A VARIATIONAL PRINCIPLE FOR FRACTAL KLEIN-GORDON EQUATION

by

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This paper studies the Klein-Gordon equation and two modifications in an infinite Cantor set and a fractal space-time. Their variational formulations are established and discussed, and the spatio-temporal discontinuity requires both spatio-fractal derivative and temporal fractal derivative for practical applications. Some basic properties of the local fractional derivative and the two-scale fractal derivative are elucidated, and the derivation of the Euler-Lagrange equation is illustrated.

Key words: *variational theory, fractional calculation, two-scale fractal theory*

Introduction

The Klein-Gordon equation can describe a solitary wave, it can be written [1, 2]:

$$\frac{\partial^2 \omega}{\partial t^2} - \frac{\partial^2 \omega}{\partial x^2} - \lambda\omega - \beta\omega^2 - \gamma\omega^3 = 0 \tag{1}$$

where λ, β, and γ are constants.

Equation (1) is also a useful model for string vibration [3, 4]. Introducing a complex variable η defined:

$$\eta = x - ct \tag{2}$$

where c is a constant. By the chain rule, we have:

$$\frac{\partial \omega}{\partial t} = \frac{\partial \omega}{\partial \eta}\frac{\partial \eta}{\partial t} = -c\frac{\partial \omega}{\partial \eta}, \quad \frac{\partial^2 \omega}{\partial t^2} = -c\frac{\partial}{\partial t}\left(\frac{\partial \omega}{\partial \eta}\right) = -c\frac{\partial \eta}{\partial t}\frac{\partial}{\partial \eta}\left(\frac{\partial \omega}{\partial \eta}\right) = c^2\frac{\partial^2 \omega}{\partial \eta^2}$$

$$\frac{\partial \omega}{\partial x} = \frac{\partial \omega}{\partial \eta}\frac{\partial \eta}{\partial x} = \frac{\partial \omega}{\partial \eta}, \quad \frac{\partial^2 \omega}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial \omega}{\partial \eta}\right) = \frac{\partial^2 \omega}{\partial \eta^2}$$

So we can convert eq. (1) into the following ordinary differential equation:

$$(c^2 + 1)\frac{\mathrm{d}^2 \omega}{\mathrm{d}\eta^2} + \lambda\omega + \beta\omega^2 + \gamma\omega^3 = 0 \tag{3}$$

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This is the well-known Duffing equation [5-13] when $\beta = 0$, so eq. (1) is also called as Klein-Gordon oscillator [14]. In this paper we want to study eq. (1) by the variational theory.

The variational theory plays a significant role in thermal science and mathematics, it is the cornerstone of various numerical methods, *e.g.* the finite element method [15], and various analytical methods, *e.g.*, the Ritz method, and the variational iteration method. Recently the fractal variational theory has been attracted much attention due to their feasibility for the establishment a real mathematical model for discontinuous problems [16].

The fractal variational theory is extensively useful for porous medium problems or the unsmooth boundary problems. Starting from the pioneering ideas going back to He [17, 18], Wang *et al.* [19, 20], Wang *et al.* [21-25], Wang *et al.* [26-28], Khan [29, 30], Zuo [31], Tian [32], Cao *et al.* [33], and Alex *et al.* [34, 35], the two-scale fractal method has become a fully matured theory. An introduction to its basic knowledge is referred in [36, 37].

It is well-known that the variational principle has the global property, that is the approximate solution obtained by the variational principle is valid for the whole solution property. This paper will study the fractal modifications of eq. (1) and their variational formulations.

