engineering for Industry*, *97* (1975), 2, pp. 689-701
[3] Siddall, J. N., *Analytical Decision-Making in Engineering Design*, Prentice Hall, Upper Saddle, River, N. J., USA, 1972
[4] Watanabe, K., Application of Natural Equations to the Synthesis of Curve Generating Mechanisms, *Mechanism and Machine Theory*, *27* (1992), 3, pp. 261-273
[5] Dhingra, A., *et al.*, Closed-form Displacement Analysis of 10-link 1-DOF Mechanisms: Part 1 – General Framework, *Mechanism and Machine Theory*, *36* (2001), 1, pp. 29-56
[6] Dhingra, A., *et al.*, Closed-form Displacement Analysis of 8, 9, and 10-link Mechanisms: Part II: 9-link 2-DOF and 10-link 3-DOF Mechanisms, *Mechanism and Machine Theory*, *35* (2000), 6, pp. 851-869
[7] Dhingra, A., *et al.*, Closed-form Displacement Analysis of 10-link 1-DOF Mechanisms: Part 2 – Polynomial Solutions, *Mechanism and Machine Theory*, *36* (2001), 1, pp. 57-75
[8] Dhingra, A., *et al.*, Closed-form Displacement Analysis of 8, 9, and 10-link Mechanisms: Part I: 8-link 1-DOF Mechanisms, *Mechanism and Machine Theory*, *35* (2000), 6, pp. 821-850
[9] Dhingra, A., *et al.*, A Closed-form Approach to Coupler-Curves of Multi-Loop Mechanisms, *Journal of Mechanical Design*, *122* (2000), 4, pp. 464-471
[10] Hongying, Y., *et al.*, Study on a New Computer Path Synthesis Method of a Four-Bar Linkage, *Mechanism and Machine Theory*, *42* (2007), 4, pp. 383-392
[11] Nariman-Zadeh, N., *et al*., Pareto Optimal Synthesis of Four-Bar Mechanisms for Path Generation, *Mechanism and Machine Theory*, *44* (2009), 1, pp. 180-191
[12] Buskiewicz, J., *et al.*, On the Application of the Curve Curvature in Path Synthesis, *Mechanism and Machine Theory*, *44* (2009), 6, pp. 1223-1239
[13] Acharyya, S., Mandal, M., Performance of EAs for Four-Bar Linkage Synthesis, *Mechanism and Machine Theory*, *44* (2009), 9, pp. 1784-1794
[14] Bai, S., Angeles, J., Coupler-Curve Synthesis of Four-Bar Linkages *via* a Novel Formulation, *Mechanism and Machine Theory*, *94* (2015), Dec., pp. 177-187
[15] Zhou, H., Cheung, E. H., Analysis and Optimal Synthesis of Hybrid Five-Bar Linkages, *Mechatronics*, *11* (2001), 3, pp. 283-300
[16] Akcali, I., Mutlu, H., A Novel Approach in the Direct Kinematics of Stewart Platform Mechanisms with Planar Platforms, *Journal of Mechanical Design*, *128* (2006), 1, pp. 252-263

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