VARIATIONAL METHOD TO FRACTAL LONG-WAVE MODEL WITH VARIABLE COEFFICIENTS

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Original scientific paper
https://doi.org/10.2298/TSCI2303779M

In this paper, a regularized long wave travelling along an unsmooth boundary is depicted by the fractal calculus, and its fractal variational principle is established via the fractal semi-inverse method, which is very helpful to construct the conservation laws in the fractal space and to study the structure of the analytical solution, and a fractal solitary wave solution is obtained.

Key words: *fractal two-scale transform method, fractal semi-inverse method, fractal calculus*

Introduction

The linear and non-linear differential equations are very important to describe very complex nature phenomena that occur in the real world. The concerned disciplines include engineering, mechanics, physics, and thermal science. However, it is very difficult for traditional differential models to depict the complexity of nature phenomena of discontinuous phenomena, problems in a micro-gravity condition, and porous media [1-3]. A differential model requires that each variable is differentiable, however, a discontinuous problem forbids this basic assumption, so fractal calculus [4, 5] has to be adopted. Tian *et al.* [6] and He [7] established a fractal MEMS oscillator in a fractal space. He *et al.* [8] suggested a fractal thermodynamical model for heat conduction in a porous concrete. Anjum *et al.* [9] obtained a fractal population model. He *et al.* [10] revised dynamic economics in view of fractal calculus. Recently fractal vibration systems have also been caught much attention, and many new and attractive properties were revealed [11-14].

In this paper, we mainly consider the fractal non-linear regularized long wave model with unsmooth boundaries, which reads:

$$\frac{\partial \upsilon}{\partial t^{\alpha}} + a(t^{\alpha})\frac{\partial}{\partial x^{\beta}}\left(\frac{\upsilon^2}{2}\right) = b\frac{\partial \upsilon}{\partial x^{2\beta}\partial t^{\alpha}} \tag{1}$$

where $a(t^{\alpha})$ is a function of t^{α}, α – the fractal dimension, b – a constant, and $\partial\upsilon/\partial t^{\alpha}$ and $\partial\upsilon/\partial x^{\beta}$ – the fractal derivatives, which are defined [4, 5]:

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$$\frac{\partial \upsilon}{\partial t^{\alpha}}(t_0^{\alpha}) = \Gamma(1+\alpha) \lim_{\substack{t-t_{0\to\Delta t} \\ \Delta t\neq 0}} \frac{\upsilon(t)-\upsilon(t_0)}{(t-t_0)^{\alpha}} \tag{2}$$

$$\frac{\partial \upsilon}{\partial x^{\beta}}(x_0^{\beta}) = \Gamma(1+\beta) \lim_{\substack{x-x_{0\to\Delta x} \\ \Delta x\neq 0}} \frac{\upsilon(x)-\upsilon(x_0)}{(x-x_0)^{\beta}} \tag{3}$$

Equation (1) is the classic non-linear regularized long wave equation for $\alpha = \beta = 1$, which can be used to describe the motion of transverse waves in shallow water [15]. Wang, *et al.* [16] studied a fractal regularized long-wave model for coast protection.

There are many different ways to solve differential equations, such as the homotopy perturbation method [17-21], the variational iteration method [22], and others [23-27]. In this paper, the fractal non-linear regularized long wave equation is studied by the fractal variational theory [28-30]. The variational theory is a very excellent analytical tool to dealing with complex non-linear problems, which has the advantages of giving the physical insight into the complex model and obtaining the best solution by the fractal trial function. Firstly, the fractal variational principle of the fractal non-linear regularized long wave equation is established by using the fractal semi-inverse method [31]. Secondly, the fractal solitary wave solution of the fractal non-linear regularized long wave equation is obtained by the fractal variational principle and the fractal two-scale transform method [32, 33], which is called the fractal variational transform method (FVTM). The fractal two-scale transform method is an efficient transform to convert a fractal model into its continuous partner, which has been widely employed in many fields [34-41], and can be extended to chaotic systems and control systems [42-45], the fractal chaotic systems and fractal control systems will be the future research frontiers. Finally, a numerical example is given to illustrate the proposed method is efficient and simple.

Fractal variational transform method

Variational theory is an effective mathematical tool to non-linear problems, especially for integro-differential equations [46], two-point boundary value problems [47], and imaging processing [48-51]. In this paper, we consider the following fractal differential equation

$$N\left(\upsilon, \frac{\partial \upsilon}{\partial x^{\beta}}, \frac{\partial \upsilon}{\partial t^{\alpha}}, \frac{\partial \upsilon}{\partial x^{2\beta}}, \frac{\partial \upsilon}{\partial t^{2\alpha}}, \cdots\right) = 0 \tag{4}$$

Step one. By using the fractal semi-inverse method, we construct the fractal variational principle of eq. (4):

$$J(\upsilon) = \int_0^{t^{\alpha}} \int_0^{x^{\beta}} L \mathrm{d}t^{\alpha} \mathrm{d}x^{\beta} \tag{5}$$

where L is fractal Lagrange function.

