VARIATIONAL PRINCIPLE FOR FRACTAL HIGH-ORDER LONG WATER-WAVE EQUATION

by

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In this article, we mainly consider a modification of the high-order long water-wave equation with unsmooth boundaries by adopting a new fractal derivative. Its fractal variational principles are successfully constructed by the fractal semi-inverse method, the obtained principles are helpful to study the symmetry, to discover the conserved quantity, and to have wide applications in numerical simulation.

Key words: *variational principle, fractal semi-inverse method, fractal derivative, long water-wave equation*

Introduction

Non-linear differential equations are extensively used to represent various curious phenomena engendering in biology, physics, thermodynamics, mechanics, chemistry, electrospinning and other fields, for examples, KdV-type equations [1, 2], Harry Dym equations [3], Klein-Gordon equation [4], and KdV-Burgers-Kuramoto equation [5]. Non-linear equations have attractive properties, such as chaos and bifurcation [6, 7]. The non-linear rule-based controller [8], synchronization [9], chaos suppression control [10], design of extended backstepping sliding mode controller [11] and extremum-seeking control technique [12] are the main mathematics tools in non-linear science. Non-linear differential models can be derived by Newton’s laws or the variational theory, but not each problem has a variational formulation, for example, the well-known Navier-Stokes equations have not a variational principle [13]. It is an inverse problem of the calculus of variations to search for a suitable variational formulation from a differential equation model, and it is extremely difficult for this inverse problem.

In this paper, we employ He’s semi-inverse method [14-16] to set up fractal variational principles for a subsequent higher-order long water-wave equations (HOLWWE). The variational formulations are profitable to learn the symmetry, to find the conserved quantity, and to have a broad applications in both numerical simulation methods and analytical methods.

The non-linear coupled HOLWWE are provided [17]:

$$\frac{\partial \eta}{\partial t} - \frac{\partial \eta}{\partial x}\eta - \frac{\partial \mu}{\partial x} + \lambda \frac{\partial^2 \eta}{\partial x^2} = 0 \tag{1}$$

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$$\frac{\partial\mu}{\partial t}-\frac{\partial(\eta\mu)}{\partial x}-\lambda\frac{\partial^2\mu}{\partial x^2}=0 \tag{2}$$

When λ = 1/2, eqs. (1) and (2) were discussed in [18], but the fractals variational principles for the investigated problem has not been dealt with in any literature.

When HOLWWE with unsmooth boundaries, the fractal derivative will be employed to represent the model:

$$\frac{\partial\eta}{\partial t^\alpha}-\eta\frac{\partial\eta}{\partial x^\beta}-\frac{\partial\mu}{\partial x^\beta}+\lambda\frac{\partial^2\eta}{\partial x^{2\beta}}=0 \tag{3}$$

$$\frac{\partial\mu}{\partial t^\alpha}-\frac{\partial(\eta\mu)}{\partial x^\beta}-\lambda\frac{\partial^2\mu}{\partial x^{2\beta}}=0 \tag{4}$$

where $\partial\eta/\partial t^\alpha$ and $\partial\eta/\partial x^\beta$ are He's fractal derivatives with regard to t^α and x^β, respectively, and they are delimited [19-21]:

$$\frac{\partial\eta}{\partial t^\alpha}(t_0,x)=\Gamma(1+\alpha)\lim_{\substack{t-t_0=\Delta t\\ \Delta t\neq 0}}\frac{\eta(t,x)-\eta(t_0,x)}{(t-t_0)^\alpha} \tag{5}$$

$$\frac{\partial\eta}{\partial x^\beta}(t,x_0)=\Gamma(1+\alpha)\lim_{\substack{x-x_0=\Delta x\\ \Delta x\neq 0}}\frac{\eta(t,x)-\eta(t,x_0)}{(x-x_0)^\beta} \tag{6}$$

where Δx is the smallest spatial scale of the discontinuous boundaries and Δt – the smallest time scale for studying fractal FHOLWWE. Recently the solitary waves travelling along an unsmooth boundary has been caught much attention, and a new branch of mathematics was born, that is the fractal solitary theory [22-24], which is to study the solitary wave properties in a fractal space, and some attractive findings were found, for example, the unsmooth boundary can greatly affect the travelling velocity, but it rarely affects its shape of the solitary wave [23]. Tian *et al.* [25-27] found that the instability of a micro-electromechanical system in a fractal dimension space is totally different from that in a smooth space.

