NON-POLYNOMIAL CUBIC SPLINE METHOD USED TO FATHOM SINE GORDON EQUATIONS IN 3+1 DIMENSIONS

by

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This study contains an algorithmic solution of the Sine Gordon equation in three space and time dimensional problems. For discretization, the central difference formula is used for the time variable. In contrast, space variable x, y, and z are discretized using the non-polynominal cubic spline functions for each. The proposed scheme brings the accuracy of order $O(h^2 + k^2 + \sigma^2 + \tau^2h^2 + \tau^2k^2 + \tau^2\sigma^2)$ *by electing suitable parametric values. The paper also discussed the truncation error of the proposed method and obtained the stability analysis. Numerical problems are elucidated by this method and compared to results taken from the literature.*

Key words: Sine *gordon equation, 3-D wave equations, truncation error, non-polynomial cubic spline function, stability analysis*

Introduction

In scientific research, there are numerous problems that are mathematically expressed by PDE. Non-linear phenomena also play a significant part in various fields of science like engineering, mathematics and physics. Many experiments are modeled by non-linear PDE [1]. One, two and three space-dimensional models are discussed in literature boundlessly. Three space-dimensional versions of PDE have the capability to illustrate diverse wavelike phenomena, *e.g.*, atmospheric waves, sound waves, electromagnetic and gravitational waves. For many grounds, *e.g.*, modeling sound propagation of sound in a solid, this PDE presents a

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convenient discrete model of a 3-D solid. Solitonary solutions of non-linear evolution equations offer improved understanding. The acquaintance of approximate solution of the non-linear evolution equations smooths the progress of the testing of numerical solvers, support the study of stability analysis and concludes with a better understanding that are modeled by these equations [2]. Accurate numerical solution for these problems is helpful for their application but there are many challenges are ssociated with computing these solutions especially in 3-D cases. Furthermore, finding the exact solution for non-linear PDE is very tough. Consequently, numerical methods are very supportive of solving these equations. Appropriate numerical techniques are always required to get more accurate results in this context, Soysal *et al.* [3] presented a study to improve the design of blades used to distribute the fertiliter in the wide area field. The discrete element method is utilied for the numerical simulation of their new design and successfully verified by experiment. Another study presented in provides the instantaneous data study for the vibration of the harrow disc in the farmland [4].

The PDE that illustrate, the non-linear waves related to the Sine Gordon equation has substantial significance in the research. The Sine Gordon equation emerges in the propagation of fluxions in dislocations in crystals, Josephson junctions and in non-linear optics *etc*. [5-7]. In this paper, we are considering one of the non linear equations in the domain, the Gordon equation, that has been persistently explored and methodically solved in recent years [8-12] particularly in three space dimensions.

We consider a three space-dimensional Sine Gordon equation:

$$\frac{\partial^2 u}{\partial t^2} = c^2\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right) - \sin(u), \quad t>0 \tag{1}$$

where the solution domain is given by:

$$R = \{(x,y,z); L_{x0.} < x < L_{x1}, L_{y0} < y < L_{y1}, L_{z0} < z < L_{z1}\}$$

along with initial conditions:

$$\begin{aligned} u(x,y,z,0) &= \varphi(x,y,z) \\ u_t(x,y,z,0) &= \psi(x,y,z) \end{aligned} \tag{2}$$

subject to the boundary conditions:

$$\begin{aligned} u(L_{0x},y,z,t) &= f_{L_{x0}}(y,z,t), \quad u(L_{1x},y,z,t) = f_{L_{x1}}(y,z,t) \\ u(x,L_{0y},z,t) &= f_{L_{y0}}(x,z,t), \quad u(x,L_{1y},z,t) = f_{L_{y1}}(x,z,t) \\ u(x,y,L_{0,z},t) &= f_{L_{z0}}(x,y,t), \quad u(x,y,L_{1z},t) = f_{L_{z1}}(x,y,t) \end{aligned} \tag{3}$$

where $u(x,y,z,t)$ is a function and c^2 is a constant (as the speed of a wave). The φ and ψ given in the initial condition are time independent but differentiable functions.

Furthermore, Johnson *et al.* [13] dealt with the 2-D Sine Gordon equation and presented three exact solutions. Chen and Lin [14] developed another exact solution for 2-D Sine Gordon equations. The Sine Gordon equations and it exact solution are available in literature [15-17]. Although exact solution provide brief description of physical phenomena to concomitant with the problem discussed, numerical methods are still indispensable.

Several numerical techniques have been originated throughout the preceding three decades mto obtain the solution for the Sine Gordon equation. The Local Kriging Meshless Method is developed by Guo *et al.* [18] to solution the non-linear Sine Gordon equation in two dimensions. Djidjeli *et al.* [19] produced a precise numerical technique to solve a damped Sine Gordon equation in two space variables.

Singh S., *et al.* [20] present a colocation method constructed with the help of the cubic Bspline function for 1 D Sine Gordon Equation. P. Guo *et al.* [18] present the local Kriging meshless method for (2 + 1) - dimensional nonlinear Sine Gordon equation. Ma and Wu [21] employed a meshless technique using a multi-quadric (M Q) quasi-interpolation. A. Akgul *et al.* [22] industrialized a new approach that gives a solution to the one-dimensional Sine Gordon equation. Wazwaz [23] utilized the basic form of Hirota's method to obtain one and two soliton solutions for each equation while exploring the nonlinear Sine Gordon equation in one-, two- and three-dimensions. Furthermore, wazwaz [24] also developed accurate traveling wave solutions for the double Sine Gordon equation. He used the variable separated ODE and Tanh methods to derive the generalized form.M. Dehghan *et al*. [25] presented the numerical solution of the two-dimensional Sine Gordon equation via three meshless methods. The Sine Gordon equation has been extensively employed in various sciences, for instance, non-commutative field theories, fluid dynamics, integrable quantum field theory and kink dynamics. Moreover, the Sine Gordon equation emerges in minimal surfaces in product spaces and the theory of constant mean curvature surfaces in space forms [26-28].

Two finite difference schemes, *i.e.* implicit and explicit were proposed by Guo *et al.* [29]. The cubic B-spline collocation method was adopted by Mittal and Bhatia [30] to get a solution for 1-D Sine Gordon equation. To examine 1-D Burgers' equation, a novel approach was proposed by Arora and Singh [31] which is the "modified cubic B-spline differential quadrature method (MCB-DQM)". Ilati and Dehghan [32] demonstrate an appropriate techniques based on collocation method to elucidate the system of coupled non-linear Sine Gordon equation system depends upon two space independent variables. Zagvan and Rashidinia [33] present the numerical solution to 2-D linear wave equation with the help of non-polynomial spline functions.

Non-polynomial cubic spline functions

The domain set *R* is divided into *N* subintervals in each space direction and *J* steps in time. So that, we got $(N+1)(N+1)(N+1)J$ meshs. Each grid-point is represented as $(x_\alpha, y_ß, z_\mu, t_j)$, where:

$$x_\alpha = L_{0x} + \alpha h; \quad h = \frac{L_{1x} - L_{0x}}{N+1}; \quad \alpha = 0,1,\cdots,N+1$$

$$y_\beta = L_{0y} + \beta k; \quad k = \frac{L_{1y} - L_{0y}}{N+1}; \quad \beta = 0,1,\cdots,N+1$$

$$z_\mu = L_{0z} + \mu\sigma; \quad \sigma = \frac{L_{1z} - L_{0z}}{N+1}; \quad \mu = 0,1,\cdots,N+1$$

and $\quad t_j = j\tau; \quad 0 \le j \le J; \quad N; \quad J \int Z^+$

Let $s_{1\beta,\mu}(x),\ s_{2\alpha,\mu}(y),\ s_{3\alpha,\beta}(z)$ be the non-polynomial spline function for each sub-interval in *x*-, *y*-, and *z*-directions respectively, and defined:

$$s_{1\beta,\mu}(x) = a_{1\alpha} + b_{1\alpha}(x - x_\alpha) + c_{1\alpha}\sin\lambda_1(x - x_\alpha) + d_{1\alpha}\cos\lambda_1(x - x_\alpha) \tag{4}$$

$$s_{2\alpha,\mu}(y) = a_{2\beta} + b_{2\beta}(y - y_\beta) + c_{2\beta}\sin\lambda_2(y - y_\beta) + d_{2\beta}\cos\lambda_2(y - y_\beta) \tag{5}$$

$$s_{3\alpha,\beta}(z)=a_{3\mu}+b_{3\mu}(z-z_\mu)+c_{3\mu}\sin\lambda_3(z-z_\mu)+d_{3\mu}\cos\lambda_3(z-z_\mu) \tag{6}$$

where $a_{1\alpha},b_{1\alpha},c_{1\alpha},d_{1\alpha},a_{2\beta},b_{2\beta},c_{2\beta},d_{2\beta},a_{3\mu},b_{3\mu},c_{3\mu},d_{3\mu}$ are undetermined coefficients and λ_1, λ_2, λ_3 are arbitrary controlling parameters. By using notations:

$$s_{1\beta,\mu}(x_\alpha)=u_{\alpha,\beta,\mu},\quad s''_{1\beta,\mu}(x_\alpha)=M_{1\alpha,\beta,\mu},\quad s_{2\alpha,\mu}(y_ß)=u_{\alpha,\beta,\mu}$$

$$s''_{2\alpha,\mu}(y_ß)=M_{2\alpha,\beta,\mu},\quad s_{3\alpha,\beta}(z_\mu)=u_{\alpha,\beta,\mu},\quad s''_{3\alpha,\beta}(z_\mu)=M_{3\alpha,\beta,\mu}$$

