NUMERICAL ANALYSIS OF SPACE-TIME FRACTIONAL BENJAMIN-BONA-MAHONY EQUATION

by

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This paper proposes a numerical approach based on the fractional complex transform and the homotopy perturbation method to solving the space-time fractional Benjamin-Bona-Mahony (mBBM) equation with Caputo fractional derivative. Approximated solutions with high accuracy are provided without linearization or complicated computation. Numerical examples are given to illustrate the efficiency of this method.

Key words: *space-time fractional mBBM equation, fractional complex transform, homotopy perturbation method, approximation*

Introduction

Fractional differential equations (FDE) have wide applications in computer science and engineering areas, for examples, the fractal-fractional models for MEMS oscillator [1], thermal response of a porous concrete [2], non-linear vibration systems in a fractal space [3, 4], the population dynamics [5], and new dynamics economics [6]. Due to the non-local properties of fractional derivatives, FDE can be used to model the physical phenomenon with the instant time and the time history. In this paper, we will consider the following space-time fractional mBBM equation:

$$\frac{\partial^\alpha u}{\partial t^\alpha}+a\frac{\partial^\alpha u}{\partial x^\alpha}+bu^2\frac{\partial^\alpha u}{\partial x^\alpha}-c\frac{\partial^\alpha}{\partial t^\alpha}\left[\frac{\partial^\alpha}{\partial x^\alpha}\left(\frac{\partial^\alpha u}{\partial x^\alpha}\right)\right]=0 \tag{1}$$

where u is a non-linear function of (x, t) with $x, t > 0$, a, b, and c are three given non-zero constants. Here $\partial^\alpha u/\partial t^\alpha$ is the fractional derivative in Caputo sense which can be given by [7]:

$$\frac{\partial^\alpha f}{\partial t^\alpha}=\frac{1}{\Gamma(n-\alpha)}\int_0^t (t-\tau)^{n-\alpha-1}f^{(n)}(\tau)\mathrm{d}\tau \tag{2}$$

with a constant $n-1<\alpha\le n$. Notice that $\partial^\alpha u/\partial x^\alpha$ is also defined by Caputo fractional derivative of order α with respect to x. In real applications, the space-time fractional mBBM type equations have been widely applied to model the propagation of waves in fluid dynamics, for examples, the surface long waves in non-linear dispersive media, the hydromagnetic waves in cold plasma and the acoustic gravity waves in compressible fluids [8-10]. Especially, eq. (1) reduces to the mBBM equation when $\alpha = 1$ [11, 12].

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Ege and Misirli [8] proposed the modified Kudryashov method for solving the space-time fractional mBBM equation and the space-time fractional potential Kadomtsev-Petviashvili equation. In [9], the generalized Kudryashov method was applied to give the traveling wave solutions of these fractional PDE. Reviewing these improvements, the numerical analysis of the space-time fractional mBBM type equations requires further research. The fractional complex transform (FCT) is a two-scale transformation with respect to space or time, which can be used to transform FDE [13, 14]. The homotopy perturbation method (HPM) proposed by He [15-17] has been applied for solving various linear or non-linear problems, and numerical examples were given to confirm its efficiency. Motivated by the fractional complex transform and homotopy perturbation method, we will consider an efficient numerical approach for (1). This approach is named as FCT-HPM. We illustrate the efficiency of FCT-HPM by solving an initial value problem associated with eq. (1). The numerical results are provided, and some conclusions are given.

The FCT-HPM technique

The FCT-HPM technique is a combination of the fractional complex transform and HPM. The main difficulty for solving the FDE lies in the fractional or fractal derivative arising from the fractional space or time. We first introduce the fractional complex transform to release this problem [13, 14]. In fact, the fractional complex transform is a two-scale method which can transform the fractional or fractal PDE to the ordinary PDE. Then the approximated solutions to the ODE can be provided by using homotopy perturbation method [15-18]. For clarity, we show this numerical approach below.

