STOCHASTIC STABILITY OF THE FRACTIONAL AND TRI-STABLE VAN DER VOL OSCILLATOR WITH TIME-DELAY FEEDBACK DRIVEN BY GAUSSIAN WHITE NOISE

by

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The stochastic P-bifurcation behavior of tri-stability in a fractional-order van der Pol system with time-delay feedback under additive Gaussian white noise excitation is investigated. Firstly, according to the equivalent principle, the fractional derivative and the time-delay term can be equivalent to a linear combination of damping and restoring forces, so the original system can be simplified into an equivalent integer-order system. Secondly, the stationary probability density function of the system amplitude is obtained by the stochastic averaging, and based on the singularity theory, the critical parameters for stochastic P-bifurcation of the system are found. Finally, the properties of stationary probability density function curves of the system amplitude are qualitatively analyzed by choosing corresponding parameters in each sub-region divided by the transition set curves. The consistence between numerical results obtained by Monte-Carlo simulation and analytical solutions has verified the accuracy of the theoretical analysis. The method used in this paper has a direct guidance in the design of fractional-order controller to adjust the dynamic behavior of the system.

Key words: *stochastic P-bifurcation, fractional derivative, Gaussian white noises, transition set curves, Monte-Carlo simulation*

Introduction

The classical integer-order derivative can not express the memory characteristics of the viscoelastic substances, while the fractional derivative contains convolution, which can express a memory effect and shows a cumulative effect over time. Therefore, the fractional

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derivative is a more suitable mathematical tool to description of the memory characteristics [1-4] and has become a powerful mathematical tool for study of fractal diffusion [5], non-Newtonian flows [6], viscoelastic materials [7], and soft matter physics [8]. Compared with the integer-order calculus, the fractional derivative can describe various reaction processes much accurately [9], thus, it is necessary and significant to study the dynamical characteristics and the fractional order parametric influences on such systems.

Recently, many scholars studied the dynamic behavior of non-linear multi-stable systems under different noise excitations and achieved fruitful results [10-14]. In terms of the integer order systems, the van der Pol-Duffing oscillators under Levy noise [12], color noise [13], combined harmonic and random noise [10, 11, 14] had caught rocketing interest.

Wu and Hao [15] investigated the tri-stable stochastic P-bifurcation in a generalized Duffing-Van der Pol oscillator under the multiplicative colored noise, they obtained an analytical expression of the system's stationary probability density function (PDF) of amplitude and then analyzed the influences of noise intensity and system parameters on the system's stochastic P-bifurcation. Qian and Chen [16] studied the random vibration of the modified single-DoF vibro-impact oscillators system with a recovery factor under broadband noise excitation, and obtained the steady-state probability density function of the energy and amplitude envelope of the system by using Markov approximation, furthermore the effectiveness of the proposed method was verified by some examples further. Huang and Jin [17] discussed the response and the stationary PDF of a single-DoF strongly non-linear system under Gaussian white noise excitation. Sun and Yang [18] analyzed the stability of fractional order energy acquisition system under Gaussian white noise excitation by using the random average method and generalized harmonic function method, and analyzed the influences of noise intensity, the order and coefficients of fractional derivative on the stochastic response of the system. Li *et al.* [19] studied the bi-stable stochastic P-bifurcation behavior of a Van der Pol-Duffing system with the fractional derivative under additive and multiplicative colored noise excitations and found that changes in the linear damping coefficient, the fractional derivative's order and the noise intensity could lead to stochastic P-bifurcation in the system.

Due to complexity of the fractional derivative, the parametric vibration characteristics of the fractional system can only be analyzed qualitatively [20-22], while the critical conditions of the parametric influences can not be obtained. In practice, the critical conditions of the parametric influences play a vital role for the analysis and design of the fractional order systems. In this paper, we take a generalized tri-stable Van der Pol system with a fractional damping and time-delay feedback term excited by additive Gaussian white noise excitation as the example, the non-linear vibration of the fractional order systems requires complex analysis. The transition set curves and critical parameter conditions for the system's stochastic P-bifurcation are obtained by the singularity method. The types of the system's stationary PDF curves in each area of the parameter plane are analyzed. We also compare the numerical results from Monte-Carlo simulation with analytical solutions obtained by the stochastic averaging. The comparison shows that the numerical results are in good agreement with the analytical solutions, verifying our theoretical analysis.

