SOME NOTES ON THE MAXIMUM PRINCIPLE OF SEMI-LINEAR DYNAMICAL SYSTEM

by

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Original scientific paper
https://doi.org/10.2298/TSCI2303889D

The maximum value principle plays an important role in the study of a semi-linear system within a time domain, and it uses in this paper to study the existence of solutions for systems with general non-linear terms.

Key words: *maximum principle, dynamical system, semi-linear*

Introduction

There have been many mature and graceful results in the research on the dynamic behavior of semi-linear equations [1, 2]. However, for coupled semi-linear systems [3], the study of its dynamic properties is very complicated [4, 5]. Previous studies have shown that the results of most semi-linear systems cannot be simply extended to general semi-linear systems [6-8]. One of the main reasons is that the complexity of the function structure of the semi-linear system makes it very difficult to establish its maximum value principle [9, 10], which is an important tool for the study of semi-linear systems.

For semi-linear dynamical systems, the maximum principle has a wide practical physical background, and is a very good tool for the existence and asymptotic behavior of solutions to initial value problems of viscous conservation laws, and the asymptotic stability of viscous shock solutions. In recent years, Liu and Nishihara [11], Liu, Matsumura and Nishihara [12] also proved the asymptotic stability of viscous shock wave and evacuation wave solutions and the asymptotic form of solutions for internal flow problems by using the maximum principle and energy estimation methods. Kawashima and Matsumura [13, 14] solved the situation of standing wave and evacuation wave in outflow problem.

Particularly many coupled mathematical models arising in practical applications can be considered in the framework of the following semi-linear dynamical system

$$\frac{\partial u_i}{\partial t} = L_i(t)u_i + B_i(t,x,\mathbf{u}), \quad (t,x) \in (0,\infty)\Omega$$

$$u_i(0,x) = u_{i,0}(x), \quad x \in \Omega, \quad (i = 1,\cdots,N) \tag{1}$$

where $\mathbf{u} = (u_1,\cdots,u_N),\Omega$ is a bounded region $\mathbb{R}^n (n \geq 1)$ or unbounded region (including the outer region of the full space bounded region $\mathbb{R}^n$ and other unbounded regions Ω_e), such as

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positive half space $\mathbb{R}_+^n = \{x = (x_1, \cdots, x_n) \in \mathbb{R}^n; x_n > 0\}$, $L_i(t)$ and $B_i(t)$ are the following differential operators and boundary operators, respectively:

$$L_i(t) = \sum_{j,k=1}^{n} a_{jk}^{(i)}(t,x) \frac{\partial^2 u}{\partial x_j \partial x_k} + \alpha_i \frac{\partial u}{\partial \nu} \tag{2}$$

$$B_i(t) = \sum_{j=1}^{n} b_j^{(i)}(t,x) \frac{\partial u}{\partial x_j} + \beta_i(t,x), \qquad (i = 1, \cdots, N) \tag{3}$$

where $\alpha_i = 0, \beta_i(t,x) < 1(t > 0, x \in \partial\Omega)$ (Dirichlet-type boundary operator) or $\alpha_1 > 1$, $\beta_{\mathbf{z}}(t,x) = 0(t > 0, x \in \partial\Omega)$ Neumann-Robin-type boundary operator, $\partial/\partial \nu$ represents the outer normal derivative on $\partial\Omega$. These models include the Lotka-Volterra diffusion model [15], Noyes-Field model [16] of gas-liquid reaction model, simplified biochemical enzyme reaction model, Fokker-Planck model in fractal media [17], non-linear oscillation model [18], and fractional KdV model [19].

