HE-LAPLACE METHOD FOR TIME FRACTIONAL BURGERS-TYPE EQUATIONS

by

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The time fractional Burgers-type equations with He's fractional derivative by He-Laplace method. It is a numerical approach coupled the Laplace transformation and HPM. The approximations to the initial value problem with different fractional orders are given without any discretization and complicated computation. Numerical results are provided to confirm its efficiency.

Key words: *time fractional Burgers-type equation, He-Laplace method, approximation*

Introduction

Fractional differential equations (FDE) have been widely applied in computer science and engineering areas, including physics, fluid dynamics, biology, finance, and thermodynamics. For examples, fractional advection-reaction-diffusion [1], fractional wave travelling [2-6], fractional population dynamics [7], fractional economy [8], and fractional thermodynamics [9]. Generally, FDE can be given by the modifications of the classical differential equations with the fractional time or space derivatives. Due to the non-local property of fractional derivatives, FDE can be used to model the non-linear physical phenomena that depend on the time instant and the time history [10, 11]. In past three decades, Burgers-type equations and their fractional modification equations have attracted much attention. In [12, 13], 1-D Burgers equations was used to describe the transport process of turbulence and flow in viscous fluid. The coupled Burgers equation (CBE) proposed by Esipov can be seen as a mathematical model of sedimentation or evolution of scaled volume concentrations of two kinds of particles in fluid suspensions or colloids, under the effect of gravity [14, 15]. There are also some modifications of 1-D Burgers or coupled Burgers equations, such as KdV-Burgers equation and 2-D Burgers equation [16, 17]. In real applications, the time or space fractional derivatives are suggested to modify these Burgers-type equations. Yildirim and Kelleci [18] studied the numerical behavior of CBE with time or space fractional derivatives by HPM. Albuohimad and Adibi [19] considered the time-fractional coupled Burgers equations by using a hybrid spectral exponential Chebyshev method. Fractional homotopy analysis transform method was presented [20] for the coupled system of non-homogeneous Burgers' equations with time fractional derivatives. Numerical approximations to the non-linear time-fractional coupled Burger's equations were given by the homotopy perturbation Sumudu transform method [21].

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In this paper, we will consider the time fractional Burgers-type equations with He's fractional derivative [11] as an example to investigate the numerical behavior and physical meaning of FDE. We will focus on the fractional modification of 1-D Burgers equation and the coupled Burgers equation, which can be formulated.

(**P1**) Time fractional 1-D Burgers equation:

$$\mathrm{D}_t^{\alpha} u(x,t) + \gamma u u_x - \mu u_{xx} = 0 \tag{1}$$

where γ and μ are two arbitrary constants.

(**P2**) Time fractional coupled Burgers equation:

$$\begin{aligned} \mathrm{D}_t^{\alpha} u(x,t) - u_{xx} - 2uu_x + (uv)_x = 0 \\ \mathrm{D}_t^{\beta} v(x,t) - v_{xx} - 2vv_x + (uv)_x = 0 \end{aligned} \tag{2}$$

where the fractional order α and β are two constants satisfying $0 < \alpha, \beta \le 1$. Here $(\partial^{\alpha} u)/(\partial t^{\alpha})$ is defined by He's fractional derivative [11]:

$$\mathrm{D}_t^{\alpha} u = \frac{\partial^{\alpha} u}{\partial t^{\alpha}} = \frac{1}{\Gamma(n-\alpha)} \frac{\mathrm{d}^n}{\mathrm{d}t^n} \int_{t_0}^{t} (s-t)^{n-\alpha-1} [u_0(x,s) - u(x,s)] \mathrm{d}s \tag{3}$$

with a constant $n-1 < \alpha \le n$ and a known function $u_0(x,t)$. When $\alpha = 1$, eq. (1) reduces to the classical 1-D Burgers equation [12, 13]. One can also obtain the coupled Burges equation when $\alpha = \beta = 1$ [14, 15].

