APPROXIMATE ANALYTICAL SOLUTION TO THE KUDRYASHOV-SINELSHCHIKOV EQUATION WITH HE'S FRACTIONAL DERIVATIVE

by

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In this paper, the Adomian decomposition method was employed successfully to solve the Kudryashov-Sinelshchikov equation involving He's fractional derivatives, and an approximate analytical solution was obtained.

Key word: *Kudryashov-Sinelshchikov equation, He's fractional derivative, Adomian decomposition method*

Introduction

In this paper, we use Adomian decomposition method (ADM) [1] for solving the following Kudryashov-Sinelshchikov equation involving He's fractional derivative:

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} + u(x,t)\frac{\partial^\beta}{\partial x^\beta} A(u)\frac{\partial^\beta u(x,t)}{\partial x^\beta}\frac{\partial^\beta}{\partial x^\beta} B(u) + \frac{\partial^\beta}{\partial x^\beta} C(u) = 0 \tag{1}$$

with the initial condition:

$$u(x,0) = f\left[\frac{x^\beta}{\Gamma(1+\beta)}\right] \tag{2}$$

where

$$A(u) = \lambda u(x,t) - \frac{\partial^\beta u}{\partial x^\beta} - \frac{\partial}{\partial x^\beta}\left(\frac{\partial^\beta u}{\partial x^\beta}\right) \tag{3}$$

$$B(u) = \frac{\partial^\beta}{\partial x^\beta}\left[u(x,t) + \mu\frac{\partial^\beta u}{\partial x^\beta}\right] \tag{4}$$

$$C(u) = \frac{\partial^\beta}{\partial x^\beta}\left[\frac{\partial^\beta u}{\partial x^\beta} - \sigma u(x,t)\right] \tag{5}$$

and λ, μ, and σ and are real parameters.

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In eqs. (1)-(5), the symbols $\partial^{\alpha}/\partial t^{\alpha}$ and $\partial^{\beta}/\partial x^{\beta}$ $(0<\alpha,\beta\le 1)$ denote He's time fractional derivative and space fractional derivative [2], respectively.

The classical Kudryashov-Sinelshchikov equation describes pressure waves in the liquid with gas bubbles taking into consideration the heat transfer and viscosity of liquid [3-11]. Polymer bubbles have been widely used for fabrication of nanofibers and nanoscale membranes [12, 13]. Due to solvent evaporation [14], the bubble wall becomes a porous medium, so the classical Kudryashov-Sinelshchikov equation has to be modified to take into account the porosity size and distribution. The PDE with He's fractional derivative [2] is suitable for this porous problem. He's fractional derivative was applied to the fractional Camassa-Holm equation [15], the fractional evolution equation [16], and the fractional KdV equation [17]. Now He's fractional derivative was developed into the two-scale fractal derivative [18-23]. Since the exact solutions to these non-linear problems are very rare, so researchers look for the approximate solutions by using various methods, *e.g.*, the homotopy perturbation method [24, 25], in this paper the ADM [1] is adopted, which can solve effectively wide ranges of non-linear equations [26-29].

There are many definitions of fractional derivative are given by many different mathematicians and scientists. He's fractional derivative is derived from the variational iteration method [30]. Now, we recall briefly the concept introduced by He [2].

Consider the following linear equation of n^{th} order:

$$u^{(n)}=f(t) \tag{6}$$

By the variational iteration method [30], we can construct the following iteration formulation:

$$u_{m+1}(t)=u_m(t)+\int_{t_0}^{t}\lambda[u_m^{(n)}(s)-f_m(s)]\mathrm{d}s \tag{7}$$

After identifying the multiplier, we have:

$$u_{m+1}(t)=u_m(t)+(-1)^n\int_{t_0}^{t}\frac{1}{(n-1)!}(s-t)^{n-1}[u_m^{(n)}(s)-f_m(s)]\mathrm{d}s \tag{8}$$

For a linear equation, from eq. (13), we have the following exact solution:

$$u(t)=u_0(t)+(-1)^n\int_{t_0}^{t}\frac{1}{(n-1)!}(s-t)^{n-1}[u_0^{(n)}(s)-f\ (s)]\mathrm{d}s \tag{9}$$

where u_0 satisfies the boundary initial conditions.

From eq. (9), we can define a fractional derivative in the following form, which is called He' fractional derivative [2]:

$$\frac{\mathrm{d}^{\alpha}}{\mathrm{d}t^{\alpha}}f(t)=\frac{1}{\Gamma(n-\alpha)}\frac{\mathrm{d}^n}{\mathrm{d}t^n}\int_{t_0}^{t}(s-t)^{n-\alpha-1}[f_0(s)-f(s)]\mathrm{d}s \tag{10}$$

For more details on the properties of the derivatives, see [2].