So far, many authors have used different methods to study the dynamics of the system or its corresponding specific reaction diffusion model. Wang [20] used the Lyapunov function method on the bounded region to discuss the autonomous system namely $L_i(t,x) = L_i(x)$, $f_i(t,x,\mathbf{u}) = f_i(x,\mathbf{u})$, $g_i(t,x) = g_i(x)$, $\beta_i(t,x) = \beta_i(x)$, $i = 1, \cdots, N$. Using the monotonic method [21], also known as the upper and lower solution method, the autonomous system was studied on the unbounded area, and the periodic system was studied on the unbounded area with the monotonic method (that is, the existence of constant $T > 0$, such that:

$$L_i(t,x) = L_i(t+T,x)$$

$$f_i(t,x,\mathbf{u}) = f_i(t+T,x,\mathbf{u})$$

$$g_i(t,x) = g_i(t+T,x)$$

$$\beta_i(t,x) = \beta_i(t+T,x)$$

$$i = 1, \cdots, N, \qquad u(r,t) = e^{L_1 t}(K + e^{-M_1 m})^{1/m}, \qquad m = (r-1)e^{J_{11}}, \qquad r \geq 1,\ t \geq 0$$

The linearization method is used to study the periodic systems [22-25] in the bounded area, as well as the competitive diffusion system and the periodic cooperative diffusion system [26]. We can also use the operator semigroup theory and the dynamic system method [27] to study the autonomous system in the unbounded area, including existence, boundedness and equilibrium of the overall solution and the positive definite stability of the zero solution with respect to disturbances.

From the general theory of partial differential equations, the comparison principle based on the maximum principle is a useful tool for studying parabolic partial differential equations or systems:

$$M_1 = (K+1)^{\overline{\frac{\alpha}{n(P-1)}}}, \quad M_2 = (K+1)^{\overline{\frac{\beta}{m(q-1)}}}$$

In the study of the dynamic behavior of a semi-linear system or its corresponding specific semi-linear system model. The response function:

$$K \le \max\{1-m(p-1), 1-n(q-1), mp, nq\}$$

is required to satisfy a certain monotonic property, which is the so-called quasi-monotonicity of the response function, which includes quasi-monotonic non-decreasing, quasi-monotonic non-increasing, and mixed quasi-monotonicity [28].

Preliminary

Vector function $\mathbf{f}(t,x,\mathrm{u}) := [f_1(t,x,\mathbf{u}),\cdots, f_N(t,x,\mathrm{u})]$, $f(t,x,\cdot)$ is mixed quasi-linear monotonic on Subset $\mathcal{M}$ of $\mathrm{R}^N (N \ge 1)$, meaning for every $i = 1,\cdots,N$, there are non-negative integers related to a_i and b_i, satisfying $a_i + b_i = N - 1$. The writable vector $\mathbf{u}$ is a separate form $\mathbf{u} = (u_i, [\mathbf{u}]_{a_i}, [\mathbf{u}]_{b_i})$, so that each component is monotonous and non-decreasing, and each component of $\mathbf{u} \in \mathcal{M}$, $f_i(t,x,u_i,[\mathbf{u}]_{a_i},[\mathbf{u}]_{b_i})$ is monotonic and non-increasing. In particular, if there is $[\mathbf{u}]_{a_i}$ for each one, then $\mathbf{f}(t,x,\mathbf{u})$ is said $[\mathbf{u}]_{b_i}$ to be quasi-monotonous (b_i = 0) and non-decreasing; if there is $i = 1,\cdots,N$ for each $a_i = 0$, then $\mathbf{f}(t,x,\mathbf{u})$ is said to be quasi-monotonous and non-increasing [5].

Mixed quasi-monotonic means quasi-monotonic non-decreasing or quasi-monotonic non-increasing, if the vector function *f*(*t*, *x*, ·) is mixed quasi-monotonic on a certain subset of $\mathcal{M}$ in $\mathbb{R}^N$.