Variational principle for Klein-Gordon equation

The variational formulation for eq. (1) can be easily obtained by the semi-inverse method [38], which reads:

$$J(\omega) = \iint L \mathrm{d}t\mathrm{d}x \tag{4}$$

where the Lagrange function is:

$$L = -\frac{1}{2}\left(\frac{\partial \omega}{\partial t}\right)^2 + \frac{1}{2}\left(\frac{\partial \omega}{\partial x}\right)^2 - \frac{1}{2}\lambda\omega^2 - \frac{1}{3}\beta\omega^3 - \frac{1}{4}\gamma\omega^4 \tag{5}$$

The Euler-Lagrange equation of eq. (4) is:

$$\begin{gathered}\frac{\partial L}{\partial \omega} - \frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \omega_t}\right) - \frac{\partial}{\partial x}\left(\frac{\partial L}{\partial \omega_x}\right) = 0 \\ \omega_t = \frac{\partial \omega}{\partial t}, \quad \omega_x = \frac{\partial \omega}{\partial x}\end{gathered} \tag{6}$$

It is easy to calculate the following terms:

$$\begin{gathered}\frac{\partial L}{\partial \omega} = -\lambda\omega - \beta\omega^2 - \gamma\omega^3 \\ \frac{\partial L}{\partial \omega_t} = -\frac{\partial \omega}{\partial t}, \quad \frac{\partial L}{\partial \omega_x} = \frac{\partial \omega}{\partial x}\end{gathered} \tag{7}$$

According to eq. (6), we have:

$$\frac{\partial L}{\partial \omega} - \frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \omega_t}\right) - \frac{\partial}{\partial x}\left(\frac{\partial L}{\partial \omega_x}\right) = -\lambda\omega - \beta\omega^2 - \gamma\omega^3 - \frac{\partial}{\partial t}\left(-\frac{\partial \omega}{\partial t}\right) - \frac{\partial}{\partial x}\left(\frac{\partial \omega}{\partial x}\right) = 0 \tag{8}$$

It is obvious that eq. (8) is equivalent to eq. (1).

The variatl formulation for eq. (3) is:

$$J(\omega)=\int\left[-\frac{1}{2}(c^2+1)\left(\frac{\mathrm{d}\omega}{\mathrm{d}\eta}\right)^2+\frac{1}{2}\lambda\omega^2+\frac{1}{3}\beta\omega^3+\frac{1}{4}\gamma\omega^4\right]\mathrm{d}\eta \tag{9}$$

Modified Klein-Gordon equation with local fractional derivative

Wang and Wang [39] studied the following Klein-Gordon model with local fractional derivatives:

$$\frac{\mathrm{d}^{2\alpha}\omega}{\mathrm{d}t^{2\alpha}}-\frac{\mathrm{d}^2\omega}{\mathrm{d}x^2}-\lambda\omega-\beta\omega^2-\gamma\omega^3=0 \tag{10}$$

and established the following variational principle [39]:

$$J(\omega)=\iint L\mathrm{d}t^{\alpha}\mathrm{d}x$$

$$L=-\frac{1}{2}\left(\frac{\mathrm{d}^{\alpha}\omega}{\mathrm{d}t^{\alpha}}\right)^2+\frac{1}{2}\left(\frac{\mathrm{d}\omega}{\mathrm{d}x}\right)^2-\frac{1}{2}\lambda\omega^2-\frac{1}{3}\beta\omega^3-\frac{1}{4}\gamma\omega^4 \tag{11}$$

where $\mathrm{d}^{\alpha}\omega/\mathrm{d}t^{\alpha}$ is the local fractal fractional derivative [40-42].

Remark 1. When time is of the local property, the space must have the local property as well eq. (10) has to be modified:

$$\frac{\mathrm{d}^{2\alpha}\omega}{\mathrm{d}t^{2\alpha}}-\frac{\mathrm{d}^{2\alpha}\omega}{\mathrm{d}x^{2\alpha}}-\lambda\omega-\beta\omega^2-\gamma\omega^3=0 \tag{12}$$

To understand this spatio-temporal correlation [43, 44], we consider a tree's growth. We assume that the tree stops growing at night, so its growth is spatio-temporally discontinuous, see fig. 1.