Motivated by the improvements in [18-22], we are interested in the efficient technique for solving the fractional Burgers-type equations. We consider He-Laplace method [23, 24] for the fractional problems (1) and (2). He-Laplace method is a modification of the HPM [25-28], and the couple of the Laplace transform makes the solving process much easier and simpler. In literature, the method was also called as the homotopy perturbation transform method [29] or Laplace HPM [30]. Numerical solutions to the time fractional eqs. (1) and (2) are given without any linearization or complicated computation. Numerical examples related with two initial value problems are presented to show the efficiency of He-Laplace method.

The basic idea of He-Laplace method

For clarity, we illustrate the basis ideas of He-Laplace method. Consider the following non-linear partial differential equation:

$$\mathrm{D}_t^{\alpha} u(x,t) + Ru(x,t) + Nu(x,t) = f(x,t), \; u(x,0) = g(x) \tag{4}$$

where $D_t^{\alpha} u(x,t)$ is defined by He's fractional derivative (3), R and N are two differential operators which represent the linear and non-linear parts, respectively, and $f(x,t)$ – a given non-linear function [22-24].

The first difficulty of (4) lies in the non-linear and fractional operator $\mathrm{D}_t^{\alpha} u(x,t)$. We apply the Laplace transformation to release this problem. Consider the Laplace transformation on the both sides of (4):

$$L[\mathrm{D}_t^{\alpha} u(x,t)] + L[Ru(x,t)] + L[Nu(x,t)] = L[f(x,t)]$$

Together with the property of Laplace transformation and the initial condition, we have:

$$L[u(x,t)] = \frac{g(x)}{s} + \frac{1}{s^{\alpha}} L[f(x,t)] - \frac{1}{s^{\alpha}} \{L[Ru(x,t)] + L[Nu(x,t)]\} \tag{5}$$

We further consider the inverse Laplace transform (L^{-1}) on (5), and obtain the following results:

$$u(x,t) = G(x,t) - L^{-1}\left(\frac{1}{s^{\alpha}}\{L[Ru(x,t)] + L[Nu(x,t)]\}\right) \tag{6}$$

where $G(x,t)$ is defined by $f(x,t)$ and $g(x)$.

The second issue of He-Laplace method is based on the HPM [24-28]. By HPM, one can construct the homotopy for (6):

$$u(x,t) = G(x,t) - pL^{-1}\left(\frac{1}{s^{\alpha}}\{L[Ru(x,t)] + L[Nu(x,t)]\}\right) \tag{7}$$

Assume that the exact solution to (4) can be given by:

$$u(x,t) = \sum_{n=0}^{\infty} p^n u_n(x,t) \tag{8}$$

Substituting (8) into (7) and collecting the same powers of *p*, we can obtain the non--linear system with respect to *p*-term. By solving the non-linear equations, we can have the sub-solutions, and formulate the approximated solution:

$$u(x,t) = \lim_{N\to\infty} \sum_{n=1}^{N} p^n u_n(x,t) \tag{9}$$

He-Laplace method for time fractional Burgers-type equations

We first consider the time fractional 1-D Burgers eq. (1) with the initial condition:

$$u(x,0) = \frac{c}{\gamma} + \frac{2\mu}{\gamma}\tanh(x) \tag{10}$$

where *c* is an arbitrary constant. Notice that the exact solution to the initial value problem (1) with $\alpha = 1$ is given by [16]:

$$u(x,t) = \frac{c}{\gamma} + \frac{2\mu}{\gamma}\tanh(x - ct) \tag{11}$$

According to He-Laplace method, we can construct the following homotopy for (1):

$$u(x,t) = u(x,0) - pL^{-1}\left[\frac{1}{s^{\alpha}} L(\gamma u u_x - \mu u_{xx})\right] \tag{12}$$

Assume that the solutions to (1) can be defined by:

$$u(x,t) = \sum_{n=0}^{\infty} p^n u_n(x,t) \tag{13}$$

We substitute (13) and (12) into the homotopy, and collect the coefficient related with the p-term, which results in the following system:

$$p^1 : u_1 = L^{-1}\left[\frac{1}{s^\alpha} L(-\gamma u_0 u_{0x} + \mu u_{0xx})\right]$$

$$p^2 : u_2 = L^{-1}\left[\frac{1}{s^\alpha} L(-\gamma u_0 u_{1x} - \gamma u_1 u_{0x} + \mu u_{1xx})\right]$$

$$p^3 : u_3 = L^{-1}\left[\frac{1}{s^\alpha} L(-\gamma u_0 u_{2x} - \gamma u_1 u_{1x} - \gamma u_2 u_{0x} - \mu u_{2xx})\right]$$

