APPROXIMATE ANALYTICAL SOLUTIONS FOR A CLASS OF GENERALIZED PERTURBED KdV-BURGERS EQUATION

by

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In this paper, we establish an efficient algorithm for solving a class of generalized perturbed KdV-Burgers equation with conformable time fractional derivative and He's space fractal derivative. An illustrative example is presented.

Key word: *the generalized perturbed KdV-Burgers equation, Adomian decomposition method, He's space fractal derivative, the conformable fractional derivative,*

Introduction

Many physical phenomena in the natural and engineering sciences can be modelled by non-linear PDE. The classical perturbed Burgers equation appears in the study of gas dynamics and also in free surface motion of waves in heated fluids [1-3]. In recent years, the non-linear differential equations with fractional derivative have gained considerable importance due to their varied applications in different fields of applied sciences [4, 5]. Several authors have investigated the non-linear PDE involving conformable fractional derivative [6, 7]. The derivative satisfies almost all the classical properties that the derivative holds, and was successfully applied to many problems [8-10]. In classic mechanics, we always assume that the space is continuous. But, if we study the phenomena in fractal media, the fractal derivative has to be used [11, 12]. Explanation of the two-scale fractal theory is available in [13, 14]. The fractal derivative can model porous medium problems, for examples, N/MEMS system in a porous space [15], porous heat transfer [16], non-linear vibration systems in a porous medium [17-19], thermal oscillation [20], fractal population dynamics [21], fractal economical dynamics [22], and fractal solitary theory [23, 24].

In this work, we study the following generalized perturbed Burgers equation in fractal media:

$$\mathrm{D}_t^{\alpha} u = (k_1 u + k_2 u^2)\frac{\partial u}{\partial x^{\beta}} + (k_3 u + k_4)\frac{\partial u}{\partial x^{2\beta}} + k_5 \frac{\partial u}{\partial x^{3\beta}} + k_6 \left(\frac{\partial u}{\partial x^{\beta}}\right)^2 \tag{1}$$

subject to:

$$u(x,0) = f\left\lfloor \frac{x^{\beta}}{\Gamma(1+\beta)} \right\rfloor \tag{2}$$

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where $0<\alpha,\beta\le 1$, and $k_i\,(i=1,\cdots,6)$ are constants, D_t^α – the time conformable fractional derivative [6, 7], and $\partial/(\partial x^\beta)$ D_t^α – the He's space fractal derivative [14].

In general, there exists no method that yields an exact solution for eqs. (1) and (2). So such problems must be solved by approximate analytical methods. These methods include the direct algebraic method [25-27], the tanh-function method [28], variational iteration method [29, 30], Jacobi elliptic function method [31], homotopy perturbation method [32-34], Adomian decomposition method [35, 36], the differential transform method [37], and fractional power series method [38].

The main goal of this work is to solve a class of generalized perturbed KdV-Burgers equation with conformable fractional derivative.

Conformable fractional derivative

In this section, we review some basic definitions and properties of conformable fractional calculus theory, for more details see [6, 7].

Definition 1. Let $\alpha\in(0,1)$ and $f:[0,\infty)\to R.$ The conformable fractional derivative of *J* of order *α* is defined by:

$$\mathrm{D}_t^\alpha f(t)=\lim_{\varepsilon\to 0}\frac{f(t+\varepsilon t^{1-\alpha})-f(t)}{\varepsilon} \tag{3}$$

for all $t>0$. Often, we write $f^{(\alpha)}$ instead of $\mathrm{D}_t^\alpha f(t)$ to denote the conformable fractional derivative of *f* of order *α*. If the conformable fractional derivative of *f* of order *α* exists, then we simply say that *f* is *α*-differentiable. If *f*(*x*) is *α*-differentiable in some $t\in(0,a)$, $a>0$, and $\lim_{t\to 0^+} f^{(\alpha)}(t)$ exists, then we define:

$$f^{(\alpha)}(0)=\lim_{t\to 0^+} f^{(\alpha)}(t) \tag{4}$$

Theorem 1. If a function $f(x):[0,\infty)\to R$ is *α*-differentiable at $t_0>0$, then *f* is continuous at t_0.

Theorem 2. Let $\alpha\in(0,1]$ and assume *f*, *g* to be *α*-differentiable. Then:

$$\mathrm{D}_t^\alpha(a\,f+b\,g)=a\mathrm{D}_t^\alpha f+b\mathrm{D}_t^\alpha g,\quad \text{for all}\quad a,b\in R \tag{5}$$

$$\mathrm{D}_t^\alpha(fg)=g\,\mathrm{D}_t^\alpha f+f\,\mathrm{D}_t^\alpha g \tag{6}$$

$$\mathrm{D}_t^\alpha\frac{f}{g}=\frac{g\mathrm{D}_t^\alpha f-f\,\mathrm{D}_t^\alpha g}{g^2} \tag{7}$$

If *f*(*x*) is differentiable at a point *t* > 0, then we have:

$$\mathrm{D}_t^\alpha f(t)=t^{1-\alpha}\frac{\mathrm{d}f}{\mathrm{d}t} \tag{8}$$

Remark. If one considers a function that is not differentiable at a point *t*, then the conformable derivative is not $t^{1-\alpha}f'(t)$.