Fractional complex transform

Consider the following fractional PDE:

$$f(u, u_t^{\alpha}, u_x^{\beta}, u_t^{2\alpha}, u_x^{2\beta}, \cdots) = 0 \tag{3}$$

where $u_t^{\alpha} = [\partial^{\alpha} u(x,t)]/(\partial t^{\alpha})$ denotes the Caputo fractional derivative defined by eq. (2), the function $u(x,t)$ is continuous (but not differentiable anywhere), and $0 < \alpha < 1$, $0 < \beta < 1$.

By applying the following fractional complex transform [13, 14]:

$$T = \frac{pt^{\alpha}}{\Gamma(1+\alpha)}, \quad X = \frac{qx^{\beta}}{\Gamma(1+\beta)}$$

with non-zero constants *p* and *q*, the fractional derivatives in eq. (3) can be converted to the classical derivatives [13, 14]:

$$\frac{\partial^{\alpha} u(x,t)}{\partial t^{\alpha}} = p\frac{\partial u}{\partial T}$$

$$\frac{\partial^{\beta} u(x,t)}{\partial x^{\beta}} = q\frac{\partial u}{\partial X}$$

Therefore, we can rewrite the fractional differential eq. (3) as an ordinary PDE which can be solved by the HPM [15-17]. We remark that *T* and *X* can be seen as two variables on a large scale, compared with the variables *t* and *x* on a small scale. The geometry explanation of FCT was given in [19-25].

Homotopy perturbation method

We consider the following non-linear differential equation:

$$A(u) - f(r) = 0, \quad r \in \Omega \tag{4}$$

with boundary conditions:

$$B\left(u, \frac{\mathrm{d}u}{\mathrm{d}n}\right) = 0, \quad r \in \Gamma \tag{5}$$

where A is a general differential operator, B – a boundary operator, u – a known analytic function, and Γ – the boundary of the domain.

We can divide the operator A into the linear and non-linear parts, which are denoted by L and N, respectively. Then eq. (4) can be rewritten:

$$L(u) + N(u) - f(r) = 0$$

The HPM is to construct a homotopy $v(r, p): \Omega[0,1] \to \mathrm{R}$ which satisfies [23]:

$$H(v, p) = (1-p)[L(v) - L(u_0)] + p[A(v) - f(r)] = 0 \tag{6}$$

or

$$H(v, p) = L(v) - L(u_0) + pL(u_0) + p[N(v) - f(r)] = 0 \tag{7}$$

where $r \in \Gamma$ and $p \in [0,1]$ is an embedding parameter, u_0 is an initial approximation of eq. (4), which satisfies the boundary conditions, eq. (5).

Assume that the solution of eq. (6) can be expressed as a power series in p:

$$v = v_0 + pv_1 + p^2 v_2 + ...$$

Then the approximate solution of eq. (4) can be given by:

$$u = \lim_{p \to 1} v = v_0 + v_1 + v_2 + ...$$

Analysis of space-time fractional mBBM equation

Consider the space-time fractional mBBM eq. (1) with the following initial condition:

$$u(x,0) = \sqrt{\frac{3ac}{-b(2+ck^2)}} k \tanh\left[\frac{kx^\alpha}{2\Gamma(1+\alpha)}\right] \tag{8}$$

where k is an arbitrary constant. We remark that the exact solution to the mBBM equation with $\alpha = 1$ is given by [12]:

$$u(x,t) = \sqrt{\frac{3ac}{-b(2+ck^2)}} k \tanh\left[\frac{1}{2} k \left(x - \frac{2at}{2+ck^2}\right)\right]$$

By the fractional complex transform with:

$$T = \frac{t^\alpha}{\Gamma(1+\alpha)}, \quad X = \frac{x^\alpha}{\Gamma(1+\alpha)} \tag{9}$$

it follows that:

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha}=\frac{\partial u}{\partial T}=u_T\,,\quad \frac{\partial^\alpha u(x,t)}{\partial x^\alpha}=\frac{\partial u}{\partial X}=u_X$$

Then eq. (1) can be rewritten as the following ordinary PDE:

$$u_T+au_X+bu^2u_X-cu_{XXT}=0 \tag{10}$$

Here the initial condition for eq. (10) is given by:

$$u(X,0)=\sqrt{\frac{3ac}{-b(2+ck^2)}}k\tanh\left(\frac{1}{2}kX\right) \tag{11}$$

By HPM [15-17], we construct the homotopy for u satisfying the following equation:

$$u_T-u_{0T}+pu_{0T}+p(au_X+bu^2u_X-cu_{XXT})=0 \tag{12}$$

with $u_0=u(X,0)$ given by eq. (11).

