# FABRIC COLOR FORMULATION USING A MODIFIED KUBELKA-MUNK THEORY CONSIDERING THERMAL EFFECT

by

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*The Kubelka-Munk function is simple but it ignores the film's thickness, so its applications are greatly limited. Though the exact relationship between the Kubelka-Munk function and the thickness can be derived from a differential model, it is too complex to be practically used. Here a modification is suggested by taking the thickness effect and the temperature effect into account, and the validity is widely enlarged. The modified Kubelka-Munk theory can be used as a color-matching model for colorful fabrics. If the porosity of the film is considered, a fractal modification with two-scale fractal derivative has to be adopted.*

Key words: *optical property, colorful fabrics, absorption coefficient, scattering coefficient, homotopy matching, porous film, two-scale fractal*

## Introduction

Colorful fabrics are widely used in textile engineering, especially the photochromic fabrics [1, 2] and thermochromic fabrics [3, 4] are the most used intelligent materials, and chameleon fabrics [5, 6] have been catching a rocketing interest in various fields from responsive camouflage to brand protection. Colorful fabrics can be also used as sensors [7] far behind fashion apparel, and far-reaching implications are emerging for applications including radiation protection [8] and energy harvesting [9]. Now the nanodyeing technology [10] has been making colorful fabrics extremely promising and remarkably challenging.

The Kubelka-Munk theory is widely used in textile engineering to describe light scattering and absorption of optical behavior of a fabric, it was first proposed by Franz Munk and Franz Munk in 1931 [11]. The Kubelka-Munk function is the main tool to fabric color formulation, it can be written in the form [11]:

$$\frac{K}{S}=\frac{(1-R)^2}{2R} \tag{1}$$

where $K$ and $S$ are the absorption coefficient and the scattering coefficient of the fabric, respectively, and $R$ is the reflectance.

Equation (1) is the famous equation for light absorption and scattering through a paint layer. It was derived under the assumption of infinite thickness $(L \to \infty)$, though many

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experiment showed that when $L > 2$ mm, eq. (1) can be used with relatively high accuracy [12]. The Kubelka-Munk theory is the basic tool in color matching technology [13-20], it has been caught much attention in different fields, *e.g.*, textile engineering, material science, physics and chemistry. Though there were claims on misuse of the Kubelka-Munk function [16] and many modified Kubelka-Munk functions were appeared in open literature, mathematical treatment on the original Kubelka-Munk theory was rare.

When the substrate is extremely thin ($L$<<1), for examples, micro fibers [21], nanoscale membranes [22, 23], eq. (1) leads to a large error, this is because K/S depends upon $L$, and there is a significant flaw to apply eq. (1) to films with nano/micro thickness [16-18].

The optical properties of biological tissues are extremely special at 633 nm [24], when the thickness reduces to about 220 nm, as that in the nanostructure optical surface of the moth eyes [25], eq. (1) becomes totally invalid. Now the electrospinning or the bubble electrospinning [26-30] can produce thin films with thickness of about 100 nm, and the nanodyeing [10] also asks for a modified Kubelka-Munk function.

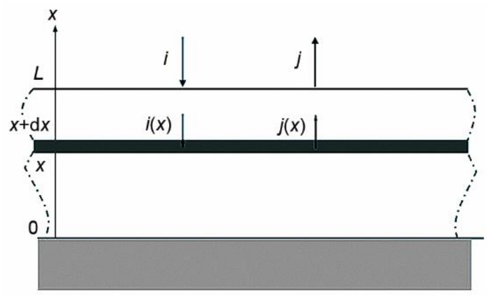


**Figure 1. Kubelka-Munk model for color formulation**

## Kubelka-Munk theory

Kubelka and Munk established a differential model to study the change of incident light intensity travelling downwards, $i$, and upwards, $j$, within a thin film [11], fig. 1.

The changes of $i$ and $j$ through an infinite distance, d$x$, are, respectively [11]:

$$\mathrm{d}i = -[-(K+S)i + Sj]\mathrm{d}x \qquad (2)$$

and

$$\mathrm{d}j = [-(K+S)j + Si]\mathrm{d}x \qquad (3)$$

where $S$ is the absorption coefficient, and $K$ the scattering coefficient.

