APPROXIMATE ANALYTICAL SOLUTIONS OF GENERALIZED FRACTIONAL KORTEWEG-DE VRIES EQUATION

by

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In this paper, a generalized Korteweg-de Vries equation involving a temporal fractional derivative and a spatial fractal derivative is studied. The temporal fractional derivative can describe the non-local property and memory property, while the spatial fractal derivative can model the space discontinuity. Its approximate analytical solution is presented using He's variational iteration method, which is extremely effective for the fractal-fractional differential equations.

Key word: *generalized fractal Korteweg-de Vries equation, fractal derivative, Caputo derivative, variational iteration method*

Introduction

The importance of the Korteweg-de Vries-type (KdV) equation is well known, it appears in a wide range of physical applications. Many physicists and mathematicians have systematically studied KdV equation and its various modifications, and much achievement was obtained [1-4]. Recently, scientists found that many non-linear phenomena in applied sciences and engineering can be described by fractional KdV-type equations [5-10], and the fractal solitary theory has become very hot recently, it has been revealed by many authors that the solitary waves are affected by the order of the fractal dimensions or the unsmooth boundary, but the wave morphology is rarely affected [11-20].

The most important advantage of making use of fractional derivative in mathematical modelling is their non-local property. Fractional derivative provides an excellent instrument for the description of memory properties of various processes. However, if we study the complex phenomena in fractal media, then the space fractal derivative has to be used [21, 22]. In this paper, we study the following generalized fractal KdV equation:

$$\mathrm{D}_t^{\alpha} u(x,t) + \lambda \frac{\partial}{\partial x^{3\beta}} u(x,t) + \xi(u) \frac{\partial u}{\partial x^{\beta}} = 0, \quad 0 < \alpha \le 1, \quad 0 < \beta \le 1 \tag{1}$$

subject to the initial condition:

$$u(x,0) = \phi\left[\frac{x^{\beta}}{\Gamma(1+\beta)}\right] \tag{2}$$

where D_t^{α} is the Caputo fractional derivative of order α, $\partial/(\partial x^{\beta})$ – the He's space fractal derivative [21], λ – an arbitrary constant, and $\xi(u)$ and $\phi(x)$ – the given functions. Physical ex-

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planation of the fractal derivative is available in [23-25], and it is now widely used for modelling discontinuous problems, for examples, fractal MEMS systems [26], fractal thermal dynamics [27], fractal vibration systems [28, 29], the two-scale population dynamics and the two-scale economics [30, 31], and the fractal diffusion [32, 33].

Usually, it is impossible to obtain exact analytical solutions of the problem of the eqs. (1) and (2). The main purpose of the present work is to solve eqs. (1) and (2) by using He's variational iteration method (VIM) [34-36], which can provide approximate analytical solutions in the form of a fractional power series with easily computed terms. For more details in using the VIM for similar problems, see [37, 38].

Fractional derivative and fractal derivative

In this section, we recall the following basic definitions of fractional calculus and fractal calculus which shall be used in this paper. For more details see [39-41].

Definition 1. A real function $f(x), x>0$ is said to be in the space $C_\lambda, \lambda \in R$ if there exists a real number $p>\lambda,$ such that $f(x)=x^p f_1(x)$ where $f_1(x)\in C[0,\infty)$ and it is said to be in the space C_n if and only if $f^{(n)}\in C_\lambda, n\in N.$

Definition 2. The Riemann-Liouville fractional integral operator of order $\alpha>0$ of a function $f(x)\in C_\lambda, \lambda\geq -1$ is defined as:

$$J^\alpha f(x)=\frac{1}{\Gamma(\alpha)}\int_0^x (x-s)^{\alpha-1} f(s)\mathrm{d}s \tag{3}$$

$$J^0 f(x)=f(x) \tag{4}$$

Properties of the operator J^α can be found in [39] and we mention only the following. For $\alpha, \beta\geq 0, x>0$ and, we have:

$$J^\alpha J^\beta f(x)=J^{\alpha+\beta} f(x) \tag{5}$$

$$J^\alpha J^\beta f(x)=J^\beta J^\alpha f(x) \tag{6}$$

$$J^\alpha (x^\lambda)=\frac{\Gamma(\lambda+1)}{\Gamma(1+\alpha+\lambda)} x^{\lambda+\alpha} \tag{7}$$

