FABRIC COLOR FORMULATION
USING A MODIFIED KUBELKA-MUNK THEORY
CONSIDERING THERMAL EFFECT

by

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The Kubelka-Munk function is simple but it ignores the film’s thickness, so its applications are greatly limited. Though the exact relationship between the Kubelka-Munk function and the thickness can be derived from a differential model, it is too complex to be practically used. Here a modification is suggested by taking the thickness effect and the temperature effect into account, and the validity is widely enlarged. The modified Kubelka-Munk theory can be used as a color-matching model for colorful fabrics. If the porosity of the film is considered, a fractal modification with two-scale fractal derivative has to be adopted.

Key words: optical property, colorful fabrics, absorption coefficient, scattering coefficient, homotopy matching, porous film, two-scale fractal

Introduction

Colorful fabrics are widely used in textile engineering, especially the photochromic fabrics [1, 2] and thermochromic fabrics [3, 4] are the most used intelligent materials, and chameleon fabrics [5, 6] have been catching a rocketing interest in various fields from responsive camouflage to brand protection. Colorful fabrics can be also used as sensors [7] far behind fashion apparel, and far-reaching implications are emerging for applications including radiation protection [8] and energy harvesting [9]. Now the nanodyeing technology [10] has been making colorful fabrics extremely promising and remarkably challenging.

The Kubelka-Munk theory is widely used in textile engineering to describe light scattering and absorption of optical behavior of a fabric, it was first proposed by Franz Munk and Franz Munk in 1931 [11]. The Kubelka-Munk function is the main tool to fabric color formulation, it can be written in the form [11]:

\[
\frac{K}{S} = \frac{(1 - R)^2}{2R}
\]  

(1)

where \( K \) and \( S \) are the absorption coefficient and the scattering coefficient of the fabric, respectively, and \( R \) is the reflectance.

Equation (1) is the famous equation for light absorption and scattering through a paint layer. It was derived under the assumption of infinite thickness \((L \rightarrow \infty)\), though many

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experiment showed that when \( L > 2 \text{ mm} \), eq. (1) can be used with relatively high accuracy [12]. The Kubelka-Munk theory is the basic tool in color matching technology [13-20], it has been caught much attention in different fields, e.g., textile engineering, material science, physics and chemistry. Though there were claims on misuse of the Kubelka-Munk function [16] and many modified Kubelka-Munk functions were appeared in open literature, mathematical treatment on the original Kubelka-Munk theory was rare.

When the substrate is extremely thin \( (L<<1) \), for examples, micro fibers [21], nanoscale membranes [22, 23], eq. (1) leads to a large error, this is because \( K/S \) depends upon \( L \), and there is a significant flaw to apply eq. (1) to films with nano/micro thickness [16-18].

The optical properties of biological tissues are extremely special at 633 nm [24], when the thickness reduces to about 220 nm, as that in the nanostructure optical surface of the moth eyes [25], eq. (1) becomes totally invalid. Now the electrospinning or the bubble electrospinning [26-30] can produce thin films with thickness of about 100 nm, and the nanodyeing [10] also asks for a modified Kubelka-Munk function.

Kubelka-Munk theory

Kubelka and Munk established a differential model to study the change of incident light intensity travelling downwards, \( i \), and upwards, \( j \), within a thin film [11], fig. 1.

The changes of \( i \) and \( j \) through an infinite distance, \( dx \), are, respectively [11]:

\[
\frac{di}{dx} = -[(K + S)i + Sj] \quad (2)
\]

and

\[
\frac{dj}{dx} = -[(K + S)j + Si] \quad (3)
\]

where \( S \) is the absorption coefficient, and \( K \) the scattering coefficient.

From eqs. (2) and (3), we have:

\[
\frac{dj}{j} - \frac{di}{i} = -2(K + S)dx + S \left( \frac{i}{j} + \frac{j}{i} \right) dx \quad (4)
\]

The reflectance is:

\[
r = \frac{j}{i} \quad (5)
\]
Equation (4) becomes:

\[ \frac{1}{2\sqrt{\lambda^2 - 1}} \ln \left[ \frac{R_r - \lambda + \sqrt{\lambda^2 - 1}}{R_r - \lambda} \left( \frac{R_s - \lambda - \sqrt{\lambda^2 - 1}}{R_s - \lambda} \right) \right] = SL \]  

(6)

where \( \lambda = (K/S) + 1 \), \( R_r \) is the reflectance when \( L = 0 \).

Equation (7) is too complex to be used for practical applications. When \( L \to \infty \), from eq. (7) we have:

\[ \left( R - \lambda + \sqrt{\lambda^2 - 1} \right) \left( R_s - \lambda - \sqrt{\lambda^2 - 1} \right) = 0 \]  

(8)

or

\[ R - \left( \frac{K}{S} + 1 \right) + \sqrt{\frac{K}{S} + 1} - 1 = 0 \]  

(9)

or

\[ \left( R - \frac{K}{S} + 1 \right)^2 = \left( \frac{K}{S} + 1 \right)^2 - 1 \]  

(10)

Solving \( K/S \) from eq. (10), we have eq. (1), so mathematically eq. (1) is valid only for \( L \to \infty \).