Derivation of the equivalent system

The initial condition of the Riemann-Liouville derivative has no physical meaning, while the initial condition of the system described by the Caputo derivative has not only clear physical meaning but also forms the same initial condition with the integer-order differential equation [23]. Therefore, in this paper we adopt the Caputo fractional derivative:

$$ {}_a^C\mathrm{D}^p[x(t)] = \frac{1}{\Gamma(m-p)}\int_a^t \frac{x^{(m)}(u)}{(t-u)^{1+p-m}}\mathrm{d}u \tag{1}$$

where $m-1 < p \le m,\ m \in N$, $t \in [a,b]$, $x^{(m)}(t)$ is the m-order derivative of $x(t)$ and $\Gamma(m)$ is the Gamma function.

For a given physical system, the initial moment of oscillators is $t = 0$ and the Caputo derivative is usually expressed:

$$ {}_0^C\mathrm{D}^p[x(t)] = \frac{1}{\Gamma(m-p)}\int_0^t \frac{x^{(m)}(u)}{(t-u)^{1+p-m}}\mathrm{d}u \tag{2}$$

where $m-1 < p \le m,\ m \in N$.

In this paper, we study the generalized Van der Pol system with the fractional damping and time-delay feedback forced by Gaussian white noise:

$$ \ddot{x} - (-\varepsilon + \alpha_1 x^2 - \alpha_2 x^4 + \alpha_3 x^6 - \alpha_4 x^8)\dot{x} + w^2 x + {}_0^C\mathrm{D}^p x + k[x(t) - x(t-\tau)] = \xi(t) \tag{3}$$

where ε is the linear damping coefficient and $\alpha_1, \alpha_2, \alpha_3$, and α_4 are the non-linear damping coefficients of the system, ${}_0^C\mathrm{D}^p[x(t)]$ is the Caputo derivative with $p(0 \le p \le 1)$ order, and $\xi(t)$ is the Gaussian white noise excitation, which satisfies

$$ E[\xi(t)] = 0, \quad E[\xi(t)\xi(t-\tau)] = 2D\delta(\tau) \tag{4}$$

where D denotes the intensity of Gaussian white noises $\xi(t)$, and $\delta(\tau)$ is the Dirac function.

Non-linear oscillation [24] has been widely studied due to its wide applications in energy harvesting [25] and controller [26], especially the three degrees-of-freedom auto-parametric systems [27], and 6-DOF systems [28].

The fractional derivative and time-delay feedback term have the contributions of damping force and restoring force [29-31], He and Liu [32] also pointed that the fractal-fractional derivative is a combination of the damping force and the inertia force. Accordingly, in this paper, we introduce the following equivalent system:

$$ \ddot{x}(t) - [-\varepsilon + \alpha_1 x^2 - \alpha_2 x^4 + \alpha_3 x^6 - \alpha_4 x^8 + C(p,\tau)]\dot{x} + [K(p,\tau) + w^2]x = \xi(t) \tag{5}$$

where $C(p,\tau)$ and $K(p,\tau)$ are the coefficients of the equivalent damping and equivalent restoring forces of the fractional derivative ${}_0^C\mathrm{D}^p[x(t)]$ and time-delay feedback $k[x(t) - x(t-\tau)]$, respectively.

The error between system (3) and (5) is:

$$ e = C(p,\tau)\dot{x} + {}_0^C\mathrm{D}^p x - K(p,\tau)x \tag{6}$$

Based on the equivalent principle [33], we can get:

$$ \frac{\partial E(e^2)}{\partial[C(p,\tau)]} = 0, \quad \frac{\partial E(e^2)}{\partial[K(p,w)]} = 0 \tag{7}$$

Assuming that the system (3) has the steady state solution of the form:

$$ x(t) = a(t)\cos\varphi(t) \tag{8}$$

where $\varphi(t) = wt + \theta$, we can obtain:

$$\dot{x}(t) = -wa(t)\sin\varphi(t), \qquad \ddot{x}(t) = -w^2 a(t)\cos\varphi(t) \tag{9}$$