Definition 3. The time fractional derivative of $u(x, t)$ in Caputo sense is defined:

$$\mathrm{D}_t^\alpha u(x,t)=\frac{1}{\Gamma(m-\alpha)}\int_0^t (t-s)^{m-\alpha-1} u_s^{(m)}(x,s)\mathrm{d}s \tag{8}$$

for $m-1<\alpha\leq m, m\in N^+, x>0$ and $u(x,t)\in C_{-1}^m.$

Definition 4. The space fractal derivative of $u(x,t)$ in the He's sense is defined:

$$\frac{\partial u(x,t)}{\partial x^\beta}=\Gamma(1+\beta)\lim_{\substack{x_1-x\to\Delta x\\ \Delta x\neq 0}}\frac{u(x_1,t)-u(x,t)}{(x_1-x)^\beta}, \qquad (0<\beta\leq 1)$$

and

$$\frac{\partial u(x,t)}{\partial x^{3\beta}}=\frac{\partial}{\partial x^\beta}\left\{\frac{\partial}{\partial x^\beta}\left[\frac{\partial u(x,t)}{\partial x^\beta}\right]\right\}$$

He's variational iteration method

In this section, we briefly describe the VIM. To illustrate the basic idea of this method, we consider the following non-linear equation:

$$Lu(x,t) + \mathrm{Nu}(x,t) = g(x,t) \tag{9}$$

where L and N are linear and non-linear operators, respectively, and $g(x,t)$ is the source in homogeneous term.

The variational iteration method was proposed by He [34-36], the iteration algorithm can be constructed:

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^t \mu\{L[u_n(x,s)] + N[\tilde{u}_n(x,s)] - g(x,s)\}\mathrm{d}s \tag{10}$$

where μ is a general Lagrange multiplier, which can be identified optimally via the variational theory, and $\tilde{u}_n$ is a restricted variation which means $\delta\tilde{u}_n = 0$.

The main steps of He's VIM require first the determination of Lagrange multiplier μ that will be identified optimally. Once it is determined, then the successive approximations u_{n+1}, $n \geq 0$, of the solution u will be obtained by using any selective function u_0. Consequently, the exact solution:

$$u = \lim_{n\to\infty} u_n \tag{11}$$

The solution of the fractal KdV

To use the solution procedure of the variational iteration method, we rewrite eq. (1):

$$\mathrm{D}_t^\alpha u(x,t) + N[u(x,t)] = 0 \tag{12}$$

where

$$N(u) = \lambda \frac{\partial u}{\partial x^{3\beta}} + \xi(u)\frac{\partial u}{\partial x^\beta} \tag{13}$$

According to the VIM, we construct the following iteration algorithm:

$$\begin{aligned} u_{n+1}(x,t) &= u_n(x,t) + J_t^\alpha\{\mu[D_t^\alpha u(x,s)] + [\tilde{u}_n(x,s)]\} = \\ &= u_n(x,t) + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\mu(s)\{D_t^\alpha u(x,s) + N[\tilde{u}_n(x,s)]\}\mathrm{d}s \end{aligned} \tag{14}$$

where μ is the general Lagrange multiplier and $\tilde{u}_n$ is the restricted variation.

By making the previous functional stationary, the following conditions can be obtained:

$$\delta u_{n+1}(x,t) = \delta u_n(x,t) + \delta\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\mu(s)\{D_t^\alpha u(x,s) + N[\tilde{u}_n(x,s)]\}\,\mathrm{d}s = 0 \tag{15}$$

From eq. (15), we can get:

$$1 + \mu(s) = 0, \quad \mathrm{D}_s^\alpha \mu(s) = 0 \tag{16}$$

Thus, the generalized Lagrange multiplier can be identified as $\mu = -1$.

So, we obtain the following iteration formular:

$$u_{n+1}(x,t) = u_n(x,t) - \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\{D_t^\alpha u(x,s) + N[u_n(x,s)]\}\mathrm{d}s \tag{17}$$

where $u_0(x,t)$ is an initial approximation which can be freely chosen if it satisfies the initial conditions of the problem.