Assume that the approximations to the previous system can be formulated:

$$u(X,T)=u_0(X,T)+pu_1(X,T)+p^2u_2(X,T)+\cdots \tag{13}$$

Substituting eq. (13) into eq. (12), and collecting the coefficients of p term, we have:

$$p^1:u_{1T}+u_{0T}+au_{0X}+bu_0^2u_{0X}-cu_{0XXT}=0$$

$$p^2:u_{2T}+au_{1X}+bu_0^2u_{1X}-cu_{1XXT}=0$$

$$p^i:\cdots\cdots$$

By solving the previous systems, we have the approximated solutions:

$$u_1=\frac{\sqrt{3}a^{3/2}b\sqrt{c}}{2[-b(2+ck^2)]^{3/2}}k^2T\ \sec h^2\left(\frac{1}{2}kX\right)\left[2+ck^2-3ck^2\tan h^2\left(\frac{1}{2}kX\right)\right]$$

$$u_2=\frac{1}{16(2+ck^2)^2\sqrt{-b(2+ck^2)}}\left\{\sqrt{3}a^{3/2}\sqrt{c}k^3T\sec h^7\left(\frac{1}{2}kX\right)\left[8ck(2+11ck^2+5c^2k^4)\cdot\right.\right.$$

$$\cdot\cosh\left(\frac{1}{2}kX\right)+ck(2-49ck^2-25c^2k^4)\cosh\left(\frac{3}{2}kX\right)-ck(2-ck^2-c^2k^4)\cosh\left(\frac{5}{2}kX\right)-$$

$$-(2+14ck^2+56c^2k^4)aT\sinh\left(\frac{1}{2}kX\right)-(3+12ck^2-15c^2k^4)aT\sinh\left(\frac{3}{2}kX\right)-$$

$$\left.\left.-(1-2ck^2+c^2k^4)aT\sinh\left(\frac{5}{2}kX\right)\right]\right\}$$

Recalling the transformation (9), we have the second order approximation to eq. (1). The rest approximations can be given in a similar manner.

Numerical results

In this section, we illustrate the efficiency of FCT-HPM for the initial value problem associated with (1). We set k = 0.2, a = 1, b = –1 and c = 1 in this example.

We first consider the numerical analysis of the classical mBBM equation with $\alpha = 1$. For comparison, we also provide the third order approximation $\hat{u}_3$ by FCT-HPM. Figure 1 shows the compared results of $\hat{u}_2$ and $\hat{u}_3$ given by FCT-HPM and the exact solution $u(x, t)$. The absolute error curves for the numerical approximations are plotted in fig. 2. Figure 2 shows that the approximated solutions agree well with the exact solution. We remark that the accuracy of the approximations by FCT-HPM can be improved by considering higher order approximations.

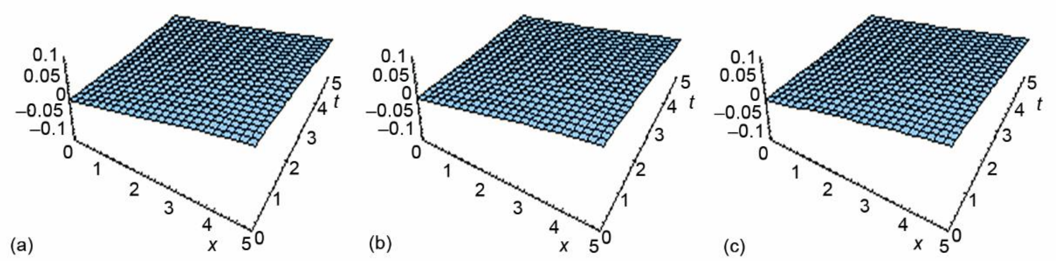

Figure 1. Numerical comparisons for the classical mBBM eq. (1) (a) $\hat{u}_2$, (b) $\hat{u}_3$, and (c) u_{exact}