Adomian decomposition method

For the convenience of the reader, we present here a brief review on the ADM [1]. Let us consider a general non-linear equation of type:

$$u(x,t) = f + N(u) \tag{11}$$

where $N: H \to H$ is a non-linear mapping from a Banach space H into itself and $f \in H$ is known.

The ADM assumes that the solution u can be expanded as a series:

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t) \tag{12}$$

and the non-linear term $N(u)$ can be decomposed:

$$N\left(\sum_{n=0}^{\infty} u_n\right) = \sum_{n=0}^{\infty} P_n(u) \tag{13}$$

for some He's polynomials $P_n(u)$ [31] that are given by:

$$P_n(u_0, u_1, \cdots, u_n) = \frac{1}{n!}\frac{\partial^n}{\partial p^n}\left[N\left(\sum_{k=0}^{n} p^k u_k\right)\right]_{p=0}, \qquad n = 0,1,2,\cdots \tag{14}$$

Substituting eqs. (12) and (13) into (11), we get:

$$\sum_{n=0}^{\infty} u_n = f + \sum_{n=0}^{\infty} P_n \tag{15}$$

The component of $u_n(x,t)$ follows immediately upon setting:

$$u_0(x,t) = f \tag{16}$$

$$u_1 = P_0 \tag{17}$$

$$u_{k+1} = P_k \quad \text{where} \quad (k = 1,2,3...) \tag{18}$$

Thus we get the k-term approximate solution of (11):

$$u = u_0 + u_1 + \cdots + u_{k-1}. \tag{19}$$

Solution of the problems (1) and (2)

By using the two-scale transform [32-35]:

$$s = \frac{t^{\alpha}}{\Gamma(1+\alpha)}, \qquad y = \frac{x^{\beta}}{\Gamma(1+\beta)} \tag{20}$$

Equation (1) can be converted into the following differential equation:

$$\frac{\partial u(y,s)}{\partial s} + u(y,s)\frac{\partial}{\partial y}A_1(u) - \frac{\partial u(y,s)}{\partial y}\frac{\partial}{\partial y}B_1(u) + \frac{\partial}{\partial y}C_1(u) = 0 \tag{21}$$

and

$$u(y,0) = f(y)$$

$$A_1(u) = \lambda u(y,s) - \frac{\partial u}{\partial y} - \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right)$$

$$B_1(u) = \frac{\partial}{\partial y}\left[u(y,s) + \mu\frac{\partial u}{\partial y}\right]$$

$$C_1(u) = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y} - \sigma u(y,s)\right)$$

We re-write eq. (21):

$$u = u(y,0) + J[L(u) + N_1(u) + N_2(u)] \tag{22}$$

where J is the integral operator with respect to s and:

$$L = \frac{\partial^2}{\partial y^2}\left(\sigma - \frac{\partial u}{\partial y}\right)$$

$$N_1(u) = u\frac{\partial}{\partial y}\left(-\lambda u + \frac{\partial}{\partial y} + \frac{\partial^2 u}{\partial y^2}\right), \qquad N_2(u) = \frac{\partial u}{\partial y}\frac{\partial}{\partial y}\left(u + \mu\frac{\partial}{\partial y}\right)$$

Suppose that the solutions take the form:

$$u(y,s) = \sum_{k=0}^{\infty} u_k(y,s) \tag{23}$$

and the non-linear terms are decomposed:

$$N_1(u) = \sum_{k=0}^{\infty} P_{1k}, \qquad N_2(u) = \sum_{k=0}^{\infty} P_{2k} \tag{24}$$

where $P_{jk}\,(j = 1,2)$ are He's polynomials and given by:

$$P_{10} = u_0\frac{\partial}{\partial y}\left(-\lambda u_0 + \frac{\partial u_0}{\partial y} + \frac{\partial^2 u_0}{\partial y^2}\right) \tag{25}$$

$$P_{11} = u_0\frac{\partial}{\partial y}\left(-\lambda u_1 + \frac{\partial u_1}{\partial y} + \frac{\partial^2 u_1}{\partial y^2}\right) + u_1\frac{\partial}{\partial y}\left(-\lambda u_0 + \frac{\partial u_0}{\partial y} + \frac{\partial^2 u_0}{\partial y^2}\right) \tag{26}$$

$$P_{12} = u_0\frac{\partial}{\partial y}\left(-\lambda u_2 + \frac{\partial u_2}{\partial y} + \frac{\partial^2 u_2}{\partial y^2}\right) + u_1\frac{\partial}{\partial y}\left(-\lambda u_1 + \frac{\partial u_1}{\partial y} + \frac{\partial^2 u_1}{\partial y^2}\right) + u_2\frac{\partial}{\partial y}\left(-\lambda u_0 + \frac{\partial u_0}{\partial y} + \frac{\partial^2 u_0}{\partial y^2}\right) \tag{27}$$