Step two. Using the fractal two-scale transform method, we assume [32, 33]:

$$T = t^{\alpha} \tag{6}$$

$$X = x^{\beta} \tag{7}$$

Therefore, eq. (4) can be converted into its traditional partner:

$$N\left(\upsilon,\frac{\partial\upsilon}{\partial X},\frac{\partial\upsilon}{\partial X},\frac{\partial\upsilon}{\partial X^2},\frac{\partial\upsilon}{\partial T^2},\cdots\right)=0 \tag{8}$$

So, the variational principle of eq. (8) is:

$$J(\upsilon)=\int_0^T\int_0^X K\mathrm{d}T\mathrm{d}X \tag{9}$$

Step three. Employ the following transform:

$$\xi=\lambda X+\gamma T \tag{10}$$

Therefore, eq. (8) can be rewritten into:

$$N\left(\upsilon,\lambda\frac{\partial\upsilon}{\partial\xi},\gamma\frac{\partial\upsilon}{\partial\xi},\lambda^2\frac{\partial\upsilon}{\partial\xi^2},\gamma^2\frac{\partial\upsilon}{\partial\xi^2},\cdots\right)=0 \tag{11}$$

Equation (9) is written into the following form:

$$J(\upsilon)=\int_0^\infty K\mathrm{d}\xi \tag{12}$$

Step four. According to He's variational method [31], we have the following solitary wave solutions:

$$\upsilon(\xi)=p\sec h(q\xi) \tag{13}$$

$$\upsilon(\xi)=p\cosh(q\xi) \tag{14}$$

$$\upsilon(\xi)=p\tanh(q\xi) \tag{15}$$

$$\upsilon(\xi)=p\coth(q\xi) \tag{16}$$

$$\upsilon(\xi)=p\sec h^2(q\xi) \tag{17}$$

……

where *p* and *q* are constants.

Substituting anyone solitary previous solution into eq. (12), obtain the following relations [31]:

$$\frac{\partial J}{\partial p}=0 \tag{18}$$

$$\frac{\partial J}{\partial q}=0 \tag{19}$$

By using eqs. (18) and (19), *p* and *q* can be easily determined. Therefore, the fractal solitary wave solution of eq. (1) is successfully obtained by eqs. (6) and (7).

Fractal variational principle for fractal regularized long wave equation

Consider the following fractal non-linear regularized long wave equation:

$$\frac{\partial \upsilon}{\partial t^{\alpha}}+a(t^{\alpha})\frac{\partial}{\partial x^{\beta}}\left(\frac{\upsilon^{2}}{2}\right)=b\frac{\partial \upsilon}{\partial x^{2\beta}\partial t^{\alpha}} \tag{20}$$

First, eq. (20) can be written into:

$$\frac{\partial \upsilon}{\partial t^{\alpha}}+a(t^{\alpha})\upsilon\frac{\partial \upsilon}{\partial x^{\beta}}-b\frac{\partial \upsilon}{\partial x^{2\beta}\partial t^{\alpha}}=0 \tag{21}$$

Employ the following transformation:

$$\upsilon(x^{\beta},t^{\alpha})=\upsilon(\xi^{\alpha,\beta}) \tag{22}$$

$$\xi^{\alpha,\beta}=\lambda t^{\alpha}+\gamma x^{\beta} \tag{23}$$

So, eq. (20) can be converted into the following equation:

$$\lambda\frac{\partial \upsilon}{\partial \xi^{\alpha,\beta}}+a(t^{\alpha})\gamma\upsilon\frac{\partial \upsilon}{\partial \xi^{\alpha,\beta}}-b\lambda\gamma^{2}\frac{\partial \upsilon}{\partial \xi^{3(\alpha,\beta)}}=0 \tag{24}$$

Integrating eq. (24), obtains:

$$\lambda\upsilon+\frac{a(t^{\alpha})\gamma\upsilon}{2}\upsilon^{2}-b\lambda\gamma^{2}\frac{\partial \upsilon}{\partial \xi^{2(\alpha,\beta)}}=0 \tag{25}$$

Second, we assume the fractal variational principle of eq. (25):

$$J(\upsilon)=\int_{0}^{\infty}L\mathrm{d}\xi \tag{26}$$

where L is the fractal Lagrange function.