From the first and second derivative continuity conditions for cubic spline functions $s_{1\beta,\mu}(x)$, $s_{2\alpha,\mu}(y)$, and $s_{3\alpha,\beta}(z)$ at $(x_\alpha,y_\beta,z_\mu,t_j)$, the following consistency relations can be obtained:

$$u_{\alpha-1,\beta,\mu}-2u_{\alpha,\beta,\mu}+u_{\alpha+1,\beta,\mu}=h^2(\alpha_1M_{1\alpha-1,\beta,\mu}+2\beta_1M_{1\alpha,\beta,\mu}+\alpha_1M_{1\alpha+1,\beta,\mu}) \tag{7}$$

$$u_{\alpha,\beta-1,\mu}-2u_{\alpha,\beta,\mu}+u_{\alpha,\beta+1,\mu}=k^2(\alpha_2M_{2\alpha,\beta-1,\mu}+2\beta_2M_{2\alpha,\beta,\mu}+\alpha_2M_{2\alpha,\beta+1,\mu}) \tag{8}$$

$$u_{\alpha,\beta,\mu-1}-2u_{\alpha,\beta,\mu}+u_{\alpha,\beta,\mu+1}=\sigma^2(\alpha_3M_{3\alpha,\beta,\mu-1}+2\beta_3M_{3\alpha,\beta,\mu}+\alpha_3M_{3\alpha,\beta,\mu+1}) \tag{9}$$

where

$$\alpha_1=\frac{1}{h^2\lambda_1^2}(h\lambda_1\csc h\lambda_1-1),\quad \alpha_2=\frac{1}{k^2\lambda_2^2}(k\lambda_2\csc k\lambda_2-1)$$

$$\alpha_3=\frac{1}{\sigma^2\lambda_3^2}(\sigma\lambda_3\csc h\lambda_3-1)$$

$$\beta_1=\frac{1}{h^2\lambda_1^2}(1-h\lambda_1\cot h\lambda_1),\quad \beta_2=\frac{1}{k^2\lambda_2^2}(1-k\lambda_2\cot k\lambda_2),\quad \beta_3=\frac{1}{\sigma^2\lambda_3^2}(1-\sigma\lambda_3\cot\sigma\lambda_3)$$

Spline numerical method

In this section we develop an approximation scheme for eq. (1), which may be discretized at the grid point $(x_\alpha,y_ß,z_\mu,t_j)$:

$$u_{tt}\big|_{\alpha,ß,\mu}^{j}=c^2\left(u_{xx}\big|_{\alpha,\beta,\mu}^{j}+u_{yy}\big|_{\alpha,\beta,\mu}^{j}+u_{zz}\big|_{\alpha,\beta,\mu}^{j}\right)-\sin(u_{\alpha,\beta,\mu}^{j}) \tag{10}$$

we use finite difference approximation for the time derivative term in eq. (10):

$$u_{tt}\big|_{\alpha,\beta,\mu}^{j}=\frac{u_{\alpha,\beta,\mu}^{j-1}-2u_{\alpha,\beta,\mu}^{j}+u_{\alpha,\beta,\mu}^{j+1}}{\tau^2}+O(\tau^2) \tag{11}$$

in addition to the non-polynomial cubic spline function for the equation's spatial derivative terms (10):

$$\begin{aligned}
u_{xx}\big|_{\alpha,\beta,\mu}^{j}&=M_{1\alpha,\beta,\mu}^{j}+O(h^2)\\
u_{yy}\big|_{\alpha,\beta,\mu}^{j}&=M_{2\alpha,\beta,\mu}^{j}+O(k^2)\\
u_{zz}\big|_{\alpha,\beta,\mu}^{j}&=M_{3\alpha,\beta,\mu}^{j}+O(\sigma^2)
\end{aligned} \tag{12}$$

Using eqs. (11)-(12), and ignoring the truncation errors, eq. (10) might be written:

$$\frac{u_{\alpha,\beta,\mu}^{j-1}-2u_{\alpha,\beta,\mu}^{j}+u_{\alpha,\beta,\mu}^{j+1}}{\tau^2}=c^2(M_{1\alpha,\beta,\mu}^{j}+M_{2\alpha,\beta,\mu}^{j}+M_{3\alpha,\beta,\mu}^{j})-\sin(u_{\alpha,\beta,\mu}^{j})$$

$$M_{1\alpha,\beta,\mu}^{j}+M_{2\alpha,\beta,\mu}^{j}+M_{3\alpha,\beta,\mu}^{j}=\frac{u_{\alpha,\beta,\mu}^{j-1}-2u_{\alpha,\beta,\mu}^{j}+u_{\alpha,\beta,\mu}^{j+1}}{c^2\tau^2}+\frac{1}{c^2}\sin(u_{\alpha,\beta,\mu}^{j}) \tag{13}$$

This is the consistency relation for a grid point $(x_\alpha, y_ß, z_\mu, t_j)$. For the entire domain in each direction, we can write:

$$M_{1\alpha\pm1,\beta,\mu}^{j}+M_{2\alpha\pm1,\beta,\mu}^{j}+M_{3\alpha\pm1,\beta,\mu}^{j}=\frac{u_{\alpha\pm1,\beta,\mu}^{j-1}-2u_{\alpha\pm1,\beta,\mu}^{j}+u_{\alpha\pm1,\beta,\mu}^{j+1}}{c^2\tau^2}+\frac{1}{c^2}\sin(u_{\alpha\pm1,\beta,\mu}^{j}) \tag{14}$$

$$M_{1\alpha,\beta\pm1,\mu}^{j}+M_{2\alpha,\beta\pm1,\mu}^{j}+M_{3\alpha,\beta\pm1,\mu}^{j}=\frac{u_{\alpha,\beta\pm1,\mu}^{j-1}-2u_{\alpha,\beta\pm1,\mu}^{j}+u_{\alpha,\beta\pm1,\mu}^{j+1}}{c^2\tau^2}+\frac{1}{c^2}\sin(u_{\alpha,\beta\pm1,\mu}^{j}) \tag{15}$$

$$M_{1\alpha,\beta,\mu\pm1}^{j}+M_{2\alpha,\beta,\mu\pm1}^{j}+M_{3\alpha,\beta,\mu\pm1}^{j}=\frac{u_{\alpha,\beta,\mu\pm1}^{j-1}-2u_{\alpha,\beta,\mu\pm1}^{j}+u_{\alpha,\beta,\mu\pm1}^{j+1}}{c^2\tau^2}+\frac{1}{c^2}\sin(u_{\alpha,\beta,\mu\pm1}^{j}) \tag{16}$$

From the consistency relation (7) and (14) we can have nine equations at each grid point of our entire domain. Similarly, for the consistency relation (8) and (15), we can have another nine equations at each grid point of our entire domain. The third set of the next nine equations can be obtained from eqs. (9) and (16).