The quasi-monotonicity of the reaction function is the basis for establishing the maximum value and comparison principle for the corresponding reaction-diffusion system, and the quasi-monotonicity of the reaction function has a profound background explanation in the application.

$$\frac{\partial u_i}{\partial t} = d_i(t,x)\Delta u_i, \quad (t,x) \in (0,\infty)\times\Omega$$
$$\mathbf{u} = (u_1,\cdots,u_N)u_i h_i(\mathbf{u})$$
$$\partial u_i = 0, \quad i = 1,2,...,N \tag{4}$$

where $d_i(t,x) > 0 (i = 1,2)$ represents the semi-linear time lag coefficient that varies with time.

The result of the comparison principle of the system, namely $\tau_i = 0(i = 1,\cdots,N))$, is extended to the semi-linear time domain system, that is, the following completely non-linear reaction diffusion system:

$$\frac{\partial u_i}{\partial t} = f_i(t,x,u,\nabla u_i,\nabla^2 u_i), \quad (t,x) \in (0,\infty)\times\Omega \tag{5}$$

$$B_i u_i = \sum_{j=1}^{N} K_{ij} \int_{\Omega} u_j(t,y)\mathrm{d}y\nabla, \quad (t,x) \in (0,\infty)\partial\times\Omega \tag{5}$$

$$\partial u_i(t,x) = u_{i0}, \quad x \in \Omega, \quad (i = 1,\cdots,N) \tag{6}$$

where $\mathbf{u} = (u_1,\cdots,u_N), \nabla u_i$, and $\nabla^2 u_i$ represent the gradient and Hessian matrix of spatial variables, respectively, and $f(t,x,u) \equiv [f_1(t,x,u,\nabla u_1,\nabla^2 u_1),\cdots, f_N(t,x,u, \quad \nabla u_N,\nabla^2 u_N)]$ is a quasi-monotonic non-decreasing vector function.

Since the early 1990's, the Cauchy problem of abstract operator differential equations with non-local initial conditions in Banach spaces was widely studied, which is:

$$u'(t) = \mathcal{A}u(t) + F[t, u(t)] + H(t_1, \cdots, t_p, u), \quad t \in (0, T] \tag{8}$$

$$u(0) = u_0 \tag{9}$$

The domain of $\mathcal{A}$ is $D(\mathcal{A})(\subset X, X), 0 < t_1 < \cdots < t_{p-1} < t_p < T(p \in N)$, $F:[0,T]X \to X$ and $H:(0,T)^p X \to Y$ are a linear operator in a complex Banach space, D and E are a subset of the given operator F and the corresponding integral-differential equations and non-autonomous equations. This kind of Cauchy problem Research has made a series of important progress.

Let $1 < m \le p < \Lambda m + 1/N, u_0 \in L^1(R^N)$. Then all nonnegative nontrivial weak solutions $u(x,t)$ do not exist globally in the whole time, and blow up in finite time, *i.e.*:

$$\lim_{t \uparrow T} \| u(t) \|_\infty = \infty \tag{10}$$

for some $T \in (0, \infty)$.

Main result

Let functions $f_i(\cdot, u, v)$ and v be continuously differentiable and a bounded kernel function, $K_{ij}(x,y)(i, j = 1, \cdots, N)$ satisfying:

$$K_{ij}(x,y) \ge 0, \quad \int_\Omega \sum_{j=1}^N K_{ij}(x,y)\mathrm{d}y < 1 \quad (x \in \partial\Omega, y \in \Omega) \tag{11}$$

$$\mathbf{f}(t, x, \mathbf{u}, \mathbf{v}) = [f_1(t, x, \mathbf{u}, \mathbf{v}), \cdots, f_N(t, x, \mathbf{u}, \mathbf{v})] \tag{12}$$

The vector function $f(t, x, \mathbf{u}, \mathbf{v})$ is mixed quasi-monotonic on the subset $R^N \times R^N$ of $\mathcal{J}$:

$$\mathbf{u} = (u_i, [\mathbf{u}]_{a_i}, [\mathbf{u}]_{b_1}), \quad \mathbf{v} = ([\mathbf{v}]_{c_i}, [\mathbf{v}]_{d_1}), \quad (\mathbf{u}, \mathbf{v}) \in \mathcal{J} \tag{13}$$