Using the fractal semi-inverse method [31], we have:

$$L=\frac{\lambda}{2}\upsilon^{2}+\frac{a(t^{\alpha})\gamma}{6}\upsilon^{3}-\frac{b\lambda\gamma^{2}}{2}\left(\frac{\partial \upsilon}{\partial \xi^{\alpha,\beta}}\right)^{2} \tag{27}$$

Hence, we have the fractal variational principle of eq. (25):

$$J(\upsilon)=\int_{0}^{\infty}\left[\frac{\lambda}{2}\upsilon^{2}+\frac{a(t^{\alpha})\gamma}{6}\upsilon^{3}-\frac{b\lambda\gamma^{2}}{2}\left(\frac{\partial \upsilon}{\partial \xi^{\alpha,\beta}}\right)^{2}\right]\mathrm{d}\xi^{\alpha,\beta} \tag{28}$$

Fractal solitary wave solution for fractal regularized long wave equation

Consider the fractal non-linear regularized long wave equation, which reads:

$$\frac{\partial \upsilon}{\partial t^{\alpha}}+\frac{\partial}{\partial x^{\beta}}\left(\frac{\upsilon^{2}}{2}\right)=b\frac{\partial \upsilon}{\partial x^{2\beta}\partial t^{\alpha}} \tag{29}$$

Apply the fractal two-scale transform method, and assume [32, 33]:

$$T = t^{\alpha} \tag{30}$$

$$X = x^{\beta} \tag{31}$$

So, we can obtain the partner of eq. (29):

$$\frac{\partial \upsilon}{\partial T} + \upsilon \frac{\partial \upsilon}{\partial X} - b\frac{\partial \upsilon}{\partial X^{2}\partial T} = 0 \tag{32}$$

Adopt the following transformation:

$$\upsilon(X,T) = \upsilon(\xi) \tag{33}$$

$$\xi = \lambda T + \gamma X \tag{34}$$

As the previous method, have:

$$\lambda\upsilon + \frac{\gamma\upsilon}{2}\upsilon^{2} - b\lambda\gamma^{2}\frac{\partial\upsilon}{\partial\xi} = 0 \tag{35}$$

Using the semi-inverse method, the variational principle of eq. (35):

$$J(\upsilon) = \int_{0}^{\infty}\left[\frac{\lambda}{2}\upsilon^{2} + \frac{\gamma}{6}\upsilon^{3} - \frac{b\lambda\gamma^{2}}{2}\left(\frac{\partial\upsilon}{\partial\xi}\right)^{2}\right]\mathrm{d}\xi \tag{36}$$

Assume the solitary wave solution is:

$$\upsilon = p\sec h(q\xi) \tag{37}$$

Substitute eq. (37) into eq. (36), and obtain:

$$J(\upsilon) = \int_{0}^{\infty}\left[\frac{\lambda}{2}p^{2}\sec h^{2}(q\xi) + \frac{\gamma p^{3}}{6}\sec h^{3}(q\xi) - \frac{pq^{2}b\lambda\gamma^{2}}{2}\sec h^{2}(q\xi)\tanh^{2}(q\xi)\right]\mathrm{d}\xi =$$

$$= -\frac{bpq^{2}\gamma^{2}\lambda}{6q} + \frac{\pi p^{2}\gamma}{24q} + \frac{p\lambda}{2q} \tag{38}$$

We have:

$$\frac{\partial J}{\partial p} = -\frac{bq^{2}\gamma^{2}\lambda}{6q} + \frac{\pi p^{2}\gamma}{8q} + \frac{p\lambda}{q} = 0 \tag{39}$$

$$\frac{\partial J}{\partial q} = -\frac{bp\gamma^{2}\lambda}{6} - \frac{\pi p^{2}\gamma}{24q^{2}} - \frac{p\lambda}{2q^{2}} = 0 \tag{40}$$

By eqs. (39) and (40), obtain:

$$p = -\frac{q\lambda}{\pi\gamma} \tag{41}$$

$$q = \frac{2\sqrt{14\lambda\pi b\gamma}}{\gamma^2 b\pi} \tag{42}$$

So, the solitary wave solution of eq. (32) is the following form:

$$\upsilon(X,T) = -\frac{2\lambda\sqrt{14\lambda\pi b\gamma}}{\pi^2 b\gamma^3}\operatorname{sec}h\left[\frac{2\sqrt{14\lambda\pi b\gamma}}{\gamma^2 b\pi}(\lambda X + \gamma T)\right] \tag{43}$$

Consequently, the fractal solitary wave solution of eq. (29) can be expressed:

$$\upsilon(x,t) = -\frac{2\lambda\sqrt{14\lambda\pi b\gamma}}{\pi^2 b\gamma^3}\operatorname{sec}h\left[\frac{2\sqrt{14\lambda\pi b\gamma}}{\gamma^2 b\pi}(\lambda x^{\beta} + \gamma t^{\alpha})\right] \tag{44}$$

Conclusion

In this paper, we successfully use the fractal semi-inverse method to obtain the fractal variational principle for the fractal non-linear regularized long wave equation, and its fractal solitary wave solution is found by the fractal variational transform method. The numerical example shows the proposed method is efficient and simple. The proposed method can be employed to find fractal solitary wave solutions for different types of fractal non-linear models.

Acknowledgment

This paper, was supported by National Natural Science Foundation of China (52174109) and Program for Innovative Research Team (in Science and Technology) in University of Henan Province.

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Paper submitted: October 15, 2021
Paper revised: July 20, 2022
Paper accepted: July 21, 2022