# GALERKIN APPROACH TO APPROXIMATE SOLUTIONS OF SOME BOUNDARY VALUE PROBLEMS

by

***Yi TIAN*** [a,c*] ***and Xiu-Qing PENG*** [b]

[a] College of Data Science and Application, Inner Mongolia University of Technology, Hohhot, China
[b] School of Economics and Management, Inner Mongolia University, Hohhot, China
[c] Inner Mongolia Autonomous Region Engineering and Technology Research Center of Big Data Based Software Service, Hohhot, China

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*This paper uses the Galerkin method to find approximate solutions of some boundary value problems. The solving process requires to solve a system of algebraic equations, which are large and difficult to be solved. According to the Groebner bases theory, an improved Buchberger's algorithm is proposed to solve the algebraic system. The results show that the Galerkin approach is simple and efficient.*

Key words: *Galerkin method, boundary value problem, improved version of Buchberger's algorithm, Groebner bases*

## Introduction

With the rapid development of natural science, various kinds of analytical methods were used to handle non-linear problems arising in mathematical, mechanics, economics, management fields and so forth, the most advanced method is the homotopy perturbation method [1-7] and its various modifications, *e.g.*, the reducing rank method [8], the higher-order HMP [9], He-Laplace method [10], Li-He’s modification [11, 12]. Other famous methods include the variational iteration method and its modifications [13, 14], and Taylor series method [15]. The variational-based method is also very attractive [16, 17], the variational principle plays a key role in both numerical and analytical analyses of a practical problem [18, 19], the idea of which is mainly reformulate the original equation as a variational problem and then to minimize the corresponding variational functional within a set of trial functions. Sometimes it is not used possibly due to the difficulty arising in establishing its variational formulation. When we apply mathematical methods to solve various practical problem, the simpler is the better [20, 21], though there are many advanced methods for theoretical analysis of the non-linear behavior for a practical problem [22-25].

The Galerkin approach [26, 27] is more flexible than the variational approach because it is applicable to a broader range of problems, for example, when a variational reformulation of the original equation is not possible.

* Corresponding author, e-mail: ttxsun@163.com

Boundary value problems arise everywhere in engineering [28], the purpose of this paper is to give a demonstration of the application of the Galerkin method to some boundary value problems.

## Groebner bases

In the following, we list the basic results of Groebner bases theory [29].

*Definition 1.* Let $f = \Sigma_\alpha a_\alpha x^\alpha$ be a non-zero polynomial in $k[x_1,\cdots,x_n]$ and let > be a monomial order.

– The multidegree of *f* is:

$$\text{multideg}(f) = \max(\alpha \in \mathbb{Z}^n_{\geq 0} : a_\alpha \neq 0)$$

(the maximum is taken with respect to >).

– The leading coefficient of *f* is:

$$\text{LC}(f) = a_{\text{multideg}(f)} \in k$$

– The leading monomial of *f* is:

$$\text{LM}(f) = x^{\text{multideg}(f)}$$

(with coefficient 1).

– The leading term of *f* is:

$$\text{LT}(f) = \text{LC}(f)\text{LM}(f)$$

*Definition 2.* Let $f, g \in k[x_1,\cdots,x_n]$ be non-zero polynomials,

– If multideg(*f*) = $\alpha$ and multideg(g) = $\beta$, then let $\gamma = (\gamma_1,\cdots,\gamma_n)$, , where $\gamma_i = \max(\alpha,\beta_i)$ for each *i*. We call $x^\gamma$ the least common multiple of LM(*f*) and LM(g), it can be written $x^\gamma = \text{LCM}[\text{LM}(f),\text{LM}(g)]$.

– The **S-polynomial** of *f* and *g* is the combination:

$$S(f,g) = \frac{x^\gamma}{\text{LT}(f)} f - \frac{x^\gamma}{\text{LT}(g)} g$$

*Definition 3.* Let $I \in k[x_1,\cdots,x_n]$ be an ideal other than $\{0\}$,

– We denote by $\text{LT}(I)$ the set of leading terms of elements of *I*. Thus:

$$\text{LT}(I) = \{cx^\alpha : \text{there exists } f \in I \text{ with } \text{LT}(f) = cx^\alpha\}$$

– We denote by $\langle \text{LT}(I) \rangle$ the ideal generated by the elements of $\text{LT}(I)$.