It is easy to obtain the approximation solutions:

$$u_1 = \frac{t^\alpha}{\gamma \Gamma(1+\alpha)} \{-4\mu^2 \sec h^2(x) \tanh(x) - 2\mu \sec h^2(x)[c + 2\mu \tanh(x)]\}$$

$$u_2 = \frac{4\mu t^{2\alpha}}{\gamma \Gamma(1+2\alpha)} \sec h^2(x)\{4\mu \sec h^3(x)[c \cosh(x) + 6\mu \sinh(x)] - \tanh(x)[c + 4\mu \tanh(x)]^2\}$$

$$u_3 = -\frac{1}{\gamma \Gamma(1+\alpha)^2 \Gamma(1+3\alpha)} (\mu t^{3\alpha} \sec h^7(x)\{-4\mu\Gamma(1+2\alpha)(-16\mu c \cosh(x) + 8\mu c \cosh(3x) +$$

$$+2[-32\mu^2 + c^2 + 16\mu^2 \cosh(2x)]\sinh(x)\} + \Gamma(1+\alpha)^2[-8c(-176\mu^2 + c^2)\cosh(x) -$$

$$-c(944\mu^2 + c^2)\cosh(3x) + 48\mu^2 c \cosh(5x) + c^3 \cosh(5x) + 11008\mu^3 \sinh(x) - 112\mu c^2 \cdot$$

$$\cdot \sinh(x) - 2368\mu^3 \sinh(3x) - 100\mu c^2 \sinh(3x) + 64\mu^3 \sinh(5x) + 12\mu c^2 \sinh(5x)]\})$$

Then we have the third order approximation by $\hat{u} = u_0 + u_1 + u_2 + u_3$. We remark that the higher order solutions can be given in a similar manner.

Analysis of time fractional CBE

We then consider the time fractional CBE (2) with the following initial condition:

$$u(x,0) = v(x,0) = \sin(x) \tag{14}$$

When the fractional derivative is defined in the Liouville-Caputo sense with $\alpha = \beta$, eq. (2) reduces to the time fractional CBE [30].

By He-Laplace method, the homotopy for (2) can be formulated by:

$$u(x,t) = u(x,0) - pL^{-1}\left\{\frac{1}{s^\alpha} L[-u_{xx} - 2uu_x + (uv)_x]\right\} \tag{15}$$

$$v(x,t) = v(x,0) - pL^{-1}\left\{\frac{1}{s^\beta} L[-v_{xx} - 2vv_x + (uv)_x]\right\} \tag{16}$$

Assume that the solutions to (2) can be defined by:

$$u(x,t) = \sum_{n=0}^{\infty} p^n u_n(x,t) \tag{17}$$

$$v(x,t) = \sum_{n=0}^{\infty} p^n v_n(x,t) \tag{18}$$

Similar to the technique for time fractional 1-D Burgers equation, we have the following system with respect to p-term:

$$p^1 : u_1 = L^{-1}\left\{\frac{1}{s^\alpha} L[u_{0xx} + 2u_0u_{0x} - (u_0v_0)_x]\right\}$$

$$p^1 : v_1 = L^{-1}\left\{\frac{1}{s^\beta} L[v_{0xx} + 2v_0v_{0x} - (u_0v_0)_x]\right\}$$

$$p^2 : u_2 = L^{-1}\left\{\frac{1}{s^\alpha} L[u_{1xx} + 2u_0u_{1x} + 2u_1u_{0x} - (u_1v_0)_x - (u_1v_0)_x]\right\}$$

$$p^2 : v_2 = L^{-1}\left\{\frac{1}{s^\beta} L[v_{1xx} + 2v_0v_{1x} + 2v_1v_{0x} - (u_0v_1)_x - (u_1v_0)_x]\right\}$$