$u(t) = U[t, u(0)]$, here:

$$U: \{(t, u_0): 0 \le t \le T(u_0), u_0 \in L^1_{\text{loc}}(R^N) \cap L^\infty_{loc}(R^N)\} \to R^+$$

for $0 \le t \le \min[T, T(u_0)]$:

$$f_i(t, x, \mathbf{u}, \mathbf{v}) \equiv f_i(\cdot, u_i, [\mathbf{u}]_{a_i}, [\mathbf{u}]_{b_i}, [\mathbf{v}]_{c_1}, [\mathbf{v}]_{d_1}) \tag{14}$$

$$\int_M K(x,y) | \mathrm{d}y \ll 1 \tag{15}$$

$$D_M = [0, M] \times \Omega, \quad \bar{D}_M = [0, M] \times \bar{\Omega}, \quad S_M = (0, M] \times \partial\Omega \tag{16}$$

$$\bar{Q}_M^{(i)} = [-\tau_i, M] \times \bar{\Omega}, \quad \bar{Q}_M = \bar{Q}_M^{(1)} \times \cdots \times \bar{Q}_M^{(N)} \tag{17}$$

Regarding each component of $[u]_{a_2}$ and $[\mathbf{v}]_{c_1}$, it is monotonous and no decreasing, while regarding $[\mathbf{u}]_{b_1}$, a component is monotonic and no decreasing [18]. In particular, if there is $b_i = d_i = 0$ for every $i = 1, \cdots, N$, then $\mathbf{f}(\cdot, \mathbf{u}, \mathbf{v})$ is said to be quasi-monotonic on $\mathcal{J}$.

Theorem. Suppose that $\mathbf{q} = (q_1, \cdots, q_N) \in \mathcal{C}^{1,2}(D_M) \cap \mathcal{C}(\bar{D}_M)$, then $\mathbf{q}(t,x) \le 0$ on $\bar{D}_M$.

Proof: We consider the following conditions:

$$1 \geq K(x,y) \geq 0, \quad R(x) > 1 \quad (x > 1, y > 2) \tag{18}$$

$$K(x,y) > 1, \quad R(x) \leq 1, \quad R(x) \neq 1 \quad (x \leq 0, y \leq 2) \tag{19}$$

If b_{ij} is bounded on D_M, and $b_{ij} \geq 0 \quad (j \neq i, i, j = 1, \cdots, N)$, and further:

$$L = \sum_{i,j=1}^{n} a_{ij}(x) \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{j=1}^{n} b_j(x) \frac{\partial}{\partial x_j} \tag{20}$$

$$\frac{\partial q_i}{\partial t} = L_i(t) q_i + \sum_{j=1}^{N} b_{ij} q_j \quad (t,x) \in D_M \tag{21}$$

$$D = (0,\infty) \times R, \quad D = [0,\infty) \times \pi, \quad S = (0,\infty) \times Q, \quad Q_0^{(i)} = [-\pi, 0] \times Q, \quad Q^{(i)} = [-\pi_{i,\infty} \infty) \times Q$$

For $k \geq 1, \quad n \in N$, the solution satisfies:

$$\frac{\partial}{\partial t} \int_{B_n} u^k \, \mathrm{d}x + \frac{4m^*(k-1)}{\Lambda^2 [m^* + (k-1)]^2}$$

$$\int_{B_n} \left| Du^{ai} \right|^2 \mathrm{d}x \leq k \int_{B_n} u^{k+p+1} \sim \mathrm{d}x, \quad i = 1, 2$$

where

$$a_1 = \frac{m^* + \Lambda(k-1)}{2}, \quad a_2 = \frac{m+k-1}{2}, \quad B_n = \{x : |x| < n\}$$

and

$$B_i q_i(t,x) = \sum_{j=1}^{N} \int_{\Omega} K_{ij}(x,y) q_i(t,y) \mathrm{d}y \quad (t,x) \in S_M \tag{22}$$