We then test the numerical behavior of the approximated solutions for the fractional mBBM equation with different α. Figures 3(a) and 3(b) present the numerical results of $\hat{u}_2$ and $\hat{u}_3$ for (1) with $\alpha = 0.3$, respectively. Numerical results of the approximations for (1) with $\alpha = 0.5$ and $\alpha = 0.8$ are presented in figs. 4 and 5, respectively. Furthermore, we also plot the numerical curves of $\hat{u}_3$ at x = 0 and x = 5 with different α in figs. 6(a) and 6(b), respectively. The propagation direction of the wave tends to be parallel to the x-axis when the value of α becomes small. By figs. 3-6, we can conclude that FCT-HPM can efficiently give the approximations without any discretization or restrict assumptions.

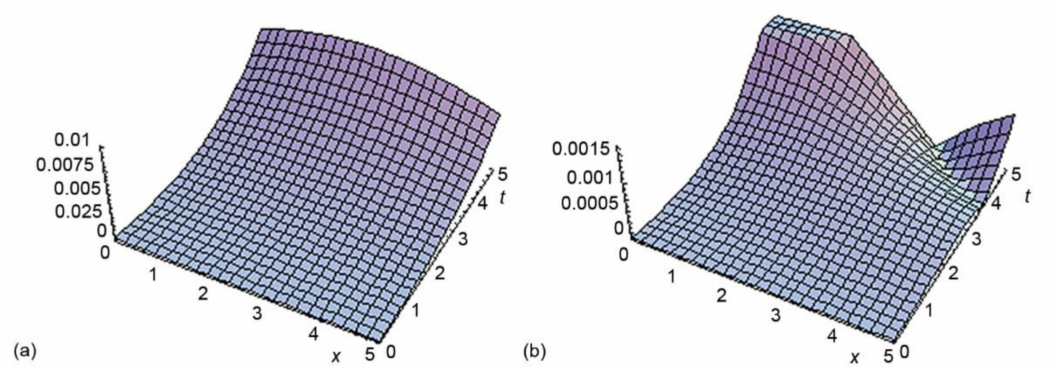

Figure 2. Errors of the approximations the classical mBBM eq. (1); (a) $\hat{u}_2$ and (b) $\hat{u}_3$

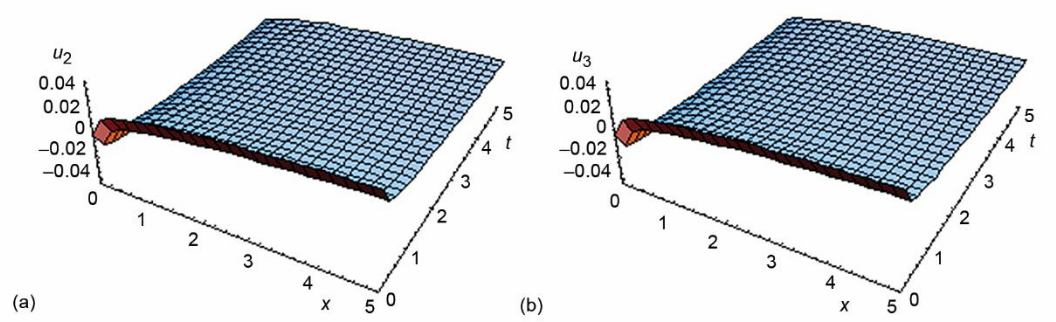

Figure 3. Numerical approximations for eq. (1) with α = 0.3; (a) $\hat{u}_2$ and (b) $\hat{u}_3$