Now the fractal derivatives are extensively used in a discontinuous problems, the two-scale fractal is simple but effective for many practical problems [28], for examples, fractal vibration theory [29-33], fractal population model [34], and fractal economics [35].

Fractals variational principles for HOLWWE

The variational principle plays a crucial role in mathematics and physics, owing to the variational formula demonstrates the possible conservation rule of energy and solution structure. Wang *et al.* [36] set-up a variational formulation for wave traveling in fractal space. Wang and He [37] extended Wang's variational principle to fractal space/time, Wang and He [38] found that the variational method is effective to two-point boundary value problems, and the variational theory is the mathematical tool to identification of the Lagrange multiplier involved in the variational iteration algorithm [38], furthermore, the variational-based algorithm is widely used to imaging processing [39-43]. In this paper, by means of He's semi-inverse method [14-16], the fractal variational principles of HOLWWE are originated.

We rewrite eqs. (3) and (4) in conservation forms:

$$\frac{\partial \eta}{\partial t^{\alpha}}+\frac{\partial}{\partial x^{\beta}}\left(-\frac{1}{2}\eta^{2}-\mu+\lambda\frac{\partial \eta}{\partial x^{\beta}}\right)=0 \tag{7}$$

$$\frac{\partial \mu}{\partial t^{\alpha}}+\frac{\partial}{\partial x^{\beta}}\left(-\eta\mu-\lambda\frac{\partial \mu}{\partial x^{\beta}}\right)=0 \tag{8}$$

In the light of eqs. (3) or (4), we can recommend a special function Θ conformed to:

$$\frac{\partial \Theta}{\partial t^{\alpha}}=-\frac{1}{2}\eta^{2}-\mu+\lambda\frac{\partial \eta}{\partial x^{\beta}} \tag{9}$$

$$\frac{\partial \Theta}{\partial x^{\beta}}=-\eta \tag{10}$$

Analogously, from eq. (4) or (8), we can introduce another special function Ξ defined:

$$\frac{\partial \Xi}{\partial t^{\alpha}}=-\eta\mu-\lambda\frac{\partial \mu}{\partial x^{\beta}} \tag{11}$$

$$\frac{\partial \Xi}{\partial x^{\alpha}}=-\mu \tag{12}$$

Our goal in this work is to build some variational formulations whose stationary conditions content eqs. (3), (11) or (4), (9), and (10). For this reason, we will apply He's semi-inverse method [14-16] to build a trial functional:

$$J(\eta,\mu,\Theta)=\iint L\mathrm{d}x^{\beta}\mathrm{d}t^{\alpha} \tag{13}$$

where L is a trial Lagrangian defined:

$$L=\mu\frac{\partial \Theta}{\partial t^{\alpha}}+\left(-\eta\mu-\lambda\frac{\partial \mu}{\partial x^{\beta}}\right)\frac{\partial \Theta}{\partial x^{\beta}}+F(\eta,\mu) \tag{14}$$

where F is an unknown function of η,μ and/or their derivatives. The superiority of the trial Lagrangian is that the stationary condition with respect to Θ is one of the governing eqs. (8) or (4).

Computing aforementioned eq. (13) stationary with respect to η and $\mu,$ we acquire the following Euler-Lagrange equations:

$$-\mu\frac{\partial \Theta}{\partial x^{\beta}}+\frac{\delta F}{\delta \eta}=0 \tag{15}$$

$$\frac{\partial \Theta}{\partial t^{\alpha}}-\eta\frac{\partial \Theta}{\partial x^{\beta}}+\lambda\frac{\partial \Theta}{\partial x^{2\beta}}+\frac{\delta F}{\delta \mu}=0 \tag{16}$$

where $\delta F/\delta\eta$ is a known He's variational derivative [44] with respect to η, which was suggested by He [44], who gave the following definition:

$$\frac{\delta F}{\delta\eta}=\frac{\partial F}{\partial\eta}-\frac{\partial}{\partial t^{\alpha}}\left[\frac{\partial F}{\partial\left(\dfrac{\partial\eta}{\partial t^{\alpha}}\right)}\right]-\frac{\partial}{\partial x^{\beta}}\left[\frac{\partial F}{\partial\left(\dfrac{\partial\eta}{\partial x^{\beta}}\right)}\right]+\cdots \tag{17}$$