Substituting eqs. (6), (8), and (9) into eq. (7), and performing the integral averaging of φ, we get the ultimate forms of $C(p,\tau)$ and $K(p,\tau)$:

$$C(p,\tau) = -w^{p-1}\sin\left(\frac{p\pi}{2}\right) - \frac{k\sin(w\tau)}{w}, \qquad K(p,\tau) = w^p\cos\left(\frac{p\pi}{2}\right) + k[1-\cos(w\tau)] \tag{10}$$

Therefore, the equivalent Van der Pol oscillator associated with system (5) can be written:

$$\ddot{x}(t) - \gamma\dot{x} + w_0^2 x = x(t)\xi(t) \tag{11}$$

where

$$\begin{gathered} \gamma = -\varepsilon + \alpha_1 x^2 - \alpha_2 x^4 + \alpha_3 x^6 - \alpha_4 x^8 - w^{p-1}\sin\left(\frac{p\pi}{2}\right) - \frac{k\sin(w\tau)}{w} \\ w_0^2 = w^2 + w^p\cos\left(\frac{p\pi}{2}\right) + k[1-\cos(w\tau)] \end{gathered} \tag{12}$$

Stationary PDF of the system amplitude

In order to obtain the steady-state probability density function of the system amplitude, we assume that the solution of system (11) has the periodic form, and we introduce the following transformation [34]:

$$\begin{gathered} X = x(t) = a(t)\cos\Phi(t) \\ Y = \dot{x} = -a(t)w_0\sin\Phi(t) \\ \Phi(t) = w_0 t + \theta(t) \end{gathered} \tag{13}$$

where w_0 is the natural frequency of the above equivalent system (11), $a(t)$ and $\theta(t)$ – the amplitude and phase processes of system response, respectively, they are both random processes.

Substituting eq. (13) into eq. (11) and using the deterministic averaging method, we can obtain:

$$\begin{gathered} \frac{\mathrm{d}a}{\mathrm{d}t} = F_{11}(a,\theta) + \mathrm{G}_{11}(a,\theta)\xi(t) \\ \frac{\mathrm{d}\theta}{\mathrm{d}t} = F_{21}(a,\theta) + G_{21}(a,\theta)\xi(t) \end{gathered} \tag{14}$$

in which:

$$F_{11}(a,\theta) = a\sin^2\Phi(-\varepsilon + \alpha_1 a^2\cos^2\Phi - \alpha_2 a^4\cos^4\Phi + \\ +\alpha_3 a^6\cos^6\Phi - \alpha_4 a^8\cos^8\Phi - w^{p-1}\sin\left(\frac{p\pi}{2}\right) - \frac{k\sin(w\tau)}{w} \\ F_{21}(a,\theta) = \sin\Phi\cos\Phi(-\varepsilon + \alpha_1 a^2\cos^2\Phi - \alpha_2 a^4\cos^4\Phi + \\ +\alpha_3 a^6\cos^6\Phi - \alpha_4 a^8\cos^8\Phi - w^{p-1}\sin\left(\frac{p\pi}{2}\right) - \frac{k\sin(w\tau)}{w} \\ G_{11} = -\frac{\sin\Phi}{w_0} \\ G_{21} = -\frac{\cos\Phi}{aw_0} \tag{15}$$

Equation (14) can be regarded as the Stratonovich stochastic differential equation [35], and by adding the relevant Wong-Zakai correction term [36], we can transform it into the Ito stochastic differential equation:

$$\mathrm{d}a = [F_{11}(a,\theta) + F_{12}(a,\theta)]\mathrm{d}t + \sqrt{2D}\,\mathrm{G}_{11}(a,\theta)\mathrm{d}B(t) \\ \mathrm{d}\theta = [F_{21}(a,\theta) + F_{22}(a,\theta)]\mathrm{d}t + \sqrt{2D}G_{21}(a,\theta)\mathrm{d}B(t) \tag{16}$$

where $B_k(t)$ are independent normalized Wiener processes and:

$$F_{12}(a,\theta) = D\frac{\partial G_{11}}{\partial a}G_{11} + D\frac{\partial G_{11}}{\partial \theta}G_{21} \\ F_{22}(a,\theta) = D\frac{\partial G_{21}}{\partial a}G_{11} + D\frac{\partial G_{21}}{\partial \theta}G_{21} \tag{17}$$