$$p^3 : u_3 = L^{-1}\left\{\frac{1}{s^\alpha} L[u_{2xx} + 2u_0u_{2x} + 2u_1u_{1x} + 2u_2u_{0x} - (u_0v_2)_x - (u_1v_1)_x - (u_2v_0)_x]\right\}$$

$$p^3 : u_3 = L^{-1}\left\{\frac{1}{s^\beta} L[v_{2xx} + 2v_0v_{2x} + 2v_1v_{1x} + 2v_2v_{0x} - (u_0v_2)_x - (u_1v_1)_x - (u_2v_0)_x]\right\}$$

It is easy to obtain the sub-solutions u_i and v_i. For clarity, we list the approximation solutions:

$$u_1 = -\frac{t^\alpha \sin(x)}{\Gamma(1+\alpha)}, \quad v_1 = -\frac{t^\beta \sin(x)}{\Gamma(1+\beta)}$$

$$u_2 = \frac{t^{\alpha+\beta}\sin(2x)}{\Gamma(1+\alpha+\beta)} + \frac{t^{2\alpha}\sin(x)[1-2\cos(x)]}{\Gamma(1+2\alpha)}$$

$$v_2 = \frac{t^{2\beta}\sin(x)[1-2\cos(x)]}{\Gamma(1+2\beta)} + \frac{t^{\alpha+\beta}\sin(2x)}{\Gamma(1+\alpha+\beta)}$$

$$u_3 = \frac{-t^{2\alpha+\beta}\sin(2x)\Gamma(1+\alpha+\beta)}{\Gamma(1+\alpha)(1+\beta)\Gamma(1+2\alpha+\beta)} + \frac{t^{3\alpha}\sin(2x)\Gamma(1+2\alpha)}{\Gamma(1+\alpha)^2\Gamma(1+3\alpha)} - \frac{1}{\Gamma(1+3\alpha)} t^{3\alpha}\sin(x)\cdot$$

$$\cdot(2 - 10\cos x + 3\cos 2x) - \frac{4}{\Gamma(1+2\alpha+\beta)} t^{2\alpha+\beta}\sin(2x) +$$

$$+\frac{1}{\Gamma(1+\alpha+2\beta)} t^{\alpha+2\beta}\sin(x)(1 - 2\cos x + 3\cos 2x)$$

$$v_3 = \frac{-t^{2\alpha+\beta}\sin(2x)\Gamma(1+\alpha+\beta)}{\Gamma(1+\alpha)\Gamma(1+\beta)\Gamma(1+\alpha+2\beta)} + \frac{t^{3\beta}\sin(2x)\Gamma(1+2\beta)}{\Gamma(1+\beta)^2\Gamma(1+3\beta)} -$$

$$-\frac{1}{\Gamma(1+3\beta)} t^{3\beta}\sin(x)(2 - 10\cos x + 3\cos 2x) - \frac{4}{\Gamma(1+\alpha+2\beta)} t^{\alpha+2\beta}\sin(2x) +$$

$$+\frac{1}{\Gamma(1+2\alpha+\beta)} t^{2\alpha+\beta}\sin(x)(1 - 2\cos x + 3\cos 2x)$$

Then we have the following third order approximations by $\hat{u} = u_0 + u_1 + u_2 + u_3$ and $\hat{v} = v_0 + v_1 + v_2 + v_3$. When $\alpha = \beta = 1$, the approximations reduce to the following solutions:

$$\hat{u} = \hat{v} = \sin(x)\left(1 - t + \frac{1}{2}t^2 - \frac{1}{6}t^3\right) \tag{19}$$

Obviously, we can obtain the exact solution $u = v = \sin(x)e^{-t}$ when $n \to \infty$ [16].

Numerical results

In this section, we consider the initial value problems associated with the time fractional differential eqs. (1) and (2). Numerical results are presented to illustrate the effectiveness of He-Laplace method.