We look for such an F so that eq. (15) is equivalent to eq. (9), and eq. (16) is equivalent to eq. (10). So in view of eqs. (9) and (10), we put up:

$$\begin{gathered}\frac{\delta F}{\delta\eta}=\mu\frac{\partial\Theta}{\partial x^{\beta}}=-\eta\mu\\ \frac{\delta F}{\delta\mu}=-\frac{\partial\Theta}{\partial t^{\alpha}}+\eta\frac{\partial\Theta}{\partial x^{\beta}}-\lambda\frac{\partial\Theta}{\partial x^{2\beta}}=-\frac{1}{2}\eta^{2}+\mu\end{gathered} \tag{18}$$

From eq. (18), the unknown F can be calculated accurately:

$$F(\eta,\mu)=-\frac{1}{2}\eta^{2}\mu+\frac{1}{2}\mu^{2} \tag{19}$$

Ultimately, we gain the following required variational principle:

$$J(\eta,\mu,\Theta)=\iint\left[\mu\frac{\partial\Theta}{\partial t^{\alpha}}+\left(-\eta\mu-\lambda\frac{\partial\mu}{\partial x^{\beta}}\right)\frac{\partial\Theta}{\partial x^{\beta}}-\frac{1}{2}\eta^{2}\mu+\frac{1}{2}\mu^{2}\right]\mathrm{d}x^{\beta}\mathrm{d}t^{\alpha} \tag{20}$$

Proof. Making the aforementioned eq. (20) stationary with respect to Θ, η and μ, we acquire the following Euler-Lagrange equations:

$$-\frac{\partial\mu}{\partial t^{\alpha}}-\frac{\partial}{\partial x^{\beta}}\left(-\eta\mu-\lambda\frac{\partial\mu}{\partial x^{\beta}}\right)=0 \tag{21}$$

$$-\mu\frac{\partial\Theta}{\partial x^{\beta}}-\eta\mu=0 \tag{22}$$

$$\frac{\partial\Theta}{\partial t^{\alpha}}-\eta\frac{\partial\Theta}{\partial x^{\beta}}+\lambda\frac{\partial\Theta}{\partial x^{2\beta}}-\frac{1}{2}\eta^{2}+\mu=0 \tag{23}$$

Equation (21) is equivalent to eq. (8), and eq. (22) is equivalent to (10), in view of (10), eq. (23) becomes eq. (9).

Analogously, we can also start with the following trial Lagrangian:

$$L_{1}(\eta,\mu,\Xi)=\eta\frac{\partial\Xi}{\partial t^{\alpha}}+\left(-\frac{1}{2}\eta^{2}-\mu+\lambda\frac{\partial\eta}{\partial x^{\beta}}\right)\frac{\partial\Xi}{\partial x^{\beta}}+H(\eta,\mu) \tag{24}$$

It is apparent that the stationary condition with respect to Ξ is equal to eq. (7) or eq. (3). Now the Euler-Lagrange equations with regard to η and μ, are:

$$\frac{\partial\Xi}{\partial t^{\alpha}}-\eta\frac{\partial\Xi}{\partial x^{\beta}}-\lambda\frac{\partial\Xi}{\partial x^{2\beta}}+\frac{\delta H}{\delta\eta}=0,\quad-\frac{\partial\Xi}{\partial x^{\beta}}+\frac{\delta H}{\delta\mu}=0 \tag{25}$$

Considering eqs. (11) and (12), we gain:

$$\frac{\delta H}{\delta \eta} = -\frac{\partial \Xi}{\partial t^{\alpha}} + \eta \frac{\partial \Xi}{\partial x^{\beta}} + \lambda \frac{\partial \Xi}{\partial x^{2\beta}} = 0, \quad \frac{\delta H}{\delta \mu} = \frac{\delta \Xi}{\delta x^{\beta}} = -\mu \tag{26}$$

In light of eq. (26), the unknown function $H(\eta,\mu)$ can be determined:

$$H(\eta,\mu) = -\frac{1}{2}\mu^2 \tag{27}$$

As a result, we acquire another needed variational formulation:

$$J_1(\eta,\mu,\Xi) = \iint \left[\eta \frac{\partial \Xi}{\partial t^{\alpha}} + \left(-\frac{1}{2}\eta^2 - \mu + \lambda \frac{\partial \eta}{\partial x^{\beta}} \right) \frac{\partial \Xi}{\partial x^{\beta}} - \frac{1}{2}\mu^2 \right] \mathrm{d}x^{\beta} \mathrm{d}t^{\alpha} \tag{28}$$

Conclusion

Zhang [17] has constructed variational formulations for HOLWWE in a continuous space. In this work, we successfully extend HOLWWE into a fractal HOLWWE based on He's fractal derivative, and establish fractal variational principles. The obtained fractal variational principles have been proved to be correct by minimizing the corresponding functionals. The paper also reveals that the semi-inverse method is influential and straightforward.