From eqs. (2) and (3), we have:

$$\frac{\mathrm{d}j}{j} - \frac{\mathrm{d}i}{i} = -2(K+S)\mathrm{d}x + S\left(\frac{i}{j} + \frac{j}{i}\right)\mathrm{d}x \qquad (4)$$

The reflectance is:

$$r = \frac{j}{i} \qquad (5)$$

Equation (4) becomes:

$$\mathrm{d}\ln r = -2(K+S)\mathrm{d}x + S\left(r+\frac{1}{r}\right)\mathrm{d}x \tag{6}$$

Solving eq. (6) exactly, we have:

$$\frac{1}{2\sqrt{\lambda^2-1}}\ln\left[\frac{\left(R_s-\lambda+\sqrt{\lambda^2-1}\right)\left(R-\lambda-\sqrt{\lambda^2-1}\right)}{\left(R-\lambda+\sqrt{\lambda^2-1}\right)\left(R_s-\lambda-\sqrt{\lambda^2-1}\right)}\right] = SL \tag{7}$$

where $\lambda = (K/S)+1$, $R_s$ is the reflectance when $L = 0$.

Equation (7) is too complex to be used for practical applications. When $L \to \infty$, from eq. (7) we have:

$$\left(R-\lambda+\sqrt{\lambda^2-1}\right)\left(R_s-\lambda-\sqrt{\lambda^2-1}\right) = 0 \tag{8}$$

or

$$R-\left(\frac{K}{S}+1\right)+\sqrt{\left(\frac{K}{S}+1\right)^2-1} = 0 \tag{9}$$

or

$$\left(R-\frac{K}{S}+1\right)^2 = \left(\frac{K}{S}+1\right)^2 - 1 \tag{10}$$

Solving *K*/*S* from eq. (10), we have eq. (1), so mathematically eq. (1) is valid only for $L \to \infty$.

#### Modified Kubelka-Munk function

We consider another case when $r \ll 1$. Under this assumption, we have:

$$r+\frac{1}{r} \approx \frac{1}{r} \tag{11}$$

Equation (6) becomes:

$$\mathrm{d}r = -2(K+S)r\mathrm{d}x + S\mathrm{d}x \tag{12}$$

or

$$\frac{\mathrm{d}r}{-2(K+S)r+S} = \mathrm{d}x \tag{13}$$

or

$$\mathrm{d}\left\{\frac{\ln[-2(K+S)r+S]}{-2(K+S)}\right\}=\mathrm{d}x \tag{14}$$

Solving eq. (14) gives the following result:

$$\frac{\ln[S-2(K+S)R_s]-\ln[S-2(K+S)R]}{-2(K+S)}=L \tag{15}$$

or

$$\ln\left[1-2\left(\frac{K}{S}+1\right)R_s\right]-\ln\left[1-2\left(\frac{K}{S}+1\right)R_s\right]=-2\left(\frac{K}{S}+1\right)SL \tag{16}$$

This simplified one is valid for small *R*, that is for the case with little scattering. Equation (16) has some potential applications to the color matching of micro/nanofibers with low scattering coefficient.

## Another Modified Kubelka-Munk function

Consider the following inequality:

$$r+\frac{1}{r}\geq 2 \tag{17}$$

Equation (6) becomes a simple relation, which reads [2]:

$$\mathrm{d}\ln r=-2(K+S)\mathrm{d}x+S\left(r+\frac{1}{r}\right)\mathrm{d}x\geq -2(K+S)\mathrm{d}x+2S\mathrm{d}x=-2K\mathrm{d}x \tag{18}$$

That means:

$$\ln\left(\frac{R}{R_s}\right)\geq -2\frac{K}{S}SL \tag{19}$$

or

$$\frac{K}{S}\geq -\frac{1}{2SL}(\ln R-\ln R_s) \tag{20}$$

Considering eq. (1) and eq. (20), we modify the Kubelka-Munk function:

$$\frac{K}{S}=\frac{\alpha(L)(1-R)^2}{2R}+\frac{1-\alpha(L)}{2SL}(\ln R_s-\ln R) \tag{21}$$

where $\alpha$ is a matching parameter, it is a function of *L*, it requires $\alpha(L > 2\ \text{mm}) = 1$ and $\alpha(L=0)=0.$ We call eq. (21) the homotopy matching, when $\alpha=0,$ it becomes eq. (21), and when $\alpha=1,$ it turns out to be eq. (1). The homotopy matching is widely applied in mathematics to solve non-linear problems [31, 32].