According to the stochastic averaging method [37] and averaging eq. (16) over Φ, we can obtain the following averaged Ito equation:

$$\mathrm{d}a = m_1(a)\mathrm{d}t + \sigma_1(a)\mathrm{d}B(t) \\ \mathrm{d}\theta = m_2(a)\mathrm{d}t + \sigma_2(a)\mathrm{d}B(t) \tag{18}$$

where $B(t)$ is a unit Wiener processes and the explicit expression for the averaged drift and diffusion coefficients can be obtained:

$$m_1(a) = -\frac{1}{2}\left[w^{p-1}\sin\left(\frac{p\pi}{2}\right) + \frac{k\sin(w\tau)}{w} + \varepsilon\right]a + \frac{1}{8}\alpha_1 a^3 \\ -\frac{1}{16}\alpha_2 a^5 + \frac{5}{128}\alpha_3 a^7 - \frac{7}{256}\alpha_4 a^9 + \frac{D}{2aw_0^2} \\ \sigma_1{}^2(a) = \frac{D}{w_0^2} \\ m_2(a) = 0 \\ \sigma_2{}^2(a) = \frac{D}{a^2 w_0^2} \tag{19}$$

where $w_0^2 = w^2 + w^p\cos(p\pi/2) + k[1 - \cos(w\tau)]$.

Equation (19) shows that the averaged Ito equation for $a(t)$ is independent of $\theta(t)$, the random process $a(t)$ is a 1-D diffusion process. Then the corresponding Fokker-Planck--Kolmogorov (FPK) equation of $a(t)$ can be written:

$$\frac{\partial p(a,t)}{\partial t} = -\frac{\partial}{\partial a}[m_1(a)p(a)] + \frac{1}{2}\frac{\partial^2}{\partial a^2}\{[\sigma_1^2(a)]p(a)\} \tag{20}$$

The boundary conditions are:

$$\begin{gathered} p = c, \quad c \in (-\infty, +\infty) \quad \text{when} \quad a = 0 \\ p \to 0, \quad \frac{\partial p}{\partial a} \to 0 \quad \text{as} \quad a \to \infty \end{gathered} \tag{21}$$

Based on the boundary conditions (21), the stationary PDF of system amplitude can be obtained:

$$p(a) = \frac{C}{\sigma_1^2(a)} \exp\left[\int_0^a \frac{2m_1(u)}{\sigma_1^2(u)} \mathrm{d}u\right] \tag{22}$$

where C is a normalized constant.

Substituting eq. (19) into eq. (22), the concrete expression for the stationary PDF of amplitude a can be obtained:

$$p(a) = \frac{Caw_0^2}{D} \exp\left[-\frac{a^2 w_0^2 \Delta}{7680D}\right]$$

in which:

$$\Delta = 3840\varepsilon + 3840\left[w^{p-1}\sin\left(\frac{p\pi}{2}\right) + \frac{k\sin(w\tau)}{w}\right] - 480\alpha_1 a^2 + 160\alpha_2 a^4 - 75\alpha_3 a^6 + 42\alpha_4 a^8 \tag{23}$$

Stochastic P-bifurcation of the system amplitude

Stochastic P-bifurcation refers to that the changes in the number of the PDF curve peaks. To obtain the critical parametric conditions for stochastic P-bifurcation of the system (3), we analyze the influence of parameters on the system stochastic P-bifurcation by using the singularity theory.

For the sake of convenience, $p(a)$ can be expressed:

$$p(a) = CR(a, D, \varepsilon, w, p, \tau, \alpha_1, \alpha_2, \alpha_3, \alpha_4)\exp[Q(a, D, \varepsilon, w, p, \tau, \alpha_1, \alpha_2, \alpha_3, \alpha_4)] \tag{24}$$

where

$$\begin{gathered} R(a, D, \varepsilon, w, p, \alpha_1, \alpha_2, \alpha_3, \alpha_4) = \frac{aw_0^2}{D} \\ Q(a, D, \varepsilon, w, p, \alpha_1, \alpha_2, \alpha_3, \alpha_4) = -\frac{a^2 w_0^2}{7680D}\left\{3840\varepsilon + 3840\left[w^{p-1}\sin\left(\frac{p\pi}{2}\right) + \frac{k\sin(w\tau)}{w}\right] - \right. \\ \left. -480\alpha_1 a^2 + 160\alpha_2 a^4 - 75\alpha_3 a^6 + 42\alpha_4 a^8\right\} \end{gathered} \tag{25}$$