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References

[1] Liu, J. *et al.*, Group Analysis of the Time Fractional (3 + 1)-Dimensional KdV-Type Equation, *Fractals, 29* (2021), 6, 2150169
[2] Liu, J. G. *et al.*, On Integrability of the Higher-Dimensional Time Fractional KdV-Type Equation, *J. Geom. Phys., 160* (2021), 2, 104000
[3] Sun, J. S., Analytical Approximate Solutions of (N+1)-Dimensional Fractal Harry Dym Equations, *Fractals, 26* (2018), 6, 1850094
[4] Sun, J. S., Approximate Analytic Solution of the Fractal Klein-Gordon Equation, *Thermal Science, 25* (2021), 2, pp. 1489-1494
[5] Sun, J. S., Traveling Wave Solution of Fractal KDV-Burgers-Kuramoto Equation Within Local Fractional Differential Operator, *Fractals, 29* (2021), 7, 2150231
[6] Yau, H. T., *et al.*, Synchronization Control for a Class of Chaotic Systems with Uncertainties, *Internetional Journal of Bifurcation and Chaos, 15* (2005), 7, pp. 2235-2246
[7] Yau, H.T., Generalized Projective Chaos Synchronization of Gyroscope Systems Subjected to Dead-Zone Non-linear Inputs, *Physics Letters A, 372* (2008), 14, pp. 2380-2385
[8] Yau, H. T., Non-linear Rule-Based Controller for Chaos Synchronization of Two Gyros With Linear-Plus-Cubic Damping, Chaos, *Solitons and Fractals, 34* (2007), 4, pp 1357-1365
[9] Lin, J.-S., *et al.*, Synchronization of Unidirectional Coupled Chaotic Systems with Unknown Channel Time-Delay: Adaptive Robust Observer-Based Approach, *Chaos, Solitons and Fractals, 26* (2005), 3, pp. 971-978.
[10] Lin, C. J.,*et al.*, Chaos Suppression Control of a Coronary Artery System with Uncertainties by Using Variable Structure Control, *Computers & Mathematics with Applications, 64* (2012), 5, pp. 988-995