*Theorem 1.* (Improved version of Buchberger's algorithm [29]) Let $I = \langle f_1,\cdots,f_s \rangle$ be a polynomial ideal. Then a Groebner basis *I* can be constructed in a finite number of steps by the following algorithm:

Input: $F = (f_1,\cdots,f_s)$

Output: ***G***, a Groebner basis for $I = \langle f_1,\cdots,f_s \rangle$

$$\{\text{initialization}\}\ B := \{(i,j): 1 \le i \le j \le s\}$$
$$G := F$$
$$t := s$$
$$\{\text{iteration}\}\ \text{WHILE}\ B \ne \varnothing\ \text{DO}$$
$$\text{Select}\ (i,j) \in B$$
$$\text{IF LCM(LT(}f_i\text{),LT(}f_i\text{))} \ne \text{LT(}f_i\text{)LT(}f_i\text{) AND}$$
$$\text{Criterion}\ (f_i, f_j, B)\ \text{is false THEN}$$
$$S := \overline{S(f_i, f_j)}^G$$
$$\text{IF}\ S \ne 0\ \text{THEN}$$
$$t := t+1;\ f_t := S$$
$$G := G \cup \{f_t\}$$
$$B := B \cup \{(i,t): 1 \le i \le t-1\}$$
$$B := B - \{(i,j)\}$$

## Galerkin method for the ODE boundary value problems

Consider the following example [30]:

$$y'' + y + x = 0 \tag{1}$$

with the boundary conditions:

$$y(0) = 0, \quad y(1) = 0 \tag{2}$$

where $y'$ is the differentiation with respect to *x*.

We choose a trial function satisfying all the boundary conditions:

$$\varphi_n(x) = x^n(x-1) \quad (n = 1, 2, \cdots) \tag{3}$$

the approximate solution can be expressed:

$$y_n(x) = \sum_{i=1}^{n} a_i \varphi_i = \sum_{i=1}^{n} a_i x^i (x-1) \tag{4}$$

*Case 1.* $n = 1$

$$y_1 = ax(x-1) \tag{5}$$

where *a* is an unknown constant to be further determined.

Substituting (5) into the following Galerkin equation:

$$\int_0^1 G_1 \varphi_i \mathrm{d}x = 0 \tag{6}$$

where $G_i = y_i'' + y_i + x (i = 1, 2, \cdots, n).$ From eq. (6), we obtain:

$$-\frac{1}{12} - \frac{3a}{10} = 0 \tag{7}$$

and we have $a = -5/18,$ and the first-order approximate solution of eq. (1) is:

$$y_1 = -\frac{5}{18}x(x-1) \tag{8}$$

*Case 2.* $n = 2$

$$y_1 = ax(x-1) + bx^2(x-1) \tag{9}$$

where $a, b$ are unknown constants to be further determined.

Substituting (9) into the following Galerkin equation:

$$\int_0^1 G_2 \varphi_i \mathrm{d}x = 0 \tag{10}$$

where $\varphi_i \in \{\varphi_1, \varphi_2\}$. From eq. (10), we obtain:

$$\begin{aligned} -\frac{1}{12} - \frac{3a}{10} - \frac{3b}{20} = 0 \\ -\frac{1}{20} - \frac{3a}{20} - \frac{13b}{105} = 0 \end{aligned} \tag{11}$$

and we have $a = -71/369, b = -7/41,$ and the second-order approximate solution of eq. (1) is:

$$y_2 = -\frac{71}{369}x(x-1) - \frac{7}{41}x^2(x-1) \tag{12}$$

*Case 3.* $n = 3$

$$y_1 = ax(x-1) + bx^2(x-1) + cx^3(x-1) \tag{13}$$

where $a, b, c$ are unknown constants to be further determined.

Substituting (13) into the following Galerkin equation:

$$\int_0^1 G_3 \varphi_i \mathrm{d}x = 0 \tag{14}$$

where $\varphi_i \in \{\varphi_1, \varphi_2, \varphi_3\}$. From eq. (14), we obtain:

$$\begin{aligned} -\frac{1}{12} - \frac{3a}{10} - \frac{3b}{20} - \frac{19c}{210} = 0 \\ -\frac{1}{20} - \frac{3a}{20} - \frac{13b}{105} - \frac{79c}{840} = 0 \\ -\frac{1}{30} - \frac{19a}{210} - \frac{79b}{840} - \frac{103c}{1260} = 0 \end{aligned} \tag{15}$$

Equation (15) seems to be large and difficult to solve by hand, even by software, such as MAPLE and MATLAB. In this paper, we use an improved version of Buchberger's algorithm to solve this problem.

Let $I$ be the ideal:

$$I = \left\langle -\frac{1}{12} - \frac{3a}{10} - \frac{3b}{20} - \frac{19c}{210}, -\frac{1}{20} - \frac{3a}{20} - \frac{13b}{105} - \frac{79c}{840}, -\frac{1}{30} - \frac{19a}{210} - \frac{79b}{840} - \frac{103c}{1260} \right\rangle \subset k[a,b,c] \tag{16}$$

corresponding to the system of eq. (15), and we want to find all points in $\mathbf{V}(I)$.