The previously said set of equations is multiplied by suitable coefficients and add them, then simplified equation can be written as:

$$p_1u_{\alpha-1,\beta-1,\mu-1}^{j+1}+p_2u_{\alpha-1,\beta-1,\mu}^{j+1}+p_1u_{\alpha-1,\beta-1,\mu+1}^{j+1}+p_3u_{\alpha-1,\beta,\mu-1}^{j+1}+$$

$$+p_5u_{\alpha-1,\beta,\mu}^{j+1}+p_3u_{\alpha-1,\beta,\mu+1}^{j+1}+p_1u_{\alpha-1,\beta+1,\mu-1}^{j+1}+p_2u_{\alpha-1,\beta+1,\mu}^{j+1}+p_1u_{\alpha-1,\beta+1,\mu+1}^{j+1}+$$

$$+p_4u_{\alpha,\beta-1,\mu-1}^{j+1}+p_6u_{\alpha,\beta-1,\mu}^{j+1}+p_4u_{\alpha,\beta-1,\mu+1}^{j+1}+p_7u_{\alpha,\beta,\mu-1}^{j+1}+p_8u_{\alpha,\beta,\mu}^{j+1}+p_7u_{\alpha,\beta,\mu+1}^{j+1}+$$

$$+p_4u_{\alpha,\beta+1,\mu-1}^{j+1}+p_6u_{\alpha,\beta+1,\mu}^{j+1}+p_4u_{\alpha,\beta+1,\mu+1}^{j+1}+p_1u_{\alpha+1,\beta-1,\mu-1}^{j+1}+p_2u_{\alpha+1,\beta-1,\mu}^{j+1}+$$

$$+p_1u_{\alpha+1,\beta-1,\mu+1}^{j+1}+p_3u_{\alpha+1,\beta,\mu-1}^{j+1}+p_5u_{\alpha+1,\beta,\mu}^{j+1}+p_3u_{\alpha+1,\beta,\mu+1}^{j+1}+p_1u_{\alpha+1,\beta+1,\mu-1}^{j+1}+$$

$$+p_2u_{\alpha+1,\beta+1,\mu}^{j+1}+p_1=-p_1u_{\alpha-1,\beta-1,\mu-1}^{j-1}-p_2u_{\alpha-1,\beta-1,\mu}^{j-1}-p_1u_{\alpha-1,\beta-1,\mu+1}^{j-1}-$$

$$-p_3u_{\alpha-1,\beta,\mu-1}^{j-1}-p_5u_{\alpha-1,\beta,\mu}^{j-1}-p_3u_{\alpha-1,\beta,\mu+1}^{j-1}-p_1u_{\alpha-1,\beta+1,\mu-1}^{j-1}-p_2u_{\alpha-1,\beta+1,\mu}^{j-1}-$$

$$-p_1u_{\alpha-1,\beta+1,\mu+1}^{j-1}-p_4u_{\alpha,\beta-1,\mu-1}^{j-1}-p_6u_{\alpha,\beta-1,\mu}^{j-1}-p_4u_{\alpha,\beta-1,\mu+1}^{j-1}-p_7u_{\alpha,\beta,\mu-1}^{j-1}-p_8u_{\alpha,\beta,\mu}^{j-1}-$$

$$-p_7u_{\alpha,\beta,\mu+1}^{j-1}-p_4u_{\alpha,\beta+1,\mu-1}^{j-1}-p_6u_{\alpha,\beta+1,\mu}^{j-1}-p_4u_{\alpha,\beta+1,\mu+1}^{j-1}-p_1u_{\alpha+1,\beta-1,\mu-1}^{j-1}-$$

$$-p_2u^{j-1}_{\alpha+1,\beta-1,\mu}-p_1u^{j-1}_{\alpha+1,\beta-1,\mu+1}-p_3u^{j-1}_{\alpha+1,\beta,\mu-1}-p_5u^{j-1}_{\alpha+1,\beta,\mu}-p_3u^{j-1}_{\alpha+1,\beta,\mu+1}-p_1u^{j-1}_{\alpha+1,\beta+1,\mu-1}-$$

$$-p_2u^{j-1}_{\alpha+1,\beta+1,\mu}-p_1u^{j-1}_{\alpha+1,\beta+1,\mu+1}+p_{16}u^{j}_{\alpha,\beta,\mu}+p_{13}u^{j}_{\alpha+1,\beta,\mu}+p_{13}u^{j}_{\alpha-1,\beta,\mu}+p_{14}u^{j}_{\alpha,\beta+1,\mu}+$$

$$+p_{14}u^{j}_{\alpha,\beta-1,\mu}+p_{15}u^{j}_{\alpha,\beta,\mu+1}+p_{15}u^{j}_{\alpha,\beta,\mu-1}+p_{10}u^{j}_{\alpha+1,\beta+1,\mu}+p_{10}u^{j}_{\alpha+1,\beta-1,\mu}+p_{10}u^{j}_{\alpha-1,\beta+1,\mu}+$$

$$+p_{10}u^{j}_{\alpha-1,\beta-1,\mu}+p_{12}u^{j}_{\alpha,\beta+1,\mu+1}+p_{12}u^{j}_{\alpha,\beta+1,\mu-1}+p_{12}u^{j}_{\alpha,\beta-1,\mu+1}+p_{12}u^{j}_{\alpha,\beta-1,\mu-1}+$$

$$+p_9u^{j}_{\alpha+1,\beta+1,\mu+1}+p_9u^{j}_{\alpha+1,\beta+1,\mu-1}+p_9u^{j}_{\alpha+1,\beta-1,\mu+1}+p_9u^{j}_{\alpha+1,\beta-1,\mu-1}+p_9u^{j}_{\alpha-1,\beta+1,\mu+1}+$$

$$+p_9u^{j}_{\alpha-1,\beta+1,\mu-1}+p_9u^{j}_{\alpha-1,\beta-1,\mu+1}+p_9u^{j}_{\alpha-1,\beta-1,\mu-1}+p_11u^{j}_{\alpha+1,\beta,\mu+1}+p_11u^{j}_{\alpha+1,\beta,\mu-1}+$$

$$+p_{11}u^{j}_{\alpha-1,\beta,\mu+1}+p_{11}u^{j}_{\alpha-1,\beta,\mu-1}-\tau^2(p_1\sin(u^{j}_{\alpha-1,\beta-1,\mu-1})+p_1\sin(u^{j}_{\alpha-1,\beta-1,\mu+1})+$$

$$+p_1\sin(u^{j}_{\alpha-1,\beta+1,\mu-1})+p_1\sin(u^{j}_{\alpha-1,\beta+1,\mu+1})+p_1\sin(u^{j}_{\alpha+1,\beta-1,\mu-1})+$$

$$+p_1\sin(u^{j}_{\alpha+1,\beta-1,\mu+1})+p_1\sin(u^{j}_{\alpha+1,\beta+1,\mu-1})+p_1\sin(u^{j}_{\alpha+1,\beta+1,\mu+1})+$$

$$+p_2\sin(u^{j}_{\alpha-1,\beta-1,\mu})+p_2\sin(u^{j}_{\alpha-1,\beta+1,\mu})+p_2\sin(u^{j}_{\alpha+1,\beta-1,\mu})+p_2\sin(u^{j}_{\alpha+1,\beta+1,\mu})+$$

$$+p_3\sin(u^{j}_{\alpha-1,\beta,\mu-1})+p_3\sin(u^{j}_{\alpha-1,\beta,\mu+1})+p_3\sin(u^{j}_{\alpha+1,\beta,\mu-1})+p_3\sin(u^{j}_{\alpha+1,\beta,\mu+1})+$$

$$+p_4\sin(u^{j}_{\alpha,\beta-1,\mu-1})+p_4\sin(u^{j}_{\alpha,\beta-1,\mu+1})+p_4\sin(u^{j}_{\alpha,\beta+1,\mu-1})+p_4\sin(u^{j}_{\alpha,\beta+1,\mu+1})+$$

$$+p_5\sin(u^{j}_{\alpha-1,\beta,\mu})+p_5\sin(u^{j}_{\alpha+1,\beta,\mu})+p_6\sin(u^{j}_{\alpha,\beta-1,\mu})+p_6\sin(u^{j}_{\alpha,\beta+1,\mu})+$$

$$+p_7\sin(u^{j}_{\alpha,\beta,\mu-1})+p_7\sin(u^{j}_{\alpha,\beta,\mu+1})+p_8\sin(u^{j}_{\alpha,\beta,\mu})) \tag{17}$$