$$q_i(0,x) = 0 \quad x \in \Omega, \quad i = 1, \cdots, N$$

From $\mathbf{q}(t,x) < 0, \quad (t,x) \in \bar{D}_M$, let $r = (r_1, \cdots, r_N)$, here $r_i = q_i e^{-\beta t}$, using the Hopf strong maximum principle [20] of the second-order parabolic partial differential equation, we obtain:

$$\left. \frac{\partial r_i}{\partial \nu} \right|_{(t,\bar{x})} = \alpha_i \frac{\partial r_i}{\partial \nu} (\bar{t}, \bar{x}) > 0 \tag{23}$$

$$\alpha_i(\bar{x}) \frac{\partial r_i}{\partial \nu} (\bar{t}, \bar{x}) + r_i(\bar{t}, \bar{x}) \geq 0 \tag{24}$$

From $D_t r_i(t,x) < 0,$, it can be seen that:

$$\int_{\Omega} \sum_{j=1}^{N} K_{ij} q_i(\bar{t}, y) \mathrm{d}y \leq 0 \tag{25}$$

Let $\mathbf{p} = (p_1, \cdots, p_N)$, $p_i := q_i - \varepsilon e^{\delta t}$, $\varepsilon > 0$, using assumptions and direct calculate, one can deduce:

$$B_{ii}(t_i x) \le \int_{j=1}^{N} K(x,y)\pi_{ij}(t,y)\mathrm{d}x \tag{26}$$

$$p_i(t,x) < \int_{\Omega} p_i(t,y)\mathrm{d}y, \quad (t,x) \in S_M \tag{27}$$

$$r^* = \min(r_1, \cdots, r_N) > 0$$

$$q_T + j(t,x) = q_1(t - r^*, x) \le 0 \quad (j = 1, \cdots, N) \tag{28}$$

$$\frac{\partial q_i}{\partial t} - L_i(t) q_i \le \sum_{j=1}^{N} b_{ij}; \quad (t_{jj},\ x) \in Q_{ij} \tag{29}$$

$$q(t,x) \le 0, \quad (t,x) \in S_M \tag{30}$$

$$q_1(t - r^*, x) \ge 0, \quad (t,x) \in S_M \tag{31}$$

$$u(t,x) \ge u(t+T, x) \tag{32}$$

$$q_i(t,x) \le 0, (t,x) \in Q_{1/2r}, \quad i = 1, \cdots, N \tag{33}$$

In addition, because it satisfies the hypothesis, it is known from the lemma that for all from the *Theorem* [20], we can give the uniqueness of a solution of the reaction-diffusion system. The previous inequality is also true, so:

$$u = (u_1 + \cdots, u_N), \quad v = (v_1, \cdots, v_N) \le L^2(D_M) \times C^1(Q_M) \tag{34}$$

$$u(t,x) = v(t,x), \quad (t,x) \in S_0 \tag{35}$$

$$u(t,x) \le u^{(m)}(t,x) \le u^{(m+1)}(t,x) \tag{36}$$

$$|B(u,v)| \le \alpha |u|_{H_0^1(\Omega)} |v|_{H_0^1(\Omega)} \tag{37}$$

$$\beta \|u\|_{H_0^2} (sz_0) \le B(u,_v) + \gamma \| \ \| u |_L^2 \ (a) \tag{38}$$

$$\theta \int_R |Du|^2 \mathrm{d}x \le \int_{R_1, 1=1}^{n} e^{i_1 \pi}(x) u_{xi}, u_{xj}, \mathrm{d}x_y \mathrm{d}x$$

$$< B(u,u) - \int_0 \left[\sum_{i=1}^{n} b^i u_{x,u} + c(x) uu \right] \mathrm{d}x$$

$$\le B(u,u) + \sum_{i=1}^{n} \|b^i\|_i \ \cos^i |x| \|u| xu + \|c|_i x| u^2 \mathrm{d}x$$