Eventually, the exact solution is given by:

$$u = \lim_{n\to\infty} u_n$$

Next, we consider two typical case.

Example 1. Consider the following time-space fractal KdV-type equation:

$$\mathrm{D}_t^\alpha u(x,t) + \frac{\partial u}{\partial x^{3\beta}} + 6u^2\frac{\partial u}{\partial x^\beta} = 0$$

subject to the initial condition:

$$u(x,0) = \cosh^{-1}\left[\frac{x^\beta}{\Gamma(1+\beta)}\right]$$

Construction the iteration formula:

$$u_{n+1}(x,t) = u_n(x,t) - \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\left[\mathrm{D}_t^\alpha u_n(x,s) + \frac{\partial u_n(x,s)}{\partial x^{3\beta}} + 6{u_n}^2\frac{\partial u_n(x,s)}{\partial x^\beta}\right]\mathrm{d}s$$

Taking the initial value $u_0(x,t) = u(x,0)$, we can get:

$$u_1(x,t) = u_0(x,t) + \frac{2E^{-2}(E-E^{-1})t^\alpha}{(1+E^{-2})^2\Gamma(1+\alpha)}$$

$$u_2(x,t) = u_1(x,t) + \frac{2E^{-2}(E-6E^{-1}+E^{-3})t^{2\alpha}}{(1+E^{-2})^3\Gamma(1+2\alpha)}$$

$$\vdots$$

Thus, the approximate solution is:

$$u(x,t) = \frac{2}{E+E^{-1}} + \frac{2E^{-2}(E-E^{-1})t^\alpha}{(1+E^{-2})^2\Gamma(1+\alpha)} + \frac{2E^{-2}(E-6E^{-1}+E^{-3})t^{2\alpha}}{(1+E^{-2})^3\Gamma(1+2\alpha)}$$

where

$$E = \exp\left[\frac{x^\beta}{\Gamma(1+\beta)}\right]$$

Example 2. Consider following time-space fractal KdV equation:

$$\mathrm{D}_t^\alpha u(x,t) - \frac{\partial u}{\partial x^{3\beta}} - \ln(u)\frac{\partial u}{\partial x^\beta} = 0$$

subject to the initial condition:

$$u(x,0)=\exp\left[1-\frac{x^{\beta}}{\Gamma(1+\beta)}\right]$$

Here we construction the iteration formula:

$$u_{n+1}(x,t)=u_n(x,t)-\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\left[\mathrm{D}_t^{\alpha}u_n(x,s)-\frac{\partial u_n(x,s)}{\partial x^{3\beta}}-\ln(u)\frac{\partial u_n(x,s)}{\partial x^{\beta}}\right]\mathrm{d}s$$

Taking the initial value $u_0(x,t)=u(x,0)$, we can obtain:

$$u_1(x,t)=u_0(x,t)+\left[-2+\frac{x^{\beta}}{\Gamma(1+\beta)}\right]\frac{E_1 t^{\alpha}}{\Gamma(1+\alpha)}$$

$$u_2(x,t)=u_1(x,t)+\left[\frac{x^{2\beta}}{\Gamma(1+2\beta)}-\frac{6x^{\beta}}{\Gamma(1+\beta)}+10\right]\frac{E_1 t^{2\alpha}}{\Gamma(1+2\alpha)}$$

and so on.

Hence, the approximate solution is:

$$u(x,t)=E_1+\left[\frac{x^{\beta}}{\Gamma(1+\beta)}-2\right]\frac{E_1 t^{\alpha}}{\Gamma(1+\alpha)}+\left[\frac{x^{2\beta}}{\Gamma(1+2\beta)}-\frac{6x^{\beta}}{\Gamma(1+\beta)}+10\right]\frac{E_1 t^{2\alpha}}{\Gamma(1+2\alpha)}$$

where

$$E_1=\exp\left[1-\frac{x^{\beta}}{\Gamma(1+\beta)}\right]$$

Conclusion

In this work, the variational iteration method has been successfully applied to obtain the approximate analytical solution of the generalized fractal KdV equations. Two examples are given to illustrate the validity and accuracy of the method. The results show that the method is efficient to handle non-linear fractal differential equation on the media.

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