Where

$$p_1=\alpha_1\alpha_2\alpha_3\quad p_2=2\alpha_1\alpha_2\beta_3,\quad p_3=2\alpha_1\alpha_3\beta_2,\quad p_4=2\alpha_2\alpha_3\beta_1,\quad p_5=4\alpha_1\beta_2\beta_3,$$

$$p_6=4\alpha_2\beta_1\beta_3,\quad p_7=4\alpha_3\beta_1\beta_2,\quad p_8=8\beta_1\beta_2\beta_3,$$

$$p_9=\frac{c^2\tau^2\alpha_1\alpha_2}{\sigma^2}+\frac{c^2\tau^2\alpha_1\alpha_3}{k^2}+\frac{c^2\tau^2\alpha_2\alpha_3}{h^2}+2\alpha_1\alpha_2\alpha_3$$

$$p_{10}=-\frac{2c^2\tau^2\alpha_1\alpha_2}{\sigma^2}+\frac{2c^2\tau^2\alpha_1\alpha_3}{k^2}+\frac{2c^2\tau^2\alpha_2\beta_3}{h^2}+4\alpha_1\alpha_2\beta_3$$

$$p_{11}=-\frac{2c^2\tau^2\alpha_1\alpha_2}{k^2}+\frac{2c^2\tau^2\alpha_1\beta_2}{\sigma^2}+\frac{2c^2\tau^2\alpha_3\beta_2}{h^2}+4\alpha_1\alpha_3\beta_2$$

$$p_{12}=-\frac{2c^2\tau^2\alpha_2\alpha_3}{h^2}+\frac{2c^2\tau^2\alpha_2\beta_1}{\sigma^2}+\frac{2c^2\tau^2\alpha_3\beta_1}{k^2}+4\alpha_2\alpha_3\beta_1$$

$$p_{13} = -\frac{4c^2\tau^2\alpha_1\beta_2}{\sigma^2} + \frac{4c^2\tau^2\alpha_1\beta_3}{k^2} + \frac{4c^2\tau^2\beta_2\beta_3}{h^2} + 8\alpha_1\beta_2\beta_3$$

$$p_{14} = -\frac{4c^2\tau^2\alpha_2\beta_2}{\sigma^2} + \frac{4c^2\tau^2\alpha_2\beta_3}{h^2} + \frac{4c^2\tau^2\beta_1\beta_3}{k^2} + 8\alpha_2\beta_1\beta_3$$

$$p_{15} = -\frac{4c^2\tau^2\alpha_3\beta_1}{k^2} + \frac{4c^2\tau^2\alpha_3\beta_2}{h^2} + \frac{4c^2\tau^2\beta_1\beta_2}{\sigma^2} + 8\alpha_3\beta_1\beta_2$$

$$p_{16} = -\frac{8c^2\tau^2\beta_1\beta_2}{\sigma^2} + \frac{8c^2\tau^2\beta_1\beta_3}{k^2} + \frac{8c^2\tau^2\beta_2\beta_3}{h^2} + 16\beta_1\beta_2\beta_3$$

By using the traditional Taylor series expansion, a third order approximation of $u(x, y, , z, t)$ at $t = \tau$ can be written:

$$u^1_{\alpha,\beta,\mu} = u^0_{\alpha,\beta,\mu} + \tau\, u_t\big|^0_{\alpha,\beta,\mu} + \frac{\tau^2}{2!}u_{tt}\big|^0_{\alpha,\beta,\mu} + \frac{\tau^3}{3!}u_{ttt}\big|^0_{\alpha,\beta,\mu} + O(\tau^4) \tag{18}$$

Since from I.C.'s (2) and (13), we have:

$$u\big|^0_{\alpha,\beta,\mu} = \varphi_{\alpha,\beta,\mu}, \qquad u_t\big|^0_{\alpha,\beta,\mu} = \psi_{\alpha,\beta,\mu} \tag{19}$$

$$u_{tt}\big|^0_{\alpha,\beta,\mu} = c^2\left(u_{xx}\big|^0_{\alpha,\beta,\mu} + u_{yy}\big|^0_{\alpha,\beta,\mu} + u_{zz}\big|^0_{\alpha,\beta,\mu}\right) - \sin(u^0_{\alpha,\beta,\mu}) \tag{20}$$

$$u_{ttt}\big|^0_{\alpha,\beta,\mu} = c^2\left(u_{txx}\big|^0_{\alpha,\beta,\mu} + u_{tyy}\big|^0_{\alpha,\beta,\mu} + u_{tzz}\big|^0_{\alpha,\beta,\mu}\right) - \cos(u^j_{\alpha,\beta,\mu}) \tag{21}$$

using eqs. (19)-(21), in eq. (18), we obtain the approximate solution $u(x, y, z, t)$ at $t = \tau$:

$$u^1_{\alpha,\beta,\mu} = \varphi_{\alpha,\beta,\mu} + \tau\psi_{\alpha,\beta,\mu} + \frac{\tau^2}{2!}\left[c^2\left(\varphi_{xx}\big|^0_{\alpha,\beta,\mu} + \varphi_{yy}\big|^0_{\alpha,\beta,\mu} + \varphi_{zz}\big|^0_{\alpha,\beta,\mu}\right) - \sin(u^0_{\alpha,\beta,\mu})\right] +$$

$$+\frac{\tau^3}{3!}\left[c^2\left(\psi_{xx}\big|^0_{\alpha,\beta\mu} + \psi_{yy}\big|^0_{\alpha,\beta,\mu} + \psi_{zz}\big|^0_{\alpha,\beta,\mu}\right) - \cos(u^j_{\alpha,\beta,\mu})\right] \tag{22}$$

The scheme derived in eq. (17), can be expressed in the following matrix form in order to obtain the solution in each time level:

$$AU^{j+1} = -AU^{j-1} + BU^j - \tau^2 A\sin(U^j), \qquad j = 1, 2, \cdots, J \tag{23}$$

where A and B are blocked tri-diagonal matrices of order N^3 and U is the solution vector, A is a block tri-diagonal matrix of order N in the form $tri\left[\mho_1, \mho_2, \mho_1\right]$, where each $\mho_1$ and $\mho_2$ are again block tri-diagonal matrices of order N in the form $tri\left[u_1, u_2, u_1\right]$, and $tri\left[u_3, u_4, u_3\right]$ respectively. Yet again $u_1 = tri(p_1, p_2, p_1)$, $u_2 = tri(p_3, p_5, p_3)$, $u_3 = tri(p_4, p_6, p_4)$, $u_4 = tri(p_7, p_8, p_7)$ are tri-diagonal matrices of order N. Similarly B is also a nested block tri-diagonal matrix. Each N block in diagonal is a nested block tri-diagonal of order N^2 in the form of $tri[tri(p_{12}, p_{14}, p_{12}),\ tri(p_{15}, p_{16}, p_{15}),\ tri(p_{12}, p_{14}, p_{12})]$. And off-diagonal blocks of order N^2 are written as $tri[tri(p_9, p_{10}, p_9),\ tri(p_{11}, p_{13}, p_{11}),\ tri(p_9, p_{10}, p_9)]$.

Truncation error

From eq. (13), we have:

$$\sin(u^j_{\alpha,\beta,\mu}) = u_{tt}\Big|^j_{\alpha,\beta,\mu} - c^2\left(u_{xx}\Big|^j_{\alpha,\beta,\mu} + u_{yy}\Big|^j_{\alpha,\beta,\mu} + u_{zz}\Big|^j_{\alpha,\beta,\mu}\right) \tag{24}$$

we have:

$$\sin(u^j_{\alpha+\rho,\beta+\eta,\mu+\gamma}) = u_{tt}\Big|^j_{\alpha+\rho,\beta+\eta,\mu+\gamma} - c^2\left(u_{xx}\Big|^j_{\alpha+\rho,\beta+\eta,\mu+\gamma} + u_{yy}\Big|^j_{\alpha+\rho,\beta+\eta,\mu+\gamma} + u_{zz}\Big|^j_{\alpha+\rho,\beta+\eta,\mu+\gamma}\right) \tag{25}$$

where $\rho,\eta,\gamma = 0,\pm 1$. Substituting these values in eq. (17) and then expanding both sides by using the Taylor series in terms of $u^j_{\alpha,\beta,\mu}$, we obtain the truncation error:

$$T^j_{\alpha,\beta,\mu)} = (4c^2\tau^2(-1+2\alpha_1+2\beta_1)(\alpha_2+\beta_2)(\alpha_3+\beta_3)\frac{\partial^2}{\partial x^2} + 4c^2\tau^2(\alpha_1+\beta_1)\cdot$$

$$\cdot(-1+2\alpha_2+2\beta_2)(\alpha_3+\beta_3)\frac{\partial^2}{\partial y^2} + 4c^2\tau^2(\alpha_1+\beta_1)(\alpha_2+\beta_2)(-1+2\alpha_3+2\beta_3)\frac{\partial^2}{\partial z^2} +$$

$$+2c^2\tau^2(k^2\alpha_2(-1+2\beta_1)+\alpha_1(2(h^2+k^2)\alpha_2+h^2(-1+2\beta_2)))(\alpha_3+\beta_3)\frac{\partial^2}{\partial x^2}\frac{\partial^2}{\partial y^2} +$$

$$+2c^2\tau^2(\alpha_2+\beta_2)(\sigma^2\alpha_3(-1+2\beta_1)+\alpha_1(2(h^2+\sigma^2)\alpha_3+h^2(-1+2\beta_3)))\frac{\partial^2}{\partial x^2}\frac{\partial^2}{\partial z^2} +$$