→

$$\leq \int_{\pi} | Du || u | \mathrm{d}x + \in | Du |^2 \ \mathrm{d}x + \frac{1}{4\in}\int_{\pi} u^2 u^2 \mathrm{d}x, \quad \forall \varepsilon > 0$$

$$\leq \frac{\theta}{2}\int (Du|^2 \mathrm{d}x \leq B(u,u) + C\int_R u^2 \mathrm{d}x$$

$$\leq \frac{\theta}{3C_0^2}\|u\|_{L^2}(2) + \frac{\theta}{3}|Du|_{L^2}(A)$$

$$\leq \frac{\theta}{2}\int_R^{\theta} |Du|_R \mathrm{d}x + \frac{\theta}{2}\int_1^{\theta} |Du|^2 \ \mathrm{d}x \tag{39}$$

Combine some relevant results that meet the boundary conditions [20-26]:

$$u(t,x) = \int_{\Omega} K(x,y) u(t,y) \mathrm{d}y \quad (t > 0, \ x \in \partial\Omega) \tag{40}$$

$$B\left\| u'|_{H_0}^2 (a_0) \right\|_{L^2}^2 \leq B(u_0, u) + \gamma(n) \tag{41}$$

$$\alpha(x)\frac{\partial u}{\partial \nu}(t,x) + u(t,x) = \int_{\Omega} K(x,y) u(t,y) \mathrm{d}y \quad (t > 0, \ x \in \partial\Omega) \tag{42}$$

For a completely non-linear system, the comparison principle is used to study the comparison principle of the semi-linear reaction-diffusion equation that satisfies the boundary conditions, and the following boundary conditions are considered.

Assume that $u_0 \in BC(R^N)$. Then for some $T = T(u_0) > 0$, exist a weak solution $u(t) = u(x,t) : Q_T \to R^+$ satisfying $u \in C\{[0,T]; L^1_{\mathrm{loc}}(R^N)\} \cap L^{\infty}_{\mathrm{loc}}(0,T;R^N)$ and for constant C':

$$u \leq C' e^{-\frac{|x|}{m^*}}, \quad t \in [0,T) \tag{43}$$

$\{\| f \|_{L^p} : f \in X\}$ are bounded.

The X is isocratic overall continuous: $\forall \varepsilon > 0, \exists \delta > 0$, for $\forall f \in X$, only $|\mathbf{z}| < \delta$ there exists:

$$\left(\int_{\Omega} | f(\mathbf{x}+\mathbf{z}) - f(\mathbf{x}) |^p \ \mathrm{d}\mathbf{x} \right)^{\frac{1}{p}} < \varepsilon \tag{44}$$

Consistent establishment:

$$\lim_{R\to\infty} \int_{\{x\in\Omega, \ \|x\|\geq R\}} | f(x) |^p \ \mathrm{d}x = 0$$

For $f \in R^n$:

$$B_i u_i = \int_{\Omega} K_i(x,y) u_i(t,y) \mathrm{d}y + h_i(x) \quad (t > 0, \ x \in \partial\Omega, \ i = 1, \cdots, N)$$

The comparison principle of the semi-linear reaction-diffusion equation system is constructed. Based on the previous results, we can establish a general comparison principle for more complex composite systems. In this way, for the semi-linear reaction-diffusion equation, we have established the system maximum principle.