Based on the singularity theory [38], the stationary PDF of system amplitude needs to satisfy the two conditions:

$$\frac{\partial p(a)}{\partial a} = 0 , \quad \frac{\partial^2 p(a)}{\partial a^2} = 0 \tag{26}$$

Substituting eq. (24) into eq. (26), we can obtain the following condition [15, 19]:

$$H = \left\{ R' + RQ' = 0, \quad R'' + 2R'Q' + RQ'' + RQ'^2 = 0 \right\} \tag{27}$$

where H is the condition for the changes in the number of the PDF curve peaks.

Taken the parameters as $\varepsilon = -0.5$, $\alpha_1 = 1.51$, $\alpha_2 = 2.85$, $\alpha_3 = 1.693$, $\alpha_4 = 0.312$, $w = 1$, $k = 1$, $\tau = 0.1$, according to eq. (27), the transition set of stochastic P-bifurcation for the system with the unfolding parameters p and D can be obtained, as shown in fig. 1.

As can be seen from fig. 1, the intercepts of transition set curves at $D = 0$ are the bifurcation values $p_1 = 0.262$, $p_2 = 0.344$, $p_3 = 0.348$, and $p_4 = 0.361$, respectively. Under the additive noise excitation, the transition set of the system (3) is divided by two approximately triangular, and the unfolding parameter plane is divided into five sub-regions by the transition set curve. According to the singularity theory, the topological structures of the stationary PDF curves of different points (p, D) in the same region are qualitatively the same. Taking a point (p, D) in each region, all varieties of the stationary PDF curves which are qualitatively different could be obtained and for convenience, each region in fig. 1 is marked with a number.

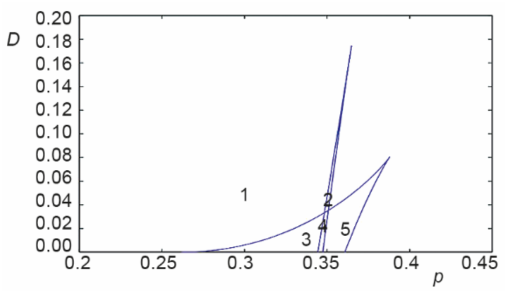

Figure 1. Transition set under multiplicative noise excitation (taking *p* and *D* as the unfolding parameters)

Taking a given point (p, D) in each of the five regions of fig. 1, the characteristics of stationary PDF curves are analyzed and the analytical results obtained are compared with the Monte-Carlo simulation results of the original system (3), the corresponding results are shown in fig. 2, respectively.

As can be seen in fig. 2, the parameter region (p, D) where the stationary PDF curve appears multimodal is surrounded by two approximately triangular regions, and the Region 4 forms a tri-modal region of the stationary PDF curve for the system.

When the parameter (p, D) is taken in the Region 1, the PDF curve $p(a)$ has a distinct peak far away from the origin, as shown in fig. 2(a). In Region 2, the PDF $p(a)$ has two distinct peaks far away from the origin, there are both the small and large limit cycles in the system simultaneously, as shown in fig. 2(b). In Region 3, the PDF curve $p(a)$ still has a distinct peak far away from the origin, but the probability is obviously not zero near the origin, there are both the equilibrium and large limit cycle in the system simultaneously, as shown in fig. 2(c). In Region 4, the PDF $p(a)$ has three peaks, it shows that the equilibrium coexists with the small and large limit cycles in the system which is tri-stable, as shown in fig. 2(d). In Region 5, $p(a)$ is qualitatively the same as in Region 3, the amplitude corresponding to the peak far away from the origin of the PDF curve is smaller than the corresponding amplitude in fig. 2(c), there are both the equilibrium and small limit cycle in the system simultaneously, as shown in fig. 2(e).