We first apply He-Laplace method for the time fractional 1-D Burgers eq. (1). The parameters $\gamma = 1$, $\mu = 0.01$, and $c = 0.1$ are set in this example. For comparison, the third order approximation $\hat{u}$ and the exact solution $u(x,t)$ are presented in fig. 1. Numerical results show that the approximated solution agrees well with the exact solution. We further test the numerical behavior of eq. (1) with different α. Figure 2 plots the curves of the approximated solutions with $\alpha = 0.1$, 0.3, 0.5, and 0.8.

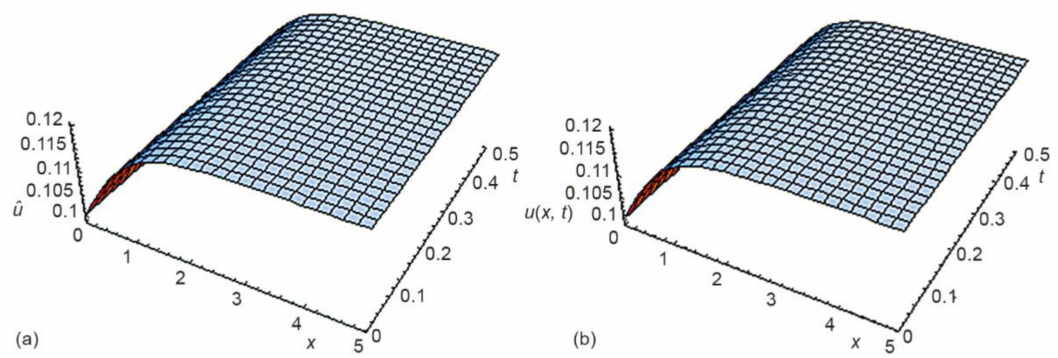

Figure 1. Comparisons of $\hat{u}$ and $u(x,t)$ for (1) with $\alpha = 1$

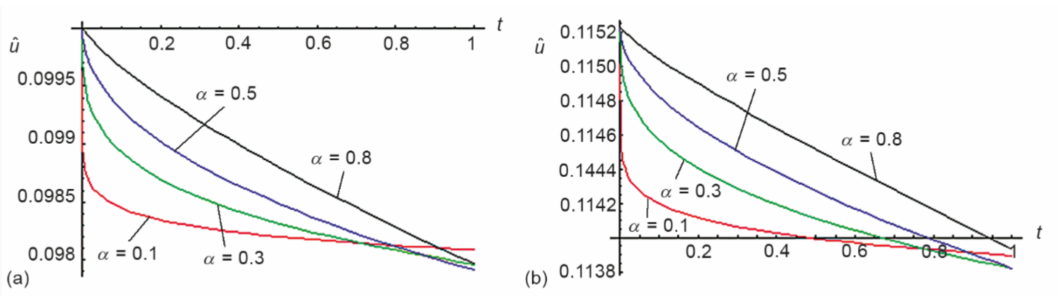

Figure 2. Numerical behavior of $\hat{u}$ with different α; (a) $x = 0$ and (b) $x = 1$

We then test He-Laplace method for the time fractional CBE (2) with different fractional order α. The comparisons between the third order approximation $\hat{u} = \hat{v}$ and the exact solution $u(x,t)$ for the classical CBE (2) are shown in fig. 3, which implies that He-Laplace performs well for CBE (2) with $\alpha = 1$. The numerical results for the time fractional CBE (2) are presented in figs. 4 and 5. Figures 4(a) and 4(b) plot the approximated solutions $\hat{u}$ and $\hat{v}$ for (2) with $\alpha = 0.3$ and $\beta = 0.5$, respectively. In order to further consider the physics behind the time fractional CBE, we also give the curves of the approximated solutions with different α at $x = 5$ and $x = 10$. Notice that the fractional order $\alpha = \beta$ is used in fig. 5, and the solutions $\hat{v}$ is equal to $\hat{u}$. The curve of the approximations tends to x-axis as the value of α becomes small, which implies the complexity of the propagation process.