Conclusion

For each $i=1,\cdots,N$, $f_i \in C[\bar{D}_M \times \mathbb{R}^N \times \mathbb{R}^n \times \mathbb{R}^{n^2}, \mathbb{R}]$ and the vector function $\mathbf{f}(t,x,\mathbf{u})$ are elliptical on (t_1,x_1) if $\mathrm{u}, P, S_{jk}, T_{jk}\,(j,k=1,\cdots,n)$ and $y_i(t_i x)=u_i(t,x)-u_i(t+T_j x)$ $\Sigma_{j,k=1}^{n}(S_{jk}-T_{jk})\lambda_j\lambda_k \le 0,\ (t_i,x)\in Q \quad i=1,\cdots,N$, for any vector $\lambda \in \mathbb{R}^n$, thus:

$$f_i(t_1,x_1,\mathbf{u},P,S) \le f_i(t_1,x_1,\mathbf{u},P,T) \tag{45}$$

If vector function $f(t,x,u)$ is elliptical on D_M for each $i=1,\cdots,N$, $\mathbf{u}=(u_i,[\mathbf{u}]_{N-1})$, $f_i(t,x,\mathbf{u}) \equiv f_i(t,x,u_i,[\mathbf{u}]_{N-1},\nabla u_i,\nabla^2 u_i)$ are all monotonous and undiminished for all $u \in \mathcal{J}$ in $\mathbb{R}^N$, the vector function $\mathbf{f}(t,x,u)$ is also quasi-monotonous.

The reaction function $f(t,x,\mathbf{u})$ and the kernel function $K_{ij}(i,j=1,\cdots,N)$ are quasi-monotonic on some given subsets of $\mathbf{R}^N$; the kernel function $K_{ij}(i,j=1,\cdots,N)$ satisfies:

$$y_i(t,x)=u_i(t,x)-u_{ri}(t+T_j x), \quad i=1,\cdots,N \tag{46}$$

$$K_{ij}(x,y)>0, \quad \sum_N K_{ij}\int T_{jk}(x,y)\mathrm{d}y \le \lambda \quad (x\in\partial\Omega,\ y\in\Omega) \tag{47}$$

$$u_k(t,x) \le u(t+kT,x) \le u_k(t,x)+\varepsilon \tag{48}$$

Acknowledgment

This work was supported by the Scientific Research Project of Henan Education Department (No. 22B110019)

References

[1] Quiros, F., Rossi, J. D., Blow-up Sets and Fujita Type Curves for a Degenerate Parabolic System with Nonlinear Boundary Conditions, *Indiana Univ. Math. J., 50* (2001), 1, pp. 629-654
[2] Wang, Z. J., *et al.*, Critical Exponents for Porous Medium Systems Coupled via Nonlinear Boundary Flux, *Nonlinear Analysis-Theory Methods & Applications, 71* (2009), 5-6, pp. 2134-2140
[3] Souplet, P., Blow-up in Non-Local Reaction-Diffusion Equations, *SIAM J. Math. Anal., 29* (2006), 6, pp. 1301-1334
[4] Wang, L. Z., Li, K. T., On Positive Solutions of the Lotka-Volterra Cooperating Models with Diffusion, *Nonlinear Analysis – Theory Methods & Applications, 53* (2003), 7-8, pp. 1115-1125
[5] Yin, H. M., On a Class of Parabolic Equations with Non-local Boundary Conditions, *J. Math. Anal. Appl., 294* (1999), 2, pp. 712-728
[6] Yang, X.-J., *Theory and Applications of Special Functions for Scientists and Engineers*, Springer Nature, New York, USA, 2021
[7] Zhang, G. B., Zhao, X. Q., Propagation Phenomena for a Two-Species Lotka–Volterra Strong Competition System with Non-local Dispersal, *Calculus of Variations and Partial Differential Equations, 59* (2019), 1, 10
[8] Hastings, S. P., Murray, J. D., The Existence of Oscillatory Solutions in the Field-Noyes Model for the Belousov-Zhabotinski Reaction, *SIAM J. Appl. Math., 28* (1975), 3, pp. 678-688
[9] Dafermos, C. M., On the Existence and the Asymptotic Stability of Solutions to the Equations of Linear Thermoelasticity, *Arch. Rat. Mech. Anal., 29* (1969), 4, pp. 241-271