We choose the matching parameter as:

$$\alpha = (1 - \mathrm{e}^{-aL})^b \tag{22}$$

where $a$ and $b$ are parameters for experimental determination. eq. (21) becomes:

$$\frac{K}{S} = \frac{(1 - \mathrm{e}^{-aL})^b (1 - R)^2}{2R} + \frac{[1 - (1 - \mathrm{e}^{-aL})^b]}{2SL} (\ln R_s - \ln R) \tag{23}$$

It is well known that reflectance is temperature-dependent [33]. If the thermal effect is considered, eq. (23) can be modified:

$$\frac{K}{S} = \frac{\{1 - [1 - (1 - \mathrm{e}^{-aL})^b]^c\}(1 - R)^2}{2R} + \frac{[1 - (1 - \mathrm{e}^{-aL})^b]^c}{2SL} (\ln R_s - \ln R) \tag{24}$$

where $c$ is a temperature-dependent parameter.

For multiple constituents, the above equation can be modified:

$$\frac{\sum_{i=1}^{N} K_i}{\sum_{i=1}^{N} S_i} = \frac{\{1 - [1 - (1 - \mathrm{e}^{-aL})^b]^c\}(1 - R)^2}{2R} + \frac{[1 - (1 - \mathrm{e}^{-aL})^b]^c}{2SL} (\ln R_s - \ln R) \tag{25}$$

where $N$ is the number of constituents, for examples, when a fabric is coated by $TiO_2$ film, $K_i (i = 1, 2)$ and $S_i (i = 1, 2)$ are the absorption coefficient and the scattering coefficient for the fabric and $TiO_2$ film, respectively.

## Discussion and conclusions

When we consider the porosity of the film, a fractal modification of the Kubelka--Munk function is needed. Eqs. (2) and (3) has to be modified:

$$\mathrm{d}i = -[-(K + S)i + Sj]\mathrm{d}x^\beta \tag{26}$$

and

$$\mathrm{d}j = [-(K + S)j + Si]\mathrm{d}x^\beta \tag{27}$$

where $\beta$ is the fractal dimensions of the film. The two-scale fractal derivative is [34-36]:

$$\frac{\mathrm{d}i}{\mathrm{d}x^\beta}(x_0) = \Gamma(1 + \beta) \lim_{\substack{x - x_0 \to \Delta x \\ \Delta x \neq 0}} \frac{i(x) - i(x_0)}{(x - x_0)^\beta} \tag{28}$$

$$\frac{\mathrm{d}j}{\mathrm{d}x^\beta}(x_0) = \Gamma(1 + \beta) \lim_{\substack{x - x_0 \to \Delta x \\ \Delta x \neq 0}} \frac{j(x) - j(x_0)}{(x - x_0)^\beta} \tag{29}$$

where $\Gamma$ is the gamma function. The two-scale fractal theory is now widely applied for porous problems, see for examples, the fractal diffusion [37], the fractal solitary wave [38], the fractal micro-electro-mechanical devices [39], the fractal concrete [40-43], and the fractal composite [44].

By a similar operation as previous, eq. (6) becomes:

$$\mathrm{d}\ln r = -2(K+S)\mathrm{d}x^{\beta} + S\left(r + \frac{1}{r}\right)\mathrm{d}x^{\beta} \tag{30}$$

The solution of eq. (30) reads:

$$\frac{1}{2\sqrt{\lambda^2-1}}\ln\left[\frac{\left(R_s-\lambda+\sqrt{\lambda^2-1}\right)\left(R-\lambda-\sqrt{\lambda^2-1}\right)}{\left(R-\lambda+\sqrt{\lambda^2-1}\right)\left(R_s-\lambda-\sqrt{\lambda^2-1}\right)}\right] = SL^{\beta} \tag{31}$$

We will further discuss this fractal model in a forthcoming article.

To be concluded, this paper suggests a modified Kubelka-Munk function, eq. (24), which is also simple and considers the thickness of the film and temperature-dependent reflectance. If the porosity of the film is considered, the two-scale fractal modification is a must.

The parameters *a* and *b* and *c* involved in eq. (24) can be estimated experimentally. The modified Kubelka-Munk function can be used for a color-matching model for colorful fabrics, and far-reaching implications are emerging for applications include colorful image technology [45, 46] and microelectromechanical systems [47, 48].

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