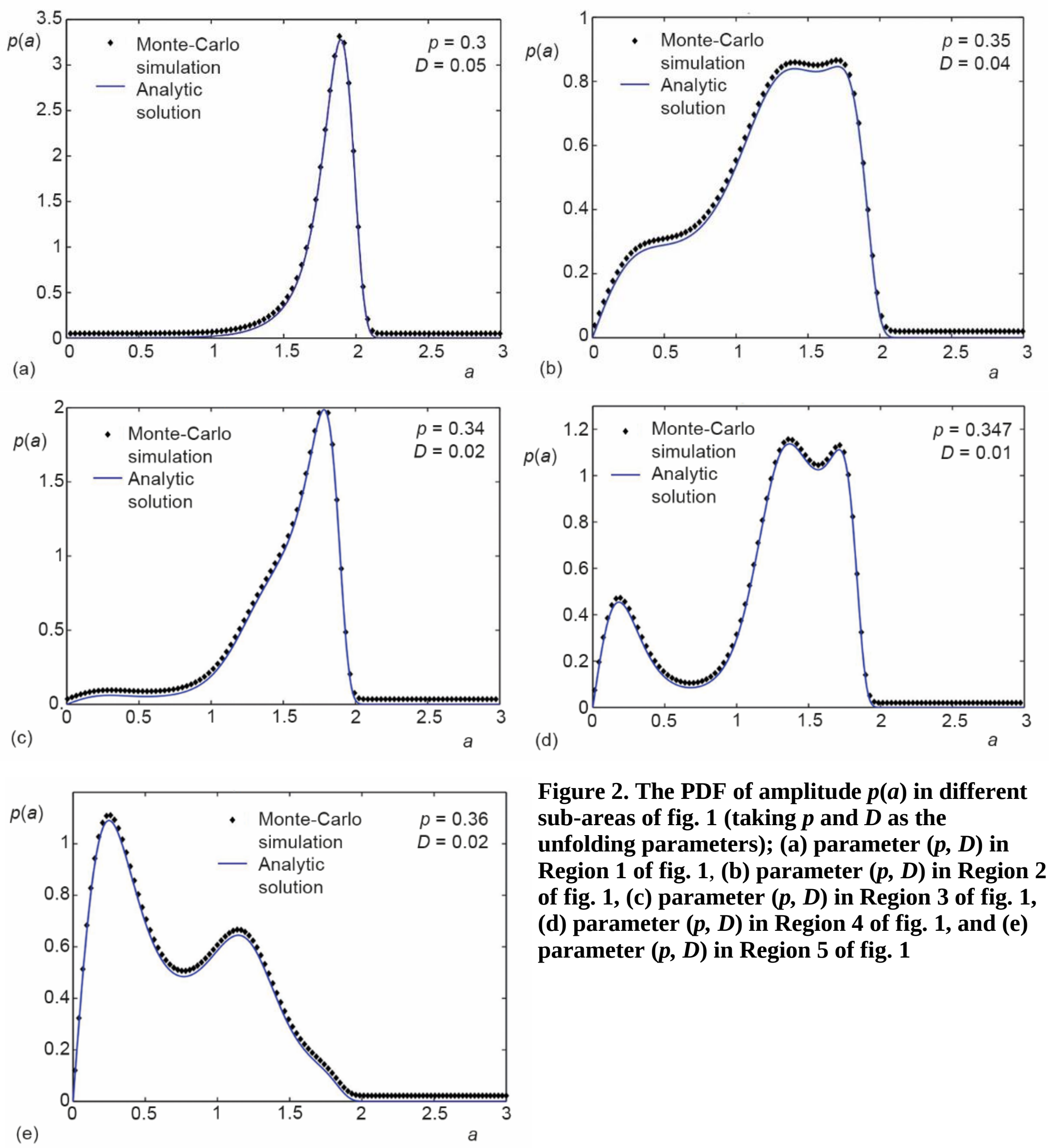

Figure 2. The PDF of amplitude *p*(*a*) in different sub-areas of fig. 1 (taking *p* and *D* as the unfolding parameters); (a) parameter (*p*, *D*) in Region 1 of fig. 1, (b) parameter (*p*, *D*) in Region 2 of fig. 1, (c) parameter (*p*, *D*) in Region 3 of fig. 1, (d) parameter (*p*, *D*) in Region 4 of fig. 1, and (e) parameter (*p*, *D*) in Region 5 of fig. 1

The results analyzed above show that the steady-state PDF curve of system amplitude can appear as different shapes while the fractional order p and noise intensity D taken as different values, which means that the steady-state PDF $p(a)$ can be regulated by adding the fractional derivative term in the system and the motion form of the system can be controlled by the fractional order p and noise intensity D, respectively.