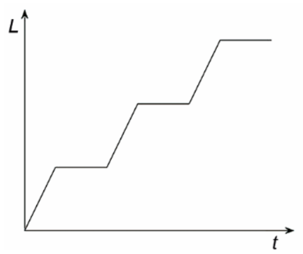

Figure 1. A tree's growing up

Remark 2. The local fractional calculus is established in an infinite Cantor set, it has the local property [40-42].

The variational principle for eq. (12) is:

$$J(\omega)=\iint L\mathrm{d}t^{\alpha}\mathrm{d}x^{\alpha}, \quad L=-\frac{1}{2}\left(\frac{\mathrm{d}^{\alpha}\omega}{\mathrm{d}t^{\alpha}}\right)^2+\frac{1}{2}\left(\frac{\mathrm{d}^{\alpha}\omega}{\mathrm{d}x^{\alpha}}\right)^2-\frac{1}{2}\lambda\omega^2-\frac{1}{3}\beta\omega^3-\frac{1}{4}\gamma\omega^4 \tag{13}$$

The Euler-Lagrange equation of eq. (13) is:

$$\begin{gathered}\frac{\partial L}{\partial\omega}-\frac{\mathrm{d}^{\alpha}}{\mathrm{d}t^{\alpha}}\left(\frac{\mathrm{d}L}{\mathrm{d}\omega_{t^{\alpha}}}\right)-\frac{\mathrm{d}^{\alpha}}{\mathrm{d}x^{\alpha}}\left(\frac{\mathrm{d}L}{\mathrm{d}\omega_{x^{\alpha}}}\right)=0\\ \omega_{t^{\alpha}}=\frac{\mathrm{d}^{\alpha}\omega}{\mathrm{d}t^{\alpha}}, \quad \omega_{x^{\alpha}}=\frac{\mathrm{d}^{\alpha}\omega}{\mathrm{d}x^{\alpha}}\end{gathered} \tag{14}$$

It is easy to calculate the following components:

$$\begin{gathered}\frac{\partial L}{\partial\omega}=-\lambda\omega-\beta\omega^2-\gamma\omega^3\\ \frac{\partial L}{\partial\omega_{t^{\alpha}}}=-\frac{\mathrm{d}^{\alpha}\omega}{\mathrm{d}t^{\alpha}}, \quad \frac{\partial L}{\partial\omega_{x^{\alpha}}}=\frac{\mathrm{d}^{\alpha}\omega}{\mathrm{d}x^{\alpha}}\end{gathered} \tag{15}$$

So the Euler-Lagrange equation is:

$$\frac{\partial L}{\partial\omega}-\frac{\mathrm{d}^{\alpha}}{\mathrm{d}t^{\alpha}}\left(\frac{\partial L}{\partial\omega_{t^{\alpha}}}\right)-\frac{\mathrm{d}^{\alpha}}{\mathrm{d}x^{\alpha}}\left(\frac{\partial L}{\partial\omega_{x^{\alpha}}}\right)=-\lambda\omega-\beta\omega^2-\gamma\omega^3-\frac{\mathrm{d}^{\alpha}}{\mathrm{d}t^{\alpha}}\left(-\frac{\mathrm{d}^{\alpha}\omega}{\mathrm{d}t^{\alpha}}\right)-\frac{\mathrm{d}^{\alpha}}{\mathrm{d}x^{\alpha}}\left(\frac{\mathrm{d}^{\alpha}\omega}{\mathrm{d}x^{\alpha}}\right)=0 \tag{16}$$

It is equivalent to eq. (12).