**Modified Kubelka-Munk function**

We consider another case when \( r \ll 1 \). Under this assumption, we have:

\[ r + \frac{1}{r} \approx \frac{1}{r} \]  

(11)

Equation (6) becomes:

\[ dr = -2(K + S)r dx + S dx \]  

(12)

or

\[ \frac{dr}{-2(K + S)r + S} = dx \]  

(13)

or
\[ \frac{\ln[-2(K + S)r + S]}{-2(K + S)} \, dx = dx \]  

(14)

Solving eq. (14) gives the following result:

\[ \frac{\ln[S - 2(K + S)R_s] - \ln[S - 2(K + S)R]}{-2(K + S)} = L \]  

(15)

or

\[ \ln \left[ 1 - 2 \left( \frac{K}{S} + 1 \right) R_s \right] - \ln \left[ 1 - 2 \left( \frac{K}{S} + 1 \right) R \right] = -2 \left( \frac{K}{S} + 1 \right) SL \]  

(16)

This simplified one is valid for small \( R \), that is for the case with little scattering. Equation (16) has some potential applications to the color matching of micro/nanofibers with low scattering coefficient.

**Another Modified Kubelka-Munk function**

Consider the following inequality:

\[ r + \frac{1}{r} \geq 2 \]  

(17)

Equation (6) becomes a simple relation, which reads [2]:

\[ d \ln r = -2(K + S)dx + S \left( r + \frac{1}{r} \right) dx \geq -2(K + S)dx + 2Sdx = -2Kdx \]  

(18)

That means:

\[ \ln \left( \frac{R}{R_s} \right) \geq -2 \frac{K}{S} SL \]  

(19)

or

\[ \frac{K}{S} \geq - \frac{1}{2SL} (\ln R - \ln R_s) \]  

(20)

Considering eq. (1) and eq. (20), we modify the Kubelka-Munk function:

\[ \frac{K}{S} = \frac{\alpha(L)(1 - R)}{2R} \rightleftharpoons \frac{1 - \alpha(L)}{2SL} (\ln R_s - \ln R) \]  

(21)

where \( \alpha \) is a matching parameter, it is a function of \( L \), it requires \( \alpha(L > 2 \text{ mm}) = 1 \) and \( \alpha(L = 0) = 0 \). We call eq. (21) the homotopy matching, when \( \alpha = 0 \), it becomes eq. (21), and when \( \alpha = 1 \), it turns out to be eq. (1). The homotopy matching is widely applied in mathematics to solve non-linear problems [31, 32].

We choose the matching parameter as:
where \( a \) and \( b \) are parameters for experimental determination. Eq. (21) becomes:

\[
\frac{K}{S} = \frac{(1 - (1 - aLb) (1 - R)^2 + [1 - (1 - aLb)] (\ln R_e - \ln R) }{2R} \]

(22)

It is well known that reflectance is temperature-dependent [33]. If the thermal effect is considered, eq. (23) can be modified:

\[
\frac{K}{S} = \frac{(1 - (1 - (1 - e^{-alb})e^{-cr}) (1 - R)^2 + [1 - (1 - (1 - e^{-alb})e^{-cr})] (\ln R_e - \ln R) }{2R} \]

(23)

where \( c \) is a temperature-dependent parameter.

For multiple constituents, the above equation can be modified:

\[
\sum_{i=1}^{N} \frac{K_i}{S_i} = \frac{(1 - (1 - (1 - e^{-alb})e^{-cr}) (1 - R)^2 + [1 - (1 - (1 - e^{-alb})e^{-cr})] (\ln R_e - \ln R) }{2R} \]

(24)

Discussion and conclusions

When we consider the porosity of the film, a fractal modification of the Kubelka-Munk function is needed. Eqs. (2) and (3) has to be modified:

\[
d_i = -(K + S)i + Sjdx^\beta
\]

(25)

and

\[
d_j = -(K + S)j + Sidx^\beta
\]

(26)

where \( \beta \) is the fractal dimensions of the film. The two-scale fractal derivative is [34-36]:

\[
\frac{di}{dx^\beta}(x_0) = \Gamma(1 + \beta) \lim_{\Delta x \to 0} \frac{i(x) - i(x_0)}{(x-x_0)^\beta}
\]

(27)

and

\[
\frac{dj}{dx^\beta}(x_0) = \Gamma(1 + \beta) \lim_{\Delta x \to 0} \frac{j(x) - j(x_0)}{(x-x_0)^\beta}
\]

(28)

where \( \Gamma \) is the gamma function. The two-scale fractal theory is now widely applied for porous problems, see for examples, the fractal diffusion [37], the fractal solitary wave [38], the fractal micro-electro-mechanical devices [39], the fractal concrete [40-43], and the fractal composite [44].
\[ d \ln r = -2(K + S)dx^\beta + S \left( r + \frac{1}{r} \right) dx^\beta \]  
\hspace{1cm} (30)

The solution of eq. (30) reads:

\[ \frac{1}{2\sqrt{\lambda^2 - 1}} \ln \left( \frac{R_x - \lambda + \sqrt{\lambda^2 - 1}}{R_x - \lambda - \sqrt{\lambda^2 - 1}} \right) = SL^\beta \]  
\hspace{1cm} (31)

We will further discuss this fractal model in a forthcoming article.

To be concluded, this paper suggests a modified Kubelka-Munk function, eq. (24), which is also simple and considers the thickness of the film and temperature-dependent reflectance. If the porosity of the film is considered, the two-scale fractal modification is a must.

The parameters \( a \) and \( b \) and \( c \) involved in eq. (24) can be estimated experimentally. The modified Kubelka-Munk function can be used for a color-matching model for colorful fabrics, and far-reaching implications are emerging for applications include colorful image technology [45, 46] and microelectromechanical systems [47, 48].

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