Definition 2. Let $\alpha\in(0,1)$ and $f:[0,\infty)\to R.$ The conformable fractional integral of *f* of order α from *a* to *t*, denoted by ${}_aI_t^\alpha(f)$, is defined by:

$$ {}_aI_t^{\alpha}(f) = \int_a^t \tau^{\alpha-1} f(\tau)\,\mathrm{d}\tau = \int_a^t f(\tau)\,\mathrm{d}_{\alpha}\tau \tag{9} $$

where the integral is usual improper Riemann integral.

Definition 3. Let $\alpha \in (0,1)$ and $f:[0,\infty) \to R$ be real valued function. Then the fractional Laplace transform of J is defined by:

$$ L_{\alpha}[f(t)](s) = \int_0^{\infty} \exp\left(-s\frac{t^{\alpha}}{\alpha}\right) f(t)\,\mathrm{d}_{\alpha}t \tag{10} $$

It is easy to show that:

$$ L_{\alpha}[\mathrm{D}_t^{\alpha} f(t)](s) = sL_{\alpha}[f(t)] - f(0) \tag{11} $$

$$ L_{\alpha}(t^C)(s) = \alpha^{\frac{C}{\alpha}} \frac{\Gamma\left(1+\frac{C}{\alpha}\right)}{s^{1+\frac{C}{\alpha}}} \quad \text{where } C \text{ is a constant} \tag{12} $$

Adomian decomposition method

The Adomain decomposition method (ADM) [35, 36] is a technique for solving non-linear equations in the form:

$$ u(x,t) = v + \Omega(u) \tag{13} $$

where $\Omega : \mathrm{M} \to \mathrm{M}$ is a non-linear mapping from a Banach space M into itself and $v \in \mathrm{M}$ is known.

The ADM assumes that the solution u can be expanded as an infinite series:

$$ u(x,t) = \sum_{n=0}^{\infty} u_n(x,t) \tag{14} $$

and the non-linear term $\Omega(u)$ can be decomposed as:

$$ \Omega\left(\sum_{n=0}^{\infty} u_n\right) = \sum_{n=0}^{\infty} H_n(u) \tag{15} $$

for some He's polynomials [39] $H_n(u)$ that are given by:

$$ H_n(u_0, u_1, \cdots, u_n) = \frac{1}{n!}\frac{\partial^n}{\partial\lambda^n}\left[\Omega\left(\sum_{k=0}^{n} \lambda^k u_k\right)\right]_{\lambda=0}, \qquad n = 0,1,2,\cdots \tag{16} $$

Substituting eqs. (14) and (15) into eq. (13) gives:

$$ \sum_{n=0}^{\infty} u_n = v + \sum_{n=0}^{\infty} H_n \tag{17} $$

which is satisfied formally if we set:

$$ u_0(x,t) = v $$

$$u_1 = H_0$$

$$u_{m+1} = H_m$$

Then k-term approximate solution of eq. (12) is given by:

$$u = u_0 + u_1 + \cdots + u_{k-1}$$

The solutions of problem (1) and (2)

In this section, we derive the main algorithms for solving the problem (1) and (2).

We rewrite the eq. (1) as:

$$\mathrm{D}_t^{\alpha} u(x,t) = P(u) + \sum_{j=1}^{4} N_j(u) \tag{18}$$

where

$$P(u) = k_4 \frac{\partial u}{\partial x^{2\beta}} + k_5 \frac{\partial u}{\partial x^{3\beta}}$$

$$N_1(u) = k_1 u \frac{\partial u}{\partial x^{\beta}}$$

$$N_2(u) = k_2 u^2 \frac{\partial u}{\partial x^{\beta}}$$

$$N_3(u) = k_3 u \frac{\partial u}{\partial x^{2\beta}}$$

$$N_4(u) = k_6 \left(\frac{\partial u}{\partial x^{\beta}} \right)^2$$

Taking the fractional Laplace transform [40] on both sides of eq. (18), we obtain:

$$L_{\alpha}[\mathrm{D}_t^{\alpha} u(x,t)] = L_{\alpha} \left[P(u) + \sum_{j=1}^{4} N_j(u) \right]$$

Using eq. (11), we have:

$$L_{\alpha}(u) = \frac{1}{s} u(x,0) + \frac{1}{s} \left\{ L_{\alpha}[P(u)] + L_{\alpha} \left[\sum_{j=1}^{4} N_j(u) \right] \right\} \tag{19}$$

Operating with the fractional Laplace inverse transform on both sides of eq. (19) gives:

$$u(x,t) = u(x,0) + L_{\alpha}^{-1} \left(\frac{1}{s} \left\{ L_{\alpha}[P(u)] + L_{\alpha} \left[\sum_{j=1}^{4} N_j(u) \right] \right\} \right) \tag{20}$$