[11] Chen, C. L., *et al.*, Design of Extended Backstepping Sliding Mode Controller for Uncertain Chaotic Systems, *International Journal of Non-linear Sciences and Numerical Simulation, 8* (2007), 2, pp. 137-145
[12] Yau, H. T., *et al.*, Comparison of Extremum-Seeking Control Techniques for Maximum Power Point Tracking in Photovoltaic Systems, *Energies, 4* (2011), 12, pp. 2180-2195
[13] Finlayson, B. A., Existence of Variational Principles for the Navier-Stokes Equation, *Physics of Fluids, 15* (1972), 6, pp. 63-967
[14] He, J.-H., *et al.*, A Fractal Modification of Chen-Lee-Liu Equation and Its Fractal Variational Principle, *International Journal of Modern Physics B, 35* (2021), 21, 2150214
[15] He, J.-H., *et al.*, Hamiltonian-Based Frequency-Amplitude Formulation for Non-linear Oscillators, *Facta Universitatis-Series Mechanical Engineering, 19* (2021), 2, pp. 199-208
[16] He, J.-H., A Fractal Variational Theory for One-Dimensional Compressible Flow in a Microgravity Space, *Fractals, 28* (2020), 2, 2050024
[17] Zhang, W., Generalized Variational Principle for Long Water-Wave Equation by He's Semi-Inverse Method, *Math. Probl. Eng., 2009* (2009), ID925187
[18] Wang, M., *et al.* Application of a Homogeneous Balance Method to Exact Solutions of Non-linear Equations In Mathematical Physics, *Phys. Lett. A, 216* (1996), 1-5, pp. 67-75
[19] Qian, M. Y., He, J. H., Two-Scale Thermal Science for Modern Life – Making the Impossible Possible, *Thermal Science, 26* (2022), 3B, pp. 2409-2412
[20] He, J.-H., Seeing with a Single Scale is Always Unbelieving: From Magic to Two-Scale Fractal, *Thermal Science, 25* (2021), 2B, pp. 1217-1219
[21] He, J.-H., El-Dib, Y. O., A Tutorial Introduction to the Two-scale Fractal Calculus and its Application to the Fractal Zhiber-Shabat Oscillator, *Fractals, 29* (2021), 8, 2150268
[22] He, J.-H., *et al.*, Variational Approach to Fractal Solitary Waves, *Fractals, 29* (2021), 7, 2150199
[23] He, J.-H., *et al.*, Solitary Waves Travelling Along an Unsmooth Boundary, *Results in Physics, 24* (2021), May, 104104
[24] Wu, P. X., *et al.*, Solitary Waves of the Variant Boussinesq-Nurgers Equation in a Fractal Dimensional Space, *Fractals, 30* (2022), 3, 2250056
[25] Tian, D., *et al.*, Fractal N/MEMS: from Pull-in Instability to Pull-in Stability, *Fractals, 29* (2021), 2, 2150030
[26] Tian, D., *et al.*, A Fractal Micro-Electromechanical System and Its Pull-In Stability, *Journal of Low Frequency Noise Vibration and Active Control, 40* (2021), 3, pp. 1380-1386
[27] Tian, D., *et al.*, Fractal Pull-in Stability Theory for Microelectromechanical Systems, *Frontiers in Physics, 9* (2021), Mar., 606011
[28] He, J.-H., When Mathematics Meets Thermal Science, The Simpler is the Better, *Thermal Science, 25* (2021), 3B, pp. 2039-2042
[29] He, C. H., *et al.*, A Fractal Model for the Internal Temperature Response of a Porous Concrete, *Applied and Computational Mathematics, 21* (2022), 1, pp. 71-77
[30] He, J.-H., *et al.*, Forced Non-linear Oscillator in a Fractal Space, *Facta Universitatis Series: Mechanical Engineering, 20* (2022), 1, pp. 1-20
[31] He, C. H., *et al.*, A Modified Frequency-Amplitude Formulation for Fractal Vibration Systems, *Fractals, 30* (2022), 3, 2250046
[32] Wang, K. L., Wei, C. F., A Powerful and Simple Frequency Formula to Non-linear Fractal Oscillators, *Journal of Low Frequency Noise Vibration and Active Control, 40* (2021), 3, pp. 1373-1379
[33] Feng, G. Q., He's Frequency Formula to Fractal Undamped Duffing Equation, *Journal of Low Frequency Noise Vibration and Active Control, 40* (2021), 4, pp. 1671-1676
[34] Anjum, N., *et al.*, Two-scale Fractal Theory for the Population Dynamics, *Fractals, 29* (2021), 7, 2150182
[35] He, J.-., *et al.*, Evans Model for Dynamic Economics Revised, *AIMS Mathematics, 6* (2021), 9, pp. 9194-9206
[36] Wang, Y., *et al.*, A Variational Formulation for Anisotropic Wave Traveling in a Porous Medium, *Fractals, 27* (2019), 4, 19500476
[37] Wang, K. L, He, C. H. A Remark on Wang's Fractal Variational Principle, *Fractals, 27* (2019), 8, 1950134
[38] Wang, S. Q., He, J.-H., Variational Iteration Method for Solving Integro-Differential Equations, *Physics letters A, 367* (2007), 3, pp. 188-191

[39] Wang, S. Q., A Variational Approach to Non-linear Two-Point Boundary Value Problems, *Computers & Mathematics with Applications, 58* (2009), 11, pp. 2452-2455
[40] Yu, W., *et al.*, Tensorizing GAN With High-Order Pooling for Alzheimer's Disease Assessment, *IEEE Transactions on Neural Networks and Learning Systems, 33* (2021), 9, pp. 4945-4959
[41] You, S., *et al.*, Fine Perceptive Gans for Brain MR Image Super-Resolution in Wavelet Domain, *IEEE transactions on neural networks and learning systems*, On-line first, https://doi.org/10.1109/TNNLS.2022.3153088, 2022
[42] Hu, S., *et al.*, Bidirectional Mapping Generative Adversarial Networks for Brain MR to PET Synthesis, *IEEE Transactions on Medical Imaging, 41* (2021), 1, pp. 145-157
[43] Yu, W., *et al*., Morphological Feature Visualization of Alzheimer's Disease *via* Multidirectional Perception GAN, *IEEE Transactions on Neural Networks and Learning Systems,* On-line first, https://doi.org/10.1109/TNNLS.2021.3118369, 2021
[44] He, J.-H., Variational Principles for Some Non-linear Partial Differential Equations with Variable Coefficients, *Chaos Solitons Fractals, 19* (2004), 4, pp. 847-851

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