Modified Klein-Gordon equation with two-scale fractal derivative

The tree grows up every day, and its growth ratio is continuous when measured on a scale of 24 hours, however, when we measure it on a scale 12 hours, it becomes discontinuous. A differential equation model can depict its growth on a large scale, but it can reveal the effect of the length of day or sunlight time on its growth. The two-scale fractal theory uses two different scales to measure the same object, the large one follows the differential equation model, and the smaller one adopts a model with the two-scale fractal derivative. The two-scale fractal modification of eq. (12) is:

$$\frac{\mathrm{d}^2\omega}{\mathrm{d}t^{2\zeta}}-\frac{\mathrm{d}^2\omega}{\mathrm{d}x^{2\zeta}}-\lambda\omega-\beta\omega^2-\gamma\omega^3=0 \tag{17}$$

The fractal derivative is [36, 37]:

$$\frac{\mathrm{d}\omega}{\mathrm{d}t^{\zeta}}=\Gamma(1+\zeta)\lim_{\substack{t-t_0=\Delta t\\ \Delta t\neq 0}}\frac{\omega-\omega_0}{(t-t_0)^{\zeta}} \tag{18}$$

$$\frac{\mathrm{d}\omega}{\mathrm{d}x^{\zeta}}=\Gamma(1+\zeta)\lim_{\substack{x-x_0=\Delta x\\ \Delta x\neq 0}}\frac{\omega-\omega_0}{(x-x_0)^{\zeta}} \tag{19}$$

where Δt is the length of day or sunlight time per day, $\Delta x=u\Delta t$, u – the average velocity on the large scale, ζ – the two-scale fractal dimensions. If the sunlight time is 8 hours each day,

$\zeta = 8/24 = 1/3$. For a complete interpretation of the concept of the two-scale fractal dimensions, audience is referred to [36, 37].

The fractal variational principle of eq. (4) is:

$$J(\omega) = \iint L \mathrm{d}t^{\zeta} \mathrm{d}x^{\zeta}$$

$$L = -\frac{1}{2}\left(\frac{\mathrm{d}\omega}{\mathrm{d}t^{\zeta}}\right)^2 + \frac{1}{2}\left(\frac{\mathrm{d}\omega}{\mathrm{d}x^{\zeta}}\right)^2 - \frac{1}{2}\lambda\omega^2 - \frac{1}{3}\beta\omega^3 - \frac{1}{4}\gamma\omega^4 \tag{20}$$

The stationary condition of eq. (20) reads:

$$\frac{\partial L}{\partial \omega} - \frac{\mathrm{d}}{\mathrm{d}t^{\zeta}}\left(\frac{\mathrm{d}L}{\mathrm{d}\omega_{t^{\zeta}}}\right) - \frac{\mathrm{d}}{\mathrm{d}x^{\zeta}}\left(\frac{\mathrm{d}L}{\mathrm{d}\omega_{x^{\zeta}}}\right) = 0$$

$$\omega_{t^{\zeta}} = \frac{\mathrm{d}\omega}{\mathrm{d}t^{\zeta}}, \quad \omega_{x^{\zeta}} = \frac{\mathrm{d}\omega}{\mathrm{d}x^{\zeta}} \tag{21}$$

It is easy to calculate the following components:

$$\frac{\partial L}{\partial \omega} = -\lambda\omega - \beta\omega^2 - \gamma\omega^3$$

$$\frac{\mathrm{d}L}{\mathrm{d}\omega_{t^{\zeta}}} = -\frac{\mathrm{d}\omega}{\mathrm{d}t^{\zeta}}, \quad \frac{\mathrm{d}L}{\mathrm{d}\omega_{x^{\zeta}}} = \frac{\mathrm{d}\omega}{\mathrm{d}x^{\zeta}} \tag{22}$$

So the Euler-Lagrange equation of eq. (20) is:

$$\frac{\partial L}{\partial \omega} - \frac{\mathrm{d}}{\mathrm{d}t^{\zeta}}\left(\frac{\mathrm{d}L}{\mathrm{d}\omega_{t^{\zeta}}}\right) - \frac{\mathrm{d}}{\mathrm{d}x^{\zeta}}\left(\frac{\mathrm{d}L}{\mathrm{d}\omega_{x^{\zeta}}}\right) = -\lambda\omega - \beta\omega^2 - \gamma\omega^3 - \frac{\mathrm{d}}{\mathrm{d}t^{\zeta}}\left(-\frac{\mathrm{d}\omega}{\mathrm{d}t^{\zeta}}\right) - \frac{\mathrm{d}}{\mathrm{d}x^{\zeta}}\left(\frac{\mathrm{d}\omega}{\mathrm{d}x^{\zeta}}\right) = 0 \tag{23}$$

This is eq. (17).