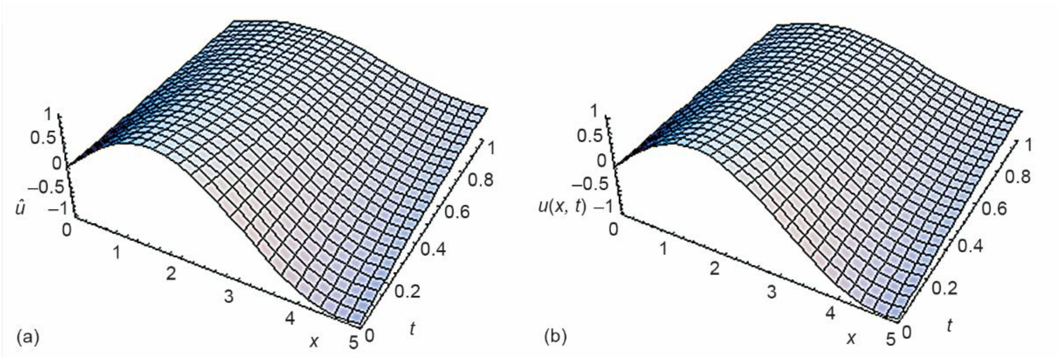

Figure 3. Compared results of $\hat{u}$ and $u(x, t)$ for (2) with $\alpha = 1$

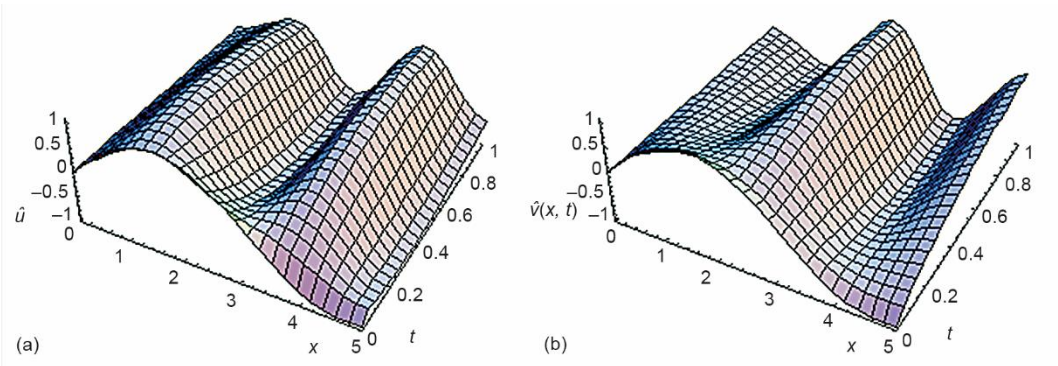

Figure 4. Numerical behavior of $\hat{u}$ and $\hat{v}$ for (2) with $\alpha = 0.3$ and $\beta = 0.5$

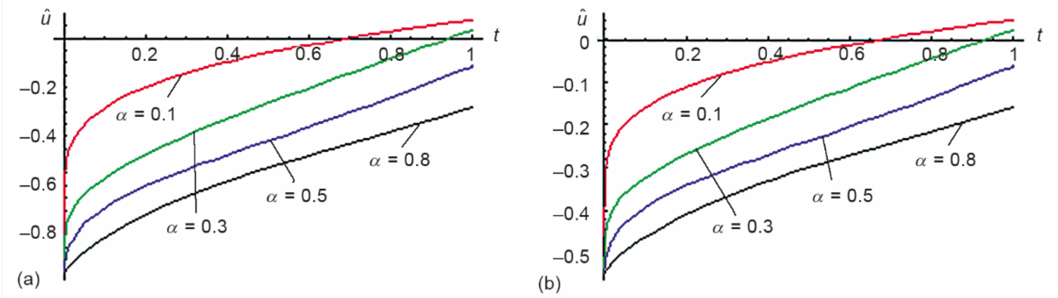

Figure 5. Propagation curves of $\hat{u}$ with different α; (a) $x = 5$ and (b) $x = 10$

Conclusion

This paper provided He-Laplace method for solving the initial value problems associated with the time fractional Burgers equations. Numerical results confirm its efficiency. In the future work, we will extend this approach to other fractional differential equations for control and oscillator systems [31-37], and its potential application in machine learning [38-40] is also very much promising.

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