[10] Sperb, R. P., *Maximum Principles and Their Applications*, Academic Press, New York, London, Toronto, Sydney, San Francisco, Cal., USA, 1981
[11] Liu, T., Nishihara, K., Asymptotic Behavior for Scalar Viscous Conservation Laws with Boundary Effect, *Diff. Eqs., 133* (1996), 2, pp. 296-320
[12] Liu, T., *et al*., Behavior of Solutions for the Burgers Equations with Boundary Corresponding to Rarefaction Waves, *SIAM J. Math. Anal., 29* (1998), 2, pp. 293-308
[13] Kawashima, S., Matsumura, A., Asymptotic Stability of Travelling Wave Solutions of Systems for One-Dimensional Gas Motion, *Comm. Math. Phys., 101* (1985), Mar., pp. 97-127
[14] Kawashima, S., Matsumura, A., Stability of Shock Profiles in Visoelasticity with Non-Convex Constitutive Relations, *Comm. Pure Appl. Math*., *47* (1994), 12, pp. 1547-1569
[15] Robinson, J. C., *Dimensions, embeddings, and attractors*, Cambridge Tracts in Mathematics, Vol. 186, Cambridge University Press, Cambridge, UK, 2010
[16] Rybakowski, K. P., Stability and Robustness for Nonlinear Systems Feedback Linearization, Ecole des Mines de Paris, Centre de Fontainebleau, 1986
[17] Deng, S., *et al.*, Fractional Fokker-Planck Equation in a Fractal Medium, *Thermal Science, 24* (2020), 4, pp. 2589-2595
[18] Devogelaere, R., On the Structure of Periodic Solutions of Conservative Systems with Applications, In Lesfchetz S. (Ed) *Contribution to the Theory of Nonlinear Oscillations*, (1958), 4, pp. 53-84
[19] Deng, S.-X., *et al.*, Approximate Analytical Solution for Modified Korteweg-de Vries Equation with Local Fractional Derivative via New Iterative Method, *Thermal Science*, *24* (2020), 6B, pp. 4027-4032
[20] Wang, M. X., Operator Semigroups and Evolutionary Equations (in Chinese), *Science Press*, Beijing, China, 2006
[21] Kawohl, B., Kutev, N., Strong Maximum Principle for Semicontinuous Viscosity Solutions of Nonlinear Partial Differential Equations, *Arch. Math., 70* (1998), 6, pp. 470-478
[22] Skrzypacz, P., *et al.,* Dynamic Pull-in and Oscillations of Current-Carrying Filaments in Magnetic Micro-Electro-Mechanical System, *Communications in Nonlinear Science and Numerical Simulation, 109* (2022), Jun., 106350
[23] He, C. H., *et al.*, Controlling the Kinematics of a Spring-Pendulum System Using an Energy Harvesting Device, *Journal of Low Frequency Noise Vibration and Active Control*, *41* (2022), 3, pp. 1234-1257
[24] He, J. H., *et al.*, Modelling of the Rotational Motion of 6-DOF Rigid Body According to the Bobylev-Steklov Conditions, *Results in Physics*, *35* (2022), Apr., 105391
[25] He, C. H., El-Dib, Y. O., A Heuristic Review on the Homotopy Perturbation Method for Non-Conservative Oscillators, *Journal of Low Frequency Noise Vibration and Active Control*, *41* (2022), 2, pp. 572-603
[26] Matsumura, A., Nishihara, K., Large-Time Behaviors of Solutions to an Inflow Problem in the Half Space for a One-Dimensional System of Compressible Viscous Gas, *Comm. Math. Phys., 222* (2001), Sept., pp. 449-474
[27] Kawashima, S., Matsumura, A., Stability of Shock Profiles in Visoelasticity with Non-Convex Constitutive Relations, *Comm. Pure Appl. Math., 47* (1994), 12, pp. 1547-1569
[28] Carter, M., Van Brunt, B., *The Lebesgue-Stieltjes Integral: a Practical Introduction*, Springer, New York, 2000

Paper submitted: March 13, 2021
Paper revised: July 20, 2022
Paper accepted: July 20, 2022