Suppose that the solutions take the form:

$$u(x,t)=\sum_{k=0}^{\infty}u_k(x,t)$$

and the non-linear terms can be decomposed as:

$$N_j(u)=\sum_{n=0}^{\infty}H_{jn},\quad (j=1,2,3,4)$$

where $H_{jn}(j=1,2,3,4)$ are some He's polynomials. By eq. (16), we can get:

$$H_{10}=k_1u_0\frac{\partial}{\partial x^{\beta}}u_0$$

$$H_{20}=k_2u_0^2\frac{\partial}{\partial x^{\beta}}u_0$$

$$H_{30}=k_3u_0\frac{\partial}{\partial x^{2\beta}}u_0$$

$$H_{40}=k_6\left(\frac{\partial}{\partial x^{\beta}}u_0\right)^2$$

$$H_{11}=k_1\left(u_0\frac{\partial u_1}{\partial x^{\beta}}+u_1\frac{\partial u_0}{\partial x^{\beta}}\right)$$

$$H_{21}=k_2\left(u_0^2\frac{\partial u_1}{\partial x^{\beta}}+2u_0u_1\frac{\partial u_0}{\partial x^{\beta}}\right)$$

$$H_{31}=k_3\left(u_0\frac{\partial u_1}{\partial x^{2\beta}}+u_1\frac{\partial u_0}{\partial x^{2\beta}}\right)$$

$$H_{41}=k_6\left(2\frac{\partial}{\partial x^{\beta}}u_0\frac{\partial}{\partial x^{\beta}}u_1\right)$$

and so on.

Therefore, by using the ADM, we have:

$$u_0(x,t)=u(x,0)\tag{21}$$

$$u_1(x,t)=L_{\alpha}^{-1}\left(\frac{1}{s}\{P[L_{\alpha}(u_0)]\}+\sum_{j=1}^{4}L_{\alpha}(H_{j0})\right)\tag{22}$$

$$u_{m+1}(x,t)=L_{\alpha}^{-1}\left(\frac{1}{s}\{P[L_{\alpha}(u_m)]\}+\sum_{j=1}^{4}L_{\alpha}(H_{jm})\right)\tag{23}$$

where $m=1,2,3,\cdots$.

Then k-term approximate solutions of eq. (18) are given by:

$$u = u_0 + u_1 + \cdots + u_{k-1}$$

To illustrate the above algorithms and to test its effectiveness, we consider the following example.

Example. Consider the problem (1) and (2) in the form:

$$\mathrm{D}_t^\alpha u = \left(-u + \frac{1}{2}u^2\right)\frac{\partial u}{\partial x^\beta} + \left(\frac{3}{2}u - 1\right)\frac{\partial u}{\partial x^{2\beta}} + \frac{1}{2}\frac{\partial u}{\partial x^{3\beta}} - \frac{1}{2}\left(\frac{\partial u}{\partial x^\beta}\right)^2 \tag{24}$$

with the initial condition:

$$u(x,0) = 2\cot(X) \tag{25}$$

where

$$X = \frac{x^\beta}{\Gamma(1+\beta)}$$

By eqs. (21)-(23), (10) and (12), we obtain:

$$u_0 = 2\cot(X)$$

$$u_1(x,t) = \frac{-4t^\alpha}{\alpha \sin^2(X)}$$

$$u_2(x,t) = \frac{8\cos(X)t^{2\alpha}}{\alpha^2 \sin^3(X)}$$

$$u_3(x,t) = \frac{-16[1+2\cos^2(X)]t^{3\alpha}}{3\alpha^3 \sin^4(X)}$$

$$u_4(x,t) = \frac{32\cos(X)[2+\cos^2(X)]t^{4\alpha}}{3\alpha^4 \sin^5(X)}$$

$$\vdots$$

Thus, the 5-therm approximate solutions of problem (24) and (25) are given by:

$$u(x,t) = u_0 + u_1(x,t) + u_2(x,t) + u_3(x,t) + u_4(x,t)$$

Remark. When $\alpha = \beta = 1,$ we get:

$$u(x,t) = 2\cot(x) + \frac{-4t}{\cos^2 x - 1} + \frac{8t^2 \cos x}{\sin^3 x} + \frac{-16t^3(1+2\cos^2 x)}{3\sin^4 x}$$

which are approximate analytical solutions of the following classical problems:

$$\frac{\partial u}{\partial t} = \left(-u + \frac{1}{2}u^2\right)\frac{\partial u}{\partial x} + \left(\frac{3}{2}u - 1\right)\frac{\partial u}{\partial x^2} + \frac{1}{2}\frac{\partial u}{\partial x^3} - \frac{1}{2}\left(\frac{\partial u}{\partial x}\right)^2$$

with initial conditions:

$$u(x,0) = 2\cot(x)$$

Conclusion

In this paper, our main goal is to propose a method for solving a class of generalized perturbed KdV-Burgers equation with conformable time fractional derivative and He's space fractal derivative. This goal has been achieved by using ADM, the fractional Laplace transform and He's polynomials. The presented example shows that our method is an efficient and reliable algorithm.

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