$$P_{20}=\frac{\partial u_0}{\partial y}\frac{\partial}{\partial y}\left(u_0+\mu\frac{\partial u_0}{\partial y}\right) \tag{28}$$

$$P_{21}=\frac{\partial u_0}{\partial y}\frac{\partial}{\partial y}\left(u_1+\mu\frac{\partial u_1}{\partial y}\right)+\frac{\partial u_1}{\partial y}\frac{\partial}{\partial y}\left(u_0+\mu\frac{\partial u_0}{\partial y}\right)$$

$$P_{22}=\frac{\partial u_0}{\partial y}\frac{\partial}{\partial y}\left(u_2+\mu\frac{\partial u_2}{\partial y}\right)+\frac{\partial u_1}{\partial y}\frac{\partial}{\partial y}\left(u_1+\mu\frac{\partial u_1}{\partial y}\right)+\frac{\partial u_2}{\partial y}\frac{\partial}{\partial y}\left(u_0+\mu\frac{\partial u_0}{\partial y}\right)$$

and so on.

Applying procedure defined in eqs. (16)-(18), we get:

$$u_0(y,s)=u(y,0) \tag{29}$$

$$u_1(y,s)=J[L(u_0)+P_{10}+P_{20}] \tag{30}$$

$$u_{m+1}(y,s)=J[L(u_m)+P_{1m}+P_{2m}] \tag{31}$$

where $m=1,2,3\cdots$.

Then, applying backward substitution to the computed components, we obtain:

$$u(x,t)=u_1+u_2+u_3+\cdots \tag{32}$$

Next, we give two examples to show the efficiency of the algorithm described previously.

Example 1. Taking $\lambda=0$, $\sigma=1$, and $\mu=-1$, we consider eq. (1) with the following initial condition:

$$u(x,0)=2+E,\quad \text{where } E=\exp\left[\frac{-2x^{\beta}}{\Gamma(1+\beta)}\right]$$

By the previous algorithms, we obtain:

$$u_0(x,t)=2+E,\qquad u_1(x,t)=\frac{4Et^{\alpha}}{\Gamma(1+\alpha)},\qquad u_2(x,t)=\frac{8Et^{2\alpha}}{\Gamma^2(1+\alpha)}$$

and so on.

Thus, the $3-term$ approximate solution in this case is:

$$u(x,t)=2+E+\frac{4Et^{\alpha}}{\Gamma(1+\alpha)}+\frac{8Et^{2\alpha}}{\Gamma^2(1+\alpha)}$$

Example 2. Taking $\lambda=-1$, $\sigma=3$, and $\mu=-1$, we consider eq. (1) with the following initial condition:

$$u(x,0)=3+\frac{x^{2\beta}E_1}{\Gamma^2(1+\beta)},\quad \text{where } E_1=\exp\left[\frac{-x^{\beta}}{\Gamma(1+\beta)}\right]$$

In a similar way as mentioned, we obtain:

$$u_0(x,t)=3+\frac{x^{2\beta}E_1}{\Gamma^2(1+\beta)},\quad u_1(x,t)=\left[\frac{x^{2\beta}}{\Gamma^2(1+\beta)}-\frac{6x^{\beta}}{\Gamma(1+\beta)}\right]\frac{t^{\alpha}E_1}{\Gamma(1+\alpha)}$$

$$u_2(x,t)=\left[\frac{x^{2\beta}}{\Gamma^2(1+\beta)}-\frac{12x^{\beta}}{\Gamma(1+\beta)}+18\right]\frac{t^{2\alpha}E_1}{2\Gamma^2(1+\alpha)}$$

and so on.

Thus, the $3-term$ approximate solution is:

$$u(x,t)=3+\frac{x^{2\beta}E_1}{\Gamma^2(1+\beta)}+\left[\frac{x^{2\beta}-6x^{\beta}\Gamma(1+\beta)}{\Gamma^2(1+\beta)\Gamma(1+\alpha)}\right]E_1t^{\alpha}+$$
$$+\left[\frac{x^{2\beta}-12x^{\beta}\Gamma(1+\beta)+18\Gamma^2(1+\beta)}{2\Gamma^2(1+\beta)\Gamma^2(1+\alpha)}\right]E_1t^{2\alpha}$$

Conclusion

In this paper, we gave a fractional modification of Kudryashov-Sinelshchikov equation, its physical understanding can be referred to [36, 37]. We used ADM to obtain the approximate analytical solution of the Kudryashov-Sinelshchikov equation involving He's fractional derivative. The method is simple in its principle and convenient for computer algorithms. Our examples show that the results obtained by the method are very accurate, and the method can be extended to more complex problems like fractional chaos synchronization [38, 39] and fractional control [40, 41].

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