Using the improved version of Buchberger's algorithm with the lexicographic order $a > b > c,$ we find the reduced Groebner basis:

$$\begin{aligned} g_1 &= -7 + 299c \\ g_2 &= 2380 + 12259b \\ g_3 &= 13811 + 73554a \end{aligned} \tag{17}$$

thus, from $g_1 = g_2 = g_3 = 0,$ we have:

$$\mathbf{V}(I) = \mathbf{V}(g_1, g_2, g_3) = \left\{ a = -\frac{13811}{73554}, b = -\frac{2380}{12259}, c = \frac{7}{299} \right\} \tag{18}$$

and the third-order approximate solution of eq. (1) is:

$$y_3 = -\frac{13811}{73554} x(x-1) - \frac{2380}{12259} x^2 (x-1) + \frac{7}{299} x^3 (x-1) \tag{19}$$

## Galerkin method for the partial differential equation boundary value problems

Consider the following example:

$$u_{tt} - u_{xx} + u = x^2 + t \tag{20}$$

with the boundary conditions:

$$u(0,t) = u(1,t) = u(x,0) = u(x,1) = 0 \tag{21}$$

We choose a trial function satisfying all the boundary conditions:

$$\varphi_n(x) = x^n (x-1) t(t-1) \quad (n = 1, 2, \cdots) \tag{22}$$

the approximate solution can be expressed:

$$u_n(x,t) = \sum_{i=1}^{n} a_i \varphi_i = \sum_{i=1}^{n} a_i x^i (x-1) t(t-1) \tag{23}$$

*Case 1.* $n = 1$

$$u_1 = ax(x-1)t(t-1) \tag{24}$$

where $a$ is a unknown constant to be further determined.

Substituting (24) into the following Galerkin equation:

$$\int_0^1 \int_0^1 G_1 \varphi_1 \mathrm{d}x \mathrm{d}t = 0 \tag{25}$$

where $G_i = (u_i)_{tt} - (u_i)_{xx} + (u_i) - x^2 - t, \quad (i = 1, 2, \cdots).$ From eq. (25), we obtain:

$$\frac{1}{900}(-20 + a) = 0 \tag{26}$$

and we have $a = 20,$ and the first-order approximate solution of eq. (20) is:

$$u_1 = 20x(x-1)t(t-1) \tag{27}$$

*Case 2.* $n=2$

$$u_2 = ax(x-1)t(t-1) + bx^2(x-1)t(t-1) \tag{28}$$

where $a,b$ are unknown constants to be further determined.

Substituting (28) into the following Galerkin equation:

$$\int_0^1\int_0^1 G_2\varphi_i \mathrm{d}x\mathrm{d}t = 0 \tag{29}$$

where $\varphi_i \in \{\varphi_1,\varphi_2\}$. From eq. (29), we obtain:

$$\begin{aligned} &\frac{-40+2a+b}{1800}=0 \\ &\frac{-315+14a+40b}{25200}=0 \end{aligned} \tag{30}$$

and we have $a = 1285/66, b = 35/33,$ and the second-order approximate solution of eq. (20) is:

$$u_2 = \frac{1285}{66}x(x-1)t(t-1) + \frac{35}{33}x^2(x-1)t(t-1) \tag{31}$$

*Case 3.* $n=3$

$$u_3 = ax(x-1)t(t-1) + bx^2(x-1)t(t-1) + cx^3(x-1)t(t-1) \tag{32}$$

where $a$, $b$, and $c$ are unknown constants to be further determined.

Substituting (32) into the following Galerkin equation:

$$\int_0^1\int_0^1 G_3\varphi_i \mathrm{d}x\mathrm{d}t = 0 \tag{33}$$

where $\varphi_i \in \{\varphi_1,\varphi_2,\varphi_3\}$. From eq. (33), we obtain:

$$\begin{aligned} &\frac{-280+14a+7b+6c}{12600}=0 \\ &\frac{-315+14a+40b+39c}{25200}=0 \\ &\frac{-205+12a+39b+42c}{25200}=0 \end{aligned} \tag{34}$$

Let $I$ be the ideal:

$$I = \left\langle \frac{-280+14a+7b+6c}{12600}, \frac{-315+14a+40b+39c}{25200}, \frac{-205+12a+39b+42c}{25200} \right\rangle \subset k[a,b,c] \tag{35}$$

corresponding to the original system of eq. (34), using the improved version of Buchberger's algorithm with the lexicographic order $a > b > c,$ we find the reduced Groebner basis:

$$\begin{aligned} g_1 &= 490+27c \\ g_2 &= -5705+297b \\ g_3 &= -10795+594a \end{aligned} \tag{36}$$

thus,

$$\mathbf{V}(I)=\mathbf{V}(g_1,g_2,g_3)=\left(a=\frac{10795}{594},b=\frac{5705}{297},c=\frac{490}{27}\right) \tag{37}$$

and the third-order approximate solution of eq. (20) is

$$u_3=\frac{10795}{594}x(x-1)t(t-1)+\frac{5705}{297}x^2(x-1)t(t-1)+\frac{490}{27}x^3(x-1)t(t-1) \tag{38}$$

## Conclusions

In this paper, we use Galerkin method to solve some boundary value problems. The solving process requires to solve a system of algebraic equations, which are large and difficult to solve by hand, the Groebner bases theory (the improved version of Buchberger's algorithm) is applied to solve this problem, which gives the smallest Groebner basis. In the future, we will try do some improvements on the Buchberger's algorithm [29, 31] for computing Groebner bases.

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