Conclusion

In this paper, the stochastic P-bifurcation of a fractional and tri-stable Van der Pol system with time-delay feedback subjected to additive noise excitation is investigated. Based on the equivalent principle, the original system can be transformed into an equivalent integer-order system, and we obtained the system amplitude's stationary PDF using the stochastic averaging method. Further, the critical parametric conditions for the system's stochastic P-bifurcation are obtained using the singularity theory, which can provide the theoretical guidance for system design. The consistency between the numerical results obtained by Monte-Carlo simulation and the analytical results can also verify the theoretical analysis. It shows that the fractional order p and noise intensity D can both arise the stochastic P-bifurcation of the system, and the number of peaks of the system's stationary PDF curves $p(a)$ can vary from one to three by selecting the appropriate unfolding parameters.

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References

[1] Xu, M., Tan, W., Theoretical Analysis of the Velocity Field, Stress Field and Vortex Sheet of Generalized Second Order Fluid with Fractional Anomalous Diffusion, *Sci. China Ser. A-Math., 44* (2001), Nov., pp. 1387-1399
[2] Sabatier, J., *et al.*, *Advances in Fractional Calculus*, Springer-Verlag, Berlin, Germany, 2007
[3] Podlubny, I., Fractional-Order Systems and P I^{λ} D^{μ} Controllers, *IEEE Trans Autom Contol, 44* (1999), 1, pp. 208-214
[4] Monje, C. A., *et al.*, *Fractional-order Systems and Controls: Fundamentals and Applications*, Springer--Verlag, Berlin, Germany, 2010
[5] He, J. H., Qian, M. Y., A Fractal Approach to the Diffusion Process of Red Ink in a Saline Water, *Thermal Science, 26* (2022), 3B, pp. 2447-2451
[6] Zuo, Y. T., Effect of SiC Particles on Viscosity of 3-D Print Paste: A Fractal Rheological Model and Experimental Verification, *Thermal Science, 25* (2021), 3B, pp. 2405-2409
[7] Liang, Y. H., Wang, K. J., A New Fractal Viscoelastic Element: Promise and Applications to Maxwell-Rheological Model, *Thermal Science, 25* (2021), 2B, pp. 1221-1227
[8] Chen, W., An Intuitive Study of Fractional Derivative Modeling and Fractional Quantum in Soft Matter, *Journal of Vibration and Control, 14* (2008), 9-10, pp. 1651-1657
[9] Dai, D. D., *et al.,* The Piecewise Reproducing Kernel Method for the Time Variable Fractional Order Advection-Reaction-Diffusion Equations, *Thermal Science, 25* (2021), 2B, pp. 1261-1268
[10] Rong, H. W., *et al.,* On Double-Peak Probability Density Functions of a Duffing Oscillator Under Narrow-Band Random Excitations (in Chinese), *Acta. Phys. Sin., 54* (2005), Nov., pp. 2557-2561
[11] Rong, H. W., *et al.,* On Double Peak Probability Density Functions of Duffing Oscillator to Combined deTerministic and Random Excitations, *Appl. Math. Mech-Engl. Ed., 27* (2006), Nov., pp. 1569-1576
[12] Gu, R. C., Stochastic Bifurcations in Duffing-van der Pol Oscillator with Levy Stable Noise, *Acta Phys. Sin., 60* (2011), 3-4, pp. 1466-1467
[13] Xu, Y., *et al.,* Stochastic Bifurcations in a Bistable Duffing-Van der Pol Oscillator with Colored Noise, *Phys. Rev. E, 83* (2011), May, 056215
[14] Zakharova, A., *et al.,* Stochastic Bifurcations and Coherence Like