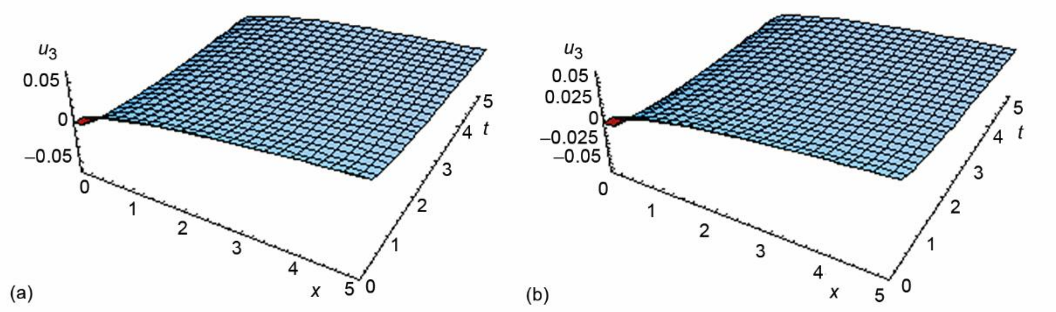

Figure 4. Numerical approximations for eq. (1) with α = 0.5; (a) $\hat{u}_2$ and (b) $\hat{u}_3$

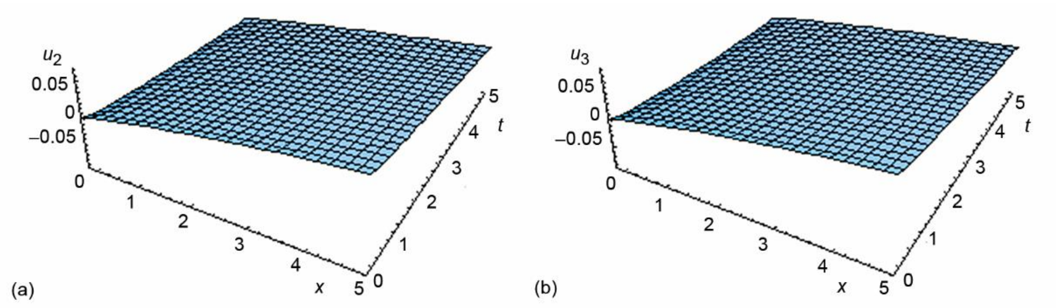

Figure 5. Numerical approximations for eq. (1) with α = 0.8; (a) $\hat{u}_2$ and (b) $\hat{u}_3$

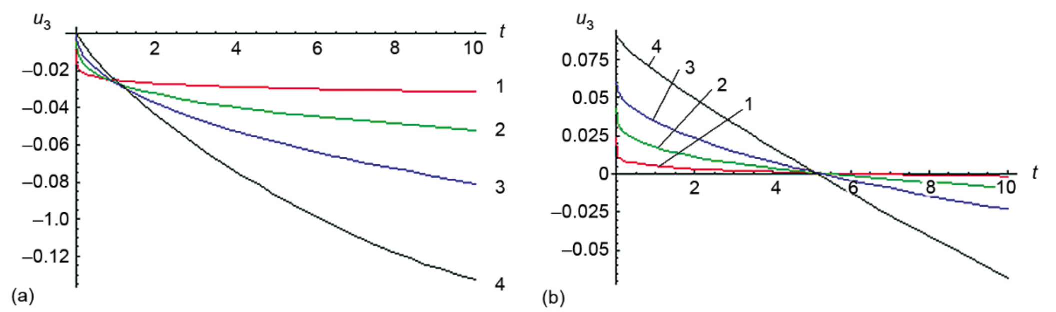

Figure 6. Numerical behavior of $\hat{u}_3$ with different α; (a) x = 0 and (b) x = 5. *1 – α = 0.1, 2 – α = 0.3, 3 – α = 0.5, and 4 – α = 0.8*

Conclusion

This paper focused on the numerical analysis of the space-time fractional mBBM equation with Caputo fractional derivative. An initial value problem of the fractional mBBM equation was solved by using FCT-HPM. Numerical approximations were given without complicated computation, and the technology has potential applications in medical imaging [26, 27], the fractal diffusion [28], the fractal solitary waves [29, 30] and fractal-fractional problems [24, 25, 31, 32]. In view of the efficiency of FCT-HPM, we will consider this method for other fractional differential equations in our future work.

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