Using the two-scale transform [45]:

$$T = t^{\zeta} \tag{24}$$

$$X = x^{\zeta} \tag{25}$$

Equation (13) becomes:

$$\frac{\mathrm{d}^{2\alpha}\omega}{\mathrm{d}T^2} - \frac{\mathrm{d}^2\omega}{\mathrm{d}X^2} - \lambda\omega - \beta\omega^2 - \gamma\omega^3 = 0 \tag{26}$$

This is the classical Klein-Gordon model on a large scale, its variational principle is:

$$J(\omega) = \iint L \mathrm{d}T \mathrm{d}X$$

$$L = -\frac{1}{2}\left(\frac{\mathrm{d}\omega}{\mathrm{d}T}\right)^2 + \frac{1}{2}\left(\frac{\mathrm{d}\omega}{\mathrm{d}X}\right)^2 - \frac{1}{2}\lambda\omega^2 - \frac{1}{3}\beta\omega^3 - \frac{1}{4}\gamma\omega^4 \tag{27}$$

The stationary condition of eq. (27) is:

$$\frac{\partial L}{\partial \omega}-\frac{\mathrm{d}}{\mathrm{d}T}\left(\frac{\partial L}{\partial \omega_T}\right)-\frac{\mathrm{d}}{\mathrm{d}X}\left(\frac{\partial L}{\partial \omega_X}\right)=0$$

$$\omega_T=\frac{\mathrm{d}\omega}{\mathrm{d}T},\quad \omega_X=\frac{\mathrm{d}\omega}{\mathrm{d}X} \tag{28}$$

It is easy to calculate the following components:

$$\frac{\partial L}{\partial \omega}=-\lambda\omega-\beta\omega^2-\gamma\omega^3$$

$$\frac{\mathrm{d}L}{\mathrm{d}\omega_T}=-\frac{\mathrm{d}\omega}{\mathrm{d}T},\quad \frac{\mathrm{d}L}{\mathrm{d}\omega_x}=\frac{\mathrm{d}\omega}{\mathrm{d}X} \tag{29}$$

So the Euler-Lagrange equation of eq. (27) is:

$$\frac{\partial L}{\partial \omega}-\frac{\mathrm{d}}{\mathrm{d}T}\left(\frac{\mathrm{d}L}{\mathrm{d}\omega_T}\right)-\frac{\mathrm{d}}{\mathrm{d}X}\left(\frac{\mathrm{d}L}{\mathrm{d}\omega_X}\right)=-\lambda\omega-\beta\omega^2-\gamma\omega^3-\frac{\mathrm{d}}{\mathrm{d}T}\left(-\frac{\mathrm{d}\omega}{\mathrm{d}T}\right)-\frac{\mathrm{d}}{\mathrm{d}X}\left(\frac{\mathrm{d}\omega}{\mathrm{d}X}\right)=0 \tag{30}$$

Equation (29) leads to eq. (26) easily.

Conclusion

This paper gives a heuristic interpretation of the fractal variational theory in view of the two-scale fractal theory, the large scale follows the continuum model, while the smaller one reveals the discontinuity of the studied problem [46]. A fractal modification of the Weierstrass function can be constructed to judge the strong minimum of a fractal variational principle as that discussed in [47, 48], the present method can be extended to integro-differential equations with fractal derivatives [49]; boundary value problems with fractal derivatives [50] and image processing [52-54].

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