$$+\frac{1}{3}c^2h^2\tau^2(-1+12\alpha_1)(\alpha_2+\beta_2)(\alpha_3+\beta_3)\frac{\partial^4}{\partial x^4} + \frac{1}{3}c^2k^2\beta^2(-1+12\alpha_2)(\alpha_1+\beta_1)$$

$$(\alpha_3+\beta_3)\frac{\partial^4}{\partial x^4} + \frac{1}{3}c^2h^2\tau^2(-1+12\alpha_1)(\alpha_2+\beta_2)(\alpha_3+\beta_3)\frac{\partial^4}{\partial x^4} + \frac{1}{3}c^2k^2\tau^2(-1+12\alpha_2)\cdot$$

$$\cdot(\alpha_1+\beta_1)(\alpha_3+\beta_3)\frac{\partial^4}{\partial y^4} + \frac{1}{3}c^2\sigma^2\tau^2(-1+12\alpha_3)(\alpha_1+\beta_1)(\alpha_2+\beta_2)\frac{\partial^4}{\partial z^4} +$$

$$+\frac{2}{3}\tau^4(\alpha_1+\beta_1)(\alpha_2+\beta_2)(\alpha_3+\beta_3)\frac{\partial^4}{\partial t^4} + \frac{1}{6}c^2h^2\tau^2(-k^2\alpha_2+\alpha_1(2(h^2+6k^2)\alpha_2 +$$

$$+h^2(-1+2\beta_2)))(\alpha_3+\beta_3)\frac{\partial^4}{\partial x^4}\frac{\partial^4}{\partial y^4} + \frac{1}{6}c^2h^2\tau^2(\alpha_2+\beta_2)(-\sigma^2\alpha_3+\alpha_1(2(h^2+6\sigma^2)\alpha_3 +$$

$$+h^2(-1+2\beta_3)))\frac{\partial^4}{\partial x^4}\frac{\partial^4}{\partial z^4} + \frac{1}{6}c^2k^2\tau^2(\alpha_1(-h^2+2(6h^2+k^2)\alpha_2)+k^2\alpha_2(-1+2\beta_1))(\alpha_3+\beta_3)\cdot$$

$$\cdot\frac{\partial^2}{\partial x^2}\frac{\partial^4}{\partial y^4} + \frac{1}{6}c^2\sigma^2\tau^2(\alpha_1(-h^2+2(6h^2+\sigma^2)\alpha_3)+\sigma^2\alpha_3(-1+2\beta_1))(\alpha_2+\beta_2)\cdot$$

$$\cdot\frac{\partial^2}{\partial x^2}\frac{\partial^4}{\partial z^4} + \frac{1}{3}h^2\tau^4\alpha_1(\alpha_2+\beta_2)(\alpha_3+\beta_3)\frac{\partial^2}{\partial x^2}\frac{\partial^4}{\partial t^4} + \frac{1}{6}c^2\sigma^2\tau^2(\alpha_1+\beta_1)\cdot$$

$$\cdot(\alpha_2(-k^2+2(6k^2+\sigma^2)\alpha_3)+\sigma^2\alpha_3(-1+2\beta_2))\frac{\partial^2}{\partial y^2}\frac{\partial^4}{\partial y^4}+\frac{1}{3}k^2\tau^4\alpha_2(\alpha_1+\beta_1)(\alpha_3+\beta_3)\cdot$$

$$\cdot\frac{\partial^2}{\partial y^2}\frac{\partial^4}{\partial t^4}+\frac{1}{3}\sigma^2\tau^4\alpha_3(\alpha_1+\beta_1)(\alpha_2+\beta_2)\frac{\partial^2}{\partial z^2}\frac{\partial^4}{\partial t^4}+\frac{1}{90}c^2h^4\tau^2(-1+30\alpha_1)(\alpha_2+\beta_2)(\alpha_3+\beta_3)\cdot$$

$$\cdot\frac{\partial^6}{\partial x^6}+\frac{1}{90}c^2k^4\tau^2(-1+30\alpha_2)(\alpha_1+\beta_1)(\alpha_3+\beta_3)\frac{\partial^6}{\partial y^6}+\frac{1}{6}c^2\sigma^4\tau^2\alpha_3(2\beta_1(\alpha_2+\beta_2)+$$

$$+\alpha_1(\alpha_2+2\beta_2))\frac{\partial^6}{\partial z^6}+\frac{1}{80}c^2h^4k^2\tau^2(-1+30\alpha_1)\alpha_2(\alpha_3+\beta_3)\frac{\partial^6}{\partial x^6}\frac{\partial^6}{\partial y^6}+$$

$$+\frac{1}{180}c^2h^4\sigma^2\tau^2(-1+30\alpha_1)\alpha_3(\alpha_2+\beta_2)\frac{\partial^6}{\partial x^6}\frac{\partial^6}{\partial z^6}+\frac{1}{180}c^2k^4\sigma^2\tau^2$$

$$(-1+30\alpha_2)\alpha_3(\alpha_1+\beta_1)\frac{\partial^6}{\partial y^6}\frac{\partial^2}{\partial z^2}+...\}u^j_{(\alpha,\beta,\mu)} \tag{26}$$

On selecting appropriate values of parameters $\alpha_1,\alpha_2,\alpha_3,\beta_1,\beta_2,$ and $\beta_3,$ methods of different classes can be obtained.

If we choose:

$$\alpha_1+\beta_1=\frac{1}{2},\quad \alpha_2+\beta_2=\frac{1}{2},\quad \alpha_3+\beta_3=\frac{1}{2} \tag{27}$$

i. We obtain a schemes of order $O(h^2+k^2+\sigma^2+\tau^2h^2+\tau^2k^2+\tau^2\sigma^2)$. Specifically if we can choose $\alpha_1=\alpha_2=\alpha_3=1/6$ and $\beta_1=\beta_2=\beta_3=1/3$

ii. A schemes of order $O(h^4+k^4+\sigma^4+\tau^2h^2+\tau^2k^2+\tau^2\sigma^2)$ is obtained by choosing

$$\alpha_1=\alpha_2=\alpha_3=1/12\quad\text{and}\quad\beta_1=\beta_2=\beta_3=5/12$$

iii. A schemes of order $O(h^6+k^6+\sigma^6+\tau^2h^2+\tau^2k^2+\tau^2\sigma^2)$ is obtained by choosing

$$\alpha_1=\alpha_2=\alpha_3=1/30\quad\text{and}\quad\beta_1=\beta_2=\beta_3=7/15$$

iv. The scheme is convergent of order $O(h^8+k^8+\sigma^8+\tau^2h^2+\tau^2k^2+\tau^2\sigma^2)$ on choosing $\alpha_1=\alpha_2=\alpha_3=1/56$ and $\beta_1=\beta_2=\beta_3=27/56,$ and so on.

Stability analysis

The stability analysis of the scheme developed in eq. (17) is discussed here, for this we consider the homogeneous part of the scheme (17), then we can have:

$$C_1Y^{j+1}_{\alpha-1,\beta-1,\mu-1}+C_4Y^{j+1}_{\alpha,\beta-1,\mu-1}+C_1Y^{j+1}_{\alpha+1,\beta-1,\mu-1}+C_1Y^{j+1}_{\alpha-1,\beta-1,\mu+1}+C_4Y^{j+1}_{\alpha,\beta-1,\mu+1}+$$

$$+C_1Y^{j+1}_{\alpha+1,\beta-1,\mu+1}+C_1Y^{j+1}_{\alpha-1,\beta+1,\mu-1}+C_4Y^{j+1}_{\alpha,\beta+1,\mu-1}+C_1Y^{j+1}_{\alpha+1,\beta+1,\mu-1}+C_1Y^{j+1}_{\alpha-1,\beta+1,\mu+1}+$$

$$+C_4Y^{j+1}_{\alpha,\beta+1,\mu+1}+C_1Y^{j+1}_{\alpha+1,\beta+1,\mu+1}+C_2Y^{j+1}_{\alpha-1,\beta-1,\mu}+C_6Y^{j+1}_{\alpha,\beta-1,\mu}+C_2Y^{j+1}_{\alpha+1,\beta-1,\mu}+$$

$$+C_2Y^{j+1}_{\alpha-1,\beta+1,\mu}+C_6Y^{j+1}_{\alpha,\beta+1,\mu}+C_2Y^{j+1}_{\alpha+1,\beta+1,\mu}+C_3Y^{j+1}_{\alpha-1,\beta,\mu-1}+C_7Y^{j+1}_{\alpha,\beta,\mu-1}+C_3Y^{j+1}_{\alpha+1,\beta,\mu-1}+$$

$$+C_3Y^{j+1}_{\alpha-1,\beta,\mu+1}+C_7Y^{j+1}_{\alpha,\beta,\mu+1}+C_3Y^{j+1}_{\alpha+1,\beta,\mu+1}+C_5Y^{j+1}_{\alpha-1,\beta,\mu}+C_7Y^{j+1}_{\alpha,\beta,\mu}+C_5Y^{j+1}_{\alpha+1,\beta,\mu}=$$
$$=D_1Y^{j}_{\alpha-1,\beta-1,\mu-1}+D_4Y^{j}_{\alpha,\beta-1,\mu-1}+D_1Y^{j}_{\alpha+1,\beta-1,\mu-1}+D_1Y^{j}_{\alpha-1,\beta-1,\mu+1}+D_4Y^{j}_{\alpha,\beta-1,\mu+1}+$$
$$+D_1Y^{j}_{\alpha+1,\beta-1,\mu+1}+D_1Y^{j}_{\alpha-1,\beta+1,\mu-1}+D_4Y^{j}_{\alpha,\beta+1,\mu-1}+D_1Y^{j}_{\alpha+1,\beta+1,\mu-1}+D_1Y^{j}_{\alpha-1,\beta+1,\mu+1}+$$
$$+D_4Y^{j}_{\alpha,\beta+1,\mu+1}+D_1Y^{j}_{\alpha+1,\beta+1,\mu+1}+D_2Y^{j}_{\alpha-1,\beta-1,\mu}+D_6Y^{j}_{\alpha,\beta-1,\mu}+D_2Y^{j}_{\alpha+1,\beta-1,\mu}+D_2Y^{j}_{\alpha-1,\beta+1,\mu}+$$
$$+D_6Y^{j}_{\alpha,\beta+1,\mu}+D_2Y^{j}_{\alpha+1,\beta+1,\mu}+D_3Y^{j}_{\alpha-1,\beta,\mu-1}+D_7Y^{j}_{\alpha,\beta,\mu-1}+D_3Y^{j}_{\alpha+1,\beta,\mu-1}+D_3Y^{j}_{\alpha-1,\beta,\mu+1}+$$
$$+D_7Y^{j}_{\alpha,\beta,\mu+1}+D_3Y^{j}_{\alpha+1,\beta,\mu+1}+D_5Y^{j}_{\alpha-1,\beta,\mu}+D_8Y^{j}_{\alpha,\beta,\mu}+D_5Y^{j}_{\alpha+1,\beta,\mu} \tag{28}$$

where

$$Y^{j}_{\alpha,\beta,\mu}=(u^{j}_{\alpha,\beta,\mu},v^{j}_{\alpha,\beta,\mu})^T \text{ and } u^{j-1}_{\alpha,\beta,\mu}=v^{j}_{\alpha,\beta,\mu}$$

$$C_1=\begin{pmatrix}p_1 & 0\\ 0 & 1\end{pmatrix},\quad C_2=\begin{pmatrix}p_2 & 0\\ 0 & 1\end{pmatrix},\quad C_3=\begin{pmatrix}p_3 & 0\\ 0 & 1\end{pmatrix},\quad C_4=\begin{pmatrix}p_4 & 0\\ 0 & 1\end{pmatrix},\quad C_5=\begin{pmatrix}p_5 & 0\\ 0 & 1\end{pmatrix}$$
$$C_6=\begin{pmatrix}p_6 & 0\\ 0 & 1\end{pmatrix},\quad C_7=\begin{pmatrix}p_7 & 0\\ 0 & 1\end{pmatrix},\quad C_8=\begin{pmatrix}p_8 & 0\\ 0 & 1\end{pmatrix}$$
$$D_1=\begin{pmatrix}p_9 & -p_1\\ 1 & 0\end{pmatrix},\quad D_2=\begin{pmatrix}p_{10} & -p_2\\ 1 & 0\end{pmatrix},\quad D_3=\begin{pmatrix}p_{11} & -p_3\\ 1 & 0\end{pmatrix},\quad D_4=\begin{pmatrix}p_{12} & -p_4\\ 1 & 0\end{pmatrix}$$
$$D_5=\begin{pmatrix}p_{13} & -p_5\\ 1 & 0\end{pmatrix},\quad D_6=\begin{pmatrix}p_{14} & -p_6\\ 1 & 0\end{pmatrix},\quad D_7=\begin{pmatrix}p_{15} & -p_7\\ 1 & 0\end{pmatrix},\quad D_8=\begin{pmatrix}p_{16} & -p_8\\ 1 & 0\end{pmatrix}$$

Let $\bar{Y}^{j}_{\alpha,\beta,\mu}$ be the numerical value of $Y^{j}_{\alpha,\beta,\mu}$ then $\varepsilon^{j}_{\alpha,\beta,\mu}=Y^{j}_{\alpha,\beta,\mu}-\bar{Y}^{j}_{\alpha,\beta,\mu}$ is the error vector at the j^{th} time level. Thus from eq. (28) we can write:

$$C_1\varepsilon^{j+1}_{\alpha-1,\beta-1,\mu-1}+C_4\varepsilon^{j+1}_{\alpha,\beta-1,\mu-1}+C_1\varepsilon^{j+1}_{\alpha+1,\beta-1,\mu-1}+C_1\varepsilon^{j+1}_{\alpha-1,\beta-1,\mu+1}+C_4\varepsilon^{j+1}_{\alpha,\beta-1,\mu+1}+C_1\varepsilon^{j+1}_{\alpha+1,\beta-1,\mu+1)}+$$
$$+C_1\varepsilon^{j+1}_{\alpha-1,\beta+1,\mu-1}+C_4\varepsilon^{j+1}_{\alpha,\beta+1,\mu-1}+C_1\varepsilon^{j+1}_{\alpha+1,\beta+1,\mu-1}+C_1\varepsilon^{j+1}_{\alpha-1,\beta+1,\mu+1}+C_4\varepsilon^{j+1}_{\alpha,\beta+1,\mu+1}+$$
$$+C_1\varepsilon^{j+1}_{\alpha+1,\beta+1,\mu+1}+C_2\varepsilon^{j+1}_{\alpha-1,\beta-1,\mu}+C_6\varepsilon^{j+1}_{\alpha,\beta-1,\mu}+C_2\varepsilon^{j+1}_{\alpha+1,\beta-1,\mu}+C_2\varepsilon^{j+1}_{\alpha-1,\beta+1,\mu}+C_6\varepsilon^{j+1}_{\alpha,\beta+1,\mu}+$$
$$+C_2\varepsilon^{j+1}_{\alpha+1,\beta+1,\mu}+C_3\varepsilon^{j+1}_{\alpha-1,\beta,\mu-1}+C_7\varepsilon^{j+1}_{\alpha,\beta,\mu-1}+C_3\varepsilon^{j+1}_{\alpha+1,\beta,\mu-1}+C_3\varepsilon^{j+1}_{\alpha-1,\beta,\mu+1}+C_7\varepsilon^{j+1}_{\alpha,\beta,\mu+1}+$$
$$+C_3\varepsilon^{j+1}_{\alpha+1,\beta,\mu+1}+C_5\varepsilon^{j+1}_{\alpha-1,\beta,\mu}+C_7\varepsilon^{j+1}_{\alpha,\beta,\mu}+C_5\varepsilon^{j+1}_{\alpha+1,\beta,\mu}=$$
$$=D_1\varepsilon^{j}_{\alpha-1,\beta-1,\mu-1}+D_4\varepsilon^{j}_{\alpha,\beta-1,\mu-1}+D_1\varepsilon^{j}_{\alpha+1,\beta-1,\mu-1}+D_1\varepsilon^{j}_{\alpha-1,\beta-1,\mu+1}+D_4\varepsilon^{j}_{\alpha,\beta-1,\mu+1}+$$
$$+D_1\varepsilon^{j}_{\alpha+1,\beta-1,\mu+1}+D_1\varepsilon^{j}_{\alpha-1,\beta+1,\mu-1}+D_4\varepsilon^{j}_{\alpha,\beta+1,\mu-1}+D_1\varepsilon^{j}_{\alpha+1,\beta+1,\mu-1}+D_1\varepsilon^{j}_{\alpha-1,\beta+1,\mu+1}+$$
$$+D_4\varepsilon^{j}_{\alpha,\beta+1,\mu+1}+D_1\varepsilon^{j}_{\alpha+1,\beta+1,\mu+1}+D_2\varepsilon^{j}_{\alpha-1,\beta-1,\mu}+D_6\varepsilon^{j}_{\alpha,\beta-1,\mu}+D_2\varepsilon^{j}_{\alpha+1,\beta-1,\mu}+$$
$$+D_2\varepsilon^{j}_{\alpha-1,\beta+1,\mu}+D_6\varepsilon^{j}_{\alpha,\beta+1,\mu}+D_2\varepsilon^{j}_{\alpha+1,\beta+1,\mu}+D_3\varepsilon^{j}_{\alpha-1,\beta,\mu-1}+D_7\varepsilon^{j}_{\alpha,\beta,\mu-1}+D_3\varepsilon^{j}_{\alpha+1,\beta,\mu-1}+$$

$$+D_3\varepsilon^j_{\alpha-1,\beta,\mu+1}+D_7\varepsilon^j_{\alpha,\beta,\mu+1}+D_3\varepsilon^j_{\alpha+1,\beta,\mu+1}+D_5\varepsilon^j_{\alpha-1,\beta,\mu}+D_8\varepsilon^j_{\alpha,\beta,\mu}+D_5\varepsilon^j_{\alpha+1,\beta,\mu} \tag{29}$$

Let the solution of eq. (29) at the grid point $(x_\alpha, y_\beta, z_\mu, t_j)$ be of the form:

$$\varepsilon^j_{\alpha,\beta,\mu}=\xi^j e^{i(\theta_1\alpha+\theta_2\beta+\theta_3\mu)} \tag{30}$$

where $=\sqrt{-1}$, $\theta_1,\theta_2,\theta_3$ are real phase angles and ξ is in general complex. Substituting eq. (28) into eq. (27) and using Euler identity $e^{i\theta}=\cos\theta+i\sin\theta,$ after simplification we get:

$$\xi\begin{pmatrix}Q_1 & 0\\ 0 & Q_2\end{pmatrix}=\begin{pmatrix}Q_3 & -Q_1\\ Q_2 & 0\end{pmatrix} \tag{31}$$

where

$$Q_1=8(\alpha_1\cos\theta_1+\beta_1)(\alpha_2\cos\theta_2+\beta_2)(\alpha_3\cos\theta_3+\beta_3)$$

$$Q_2=(1+2\cos\theta_1)(1+2\cos\theta_2)(1+2\cos\theta_3)$$

$$\begin{aligned}Q_3=\frac{8}{h^2k^2\sigma^2}(&-c^2h^2\sigma^2\tau^2\alpha_3\beta_1\cos\theta_3+c^2h^2\sigma^2\tau^2\alpha_3\beta_1\cos\theta_2\cos\theta_3-c^2k^2\sigma^2\tau^2\alpha_3\beta_2\cos\theta_3+\\&+c^2k^2\sigma^2\tau^2\alpha_3\beta_2\cos\theta_1\cos\theta_3-c^2h^2k^2\tau^2\beta_1\beta_2+c^2h^2k^2\tau^2\beta_1\beta_2\cos\theta_3+\\&+2h^2k^2\sigma^2\alpha_3\beta_1\beta_2\cos\theta_3+\sigma^2(c^2k^2\tau^2\beta_2(-1+\cos\theta_1)+h^2\beta_1(c^2\tau^2(-1+\cos\theta_2)+2k^2\beta_2))\beta_3+\\&+k^2\alpha_2\cos\theta_2(\sigma^2\cos\theta_3\alpha_3(c^2\tau^2(-1+\cos\theta_1)+2h^2\beta_1)+c^2\sigma^2\tau^2(-1+\cos\theta_1)\beta_3+\\&+h^2\beta_1(c^2\tau^2(-1+\cos\theta_3)+2\sigma^2\beta_3))+h^2\cos\theta_1\alpha_1(c^2k^2\tau^2(-1+\cos\theta_3)\beta_2+\\&+\sigma^2\alpha_3\cos\theta_3(c^2\tau^2(-1+\cos\theta_2)+2k^2\beta_2)+\sigma^2(c^2\tau^2(-1+\cos\theta_2)+2k^2\beta_2)\beta_3+\\&+k^2\alpha_2\cos\theta_2(c^2\tau^2(-1+\cos\theta_3)+2\sigma^2(\cos\theta_3\alpha_3+\beta_3))))\end{aligned}$$

Now we find the amplification matrix of the scheme developed in eq. (17):

$$G=\begin{pmatrix}Q_1 & 0\\ 0 & Q_2\end{pmatrix}^{-1}\begin{pmatrix}Q_3 & -Q_1\\ Q_2 & 0\end{pmatrix}=$$

$$=\begin{pmatrix}\left\{2-2c^2\tau^2\left(\left[\dfrac{\sin^2\frac{\theta_1}{2}}{h^2(\alpha_1\cos\theta_1+\beta_1)}+\dfrac{\sin^2\frac{\theta_2}{2}}{k^2(\alpha_2\cos\theta_2+\beta_2)}+\dfrac{\sin^2\frac{\theta_3}{2}}{\sigma^2(\alpha_3\cos\theta_3+\beta_3)}\right]\right)\right\} & -1\\ 1 & 0\end{pmatrix} \tag{32}$$

The following equation is satisfied by the eigenvalues of the matrix G:

$$\lambda^2-2b\lambda+1=0 \tag{33}$$

here $b=1-c^2\tau^2\left[\dfrac{\sin^2\frac{\theta_1}{2}}{h^2(\alpha_1\cos\theta_1+\beta_1)}+\dfrac{\sin^2\frac{\theta_2}{2}}{k^2(\alpha_2\cos\theta_2+\beta_2)}+\dfrac{\sin^2\frac{\theta_3}{2}}{\sigma^2(\alpha_3\cos\theta_3+\beta_3)}\right]$

Using the transformation $\lambda=(1+z)/(1-z)$, eq. (33) takes the form:

$$(2+b)z^2+(2-2\text{b})=0 \tag{34}$$

The previous transformation is used to map a unit circle on the left side of the plane. Therefore, the stability criterion $|\lambda|<1$, will be satisfied, when $|b|<1$.

Numerical testing

A thorough discussion of the Sine Gordon equation is presented in this section. We apply our proposed non-polynomial cubic spline methods and L_2 and L_∞ errors are tabulated in the table. The three-dimensional surface plot is also shown in the fig. 1.

Consider a 3-D linear Sine Gordon equation:

$$u_{tt}=u_{xx}+u_{yy}+u_{zz}-\sin(u), \quad -500<(x,y,z)<500, \quad t>0 \tag{35}$$

Along with the initial conditions:

$$u(x,y,z,0)=\varphi(x,y,z)=4^3\tan^{-1}\left[\frac{\mu}{\sqrt{1+\mu^2}}\sin\left(\sqrt{1}\right)\operatorname{sech}(\mu x)\right]$$

$$\tan^{-1}\left[\frac{\mu}{\sqrt{1+\mu^2}}\sin\left(\sqrt{1}\right)\operatorname{sech}(\mu y)\right]\tan^{-1}\left[\frac{\mu}{\sqrt{1+\mu^2}}\sin\left(\sqrt{1}\right)\operatorname{sech}(\mu z)\right]$$

$$u_t(x,y,z,0)=\psi(x,y,z)=\frac{-2^5\mu^3\cos\sqrt{1-\mu^2 t}}{\sqrt{1-\mu^2 t}\sqrt{1+\mu^2}}$$

$$\frac{\operatorname{sech}(\mu x)\left(\tan^{-1}\left(\frac{\mu\operatorname{sech}(\mu y)}{\sqrt{1+\mu^2}}\sin\left(\sqrt{1-\mu^2}t\right)\right)\right)\left(\tan^{-1}\left(\frac{\mu\operatorname{sech}(\mu z)}{\sqrt{1+\mu^2}}\sin\left(\sqrt{1-\mu^2}t\right)\right)\right)}{1+\left(\frac{\mu\operatorname{sech}(\mu x)}{\sqrt{1+\mu^2}}\sin\left(\sqrt{1-\mu^2}t\right)\right)^2}+$$

$$+\frac{\operatorname{sech}(\mu y)\left(\tan^{-1}\left(\frac{\mu\operatorname{sech}(\mu x)}{\sqrt{1+\mu^2}}\sin\left(\sqrt{1-\mu^2}t\right)\right)\right)\left(\tan^{-1}\left(\frac{\mu\operatorname{sech}(\mu z)}{\sqrt{1+\mu^2}}\sin\left(\sqrt{1-\mu^2}t\right)\right)\right)}{1+\left(\frac{\mu\operatorname{sech}(\mu y)}{\sqrt{1+\mu^2}}\sin\left(\sqrt{1-\mu^2}t\right)\right)^2}+$$

$$+\frac{\operatorname{sech}(\mu z)\left(\tan^{-1}\left(\frac{\mu\operatorname{sech}(\mu x)}{\sqrt{1+\mu^2}}\sin\left(\sqrt{1-\mu^2}t\right)\right)\right)\left(\tan^{-1}\left(\frac{\mu\operatorname{sech}(\mu y)}{\sqrt{1+\mu^2}}\sin\left(\sqrt{1-\mu^2}t\right)\right)\right)}{1+\left(\frac{\mu\operatorname{sech}(\mu z)}{\sqrt{1+\mu^2}}\sin\left(\sqrt{1-\mu^2}t\right)\right)^2}$$

Boundary conditions are determined from the exact solution.

Analytic solution of the previously discussed problem is taken from [31]:

$$u(x,y,z,t)=4^3\tan^{-1}\left[\frac{\mu}{\sqrt{1+\mu^2}}\sin\left(\sqrt{1-\mu^2 t}\right)\operatorname{sech}(\mu x)\right]$$

$$\cdot\tan^{-1}\left[\frac{\mu}{\sqrt{1+\mu^2}}\sin\left(\sqrt{1-\mu^2 t}\right)\operatorname{sech}(\mu y)\right]\tan^{-1}\left[\frac{\mu}{\sqrt{1+\mu^2}}\sin\left(\sqrt{1-\mu^2 t}\right)\operatorname{sech}(\mu z)\right] \quad (37)$$

We applied our proposed method to provide a numerical solution for this problem with $\mu = 0.7.$ Tab. 1 represents the error estimation for different times, *i.e.*, t = 1, 2, 3, 4, and 5 through four various time steps. For space variables, three different step sizes $(h,k,\sigma)=(75,75,75)$, $(25,25,25)$ and $(20,20,20)$ are picked to tabulate L_2 and L_∞ errors. Each spacial step size has '11 x 11 x 11', '21 x 21 x 21' and '24 x 24 x 24' grid points. From the table, L_∞ error changes from 3.25(–2) to 3.02(–2) for $\tau = 0.1$ and from 3.64(–2) to 3.24(–2) for $\tau = 0.001$, if (h,k,σ) changes from $(75,75,75)$ to (20, 20, 20). It is observed that more accurate values are obtained by minimizing the space steps rather than the time steps, which causes less iteration and low time costs. Hence, our proposed method provides better results in terms of error and time consumption, which provides the platform to discuss the linear Sine Gordon equation.

Table 1. L_2 – error and L_∞ – error in the numerical solution of the example with μ = 0.7 and $h = k = \sigma$ = 75, 25 and 20 over the domain 0 < x, y, z < 500

(h,k,σ)	Time [t]	$\tau=0.001$		$\tau=0.01$		$\tau=0.05$		$\tau=0.1$	
		L_2	L_∞	L_2	L_∞	L_2	L_∞	L_2	L_∞
(75,75,75)	1	4.07(-2)	3.64(-2)	4.03(-2)	3.61(-2)	3.85(-2)	3.45(-2)	3.63(-2)	3.25(-2)
	2	1.115(-1)	9.97(-2)	1.11(-1)	9.93(-2)	1.09(-1)	9.76(-2)	1.067(-1)	9.54(-2)
	3	1.228(-1)	1.099(-1)	1.227(-1)	1.099(-1)	1.224(-1)	1.096(-1)	1.22(-1)	1.092(-1)
	4	3.78(-2)	3.4(-2)	3.82(-2)	3.43(-2)	3.98(-2)	3.57(-2)	4.17(-2)	3.74(-2)
	5	8.68(-2)	7.75(-2)	8.64(-2)	7.71(-2)	8.43(-2)	7.53(-2)	8.19(-2)	7.31(-2)
(25,25,25)	1	3.81(-2)	3.41(-2)	3.77(-2)	3.37(-2)	3.6(-2)	3.22(-2)	3.38(-2)	3.03(-2)
	2	9.93(-2)	8.89(-2)	9.88(-2)	8.85(-2)	9.7(-2)	8.69(-2)	9.47(-2)	8.48(-2)
	3	9.55(-2)	8.56(-2)	9.54(-2)	8.57(-2)	9.51(-2)	8.55(-2)	9.48(-2)	8.52(-2)
	4	5.8(-3)	4.7(-3)	3.7(-3)	2.4(-3)	2.4(-3)	1(-3)	1.5(-3)	6.8757(-4)
	5	1.345(-1)	1.2(-1)	1.341(-1)	1.198(-1)	1.323(-1)	1.181(-1	1.3(-1)	1.16(-1)
(20,20,20)	1	3.64(-2)	3.26(-2)	3.6(-2)	3.22(-2)	3.43(-2)	3.07(-2)	3.38(-2)	3.02(-2)
	2	9.17(-2)	8.19(-2)	9.12(-2)	8.17(-2)	8.94(-2)	8.20(-2)	9.47(-2)	8.39(-2)
	3	7.86(-1)	7.05(-2)	7.84(-2)	7.07(-2)	7.82(-2)	7.06(-2)	9.48(-2)	8.52(-2)

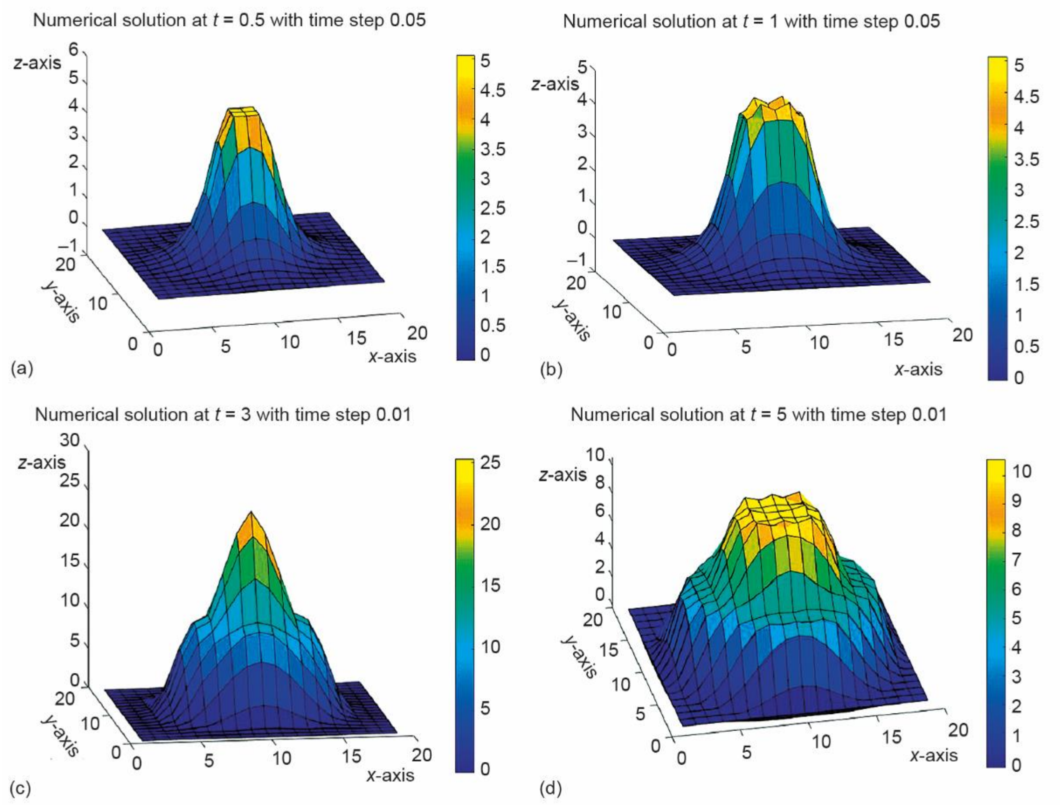

Figure 1. The surface plot for numerical solution at *t* = 0.5, 1, 3, 5 with different time steps

Conclusion

This research paper contains a new technique to solve the linear sine Gordon equation based on a non-polynomial cubic spline function to approximate the terms with space variables. Second order time derivative term is approximated by central difference approximation. Suitable values for involved parameters give efficient numerical results. The discussed method brings the order of convergence of $O(h^2 + k^2 + \sigma^2 + \tau^2 h^2 + \tau^2 k^2 + \tau^2 \sigma^2)$. This accuracy could be increased up to order $O(h^8 + k^8 + \sigma^8 + \tau^2 h^2 + \tau^2 k^2 + \tau^2 \sigma^2)$ here. The numerical problem is solved and compares these results with corresponding exact solutions. The given domain is divided into three different subintervals such that $N^3 = 11^3$, 21^3 and 24^3. Each set is discussed for time steps $\tau = 0.1$, 0.05, 0.01, and 0.001, respectively. Recon error shows that the method proposed here, gives compatible and better approximation with low time cost. It justifies the accuracy of this method, which is also easy to apply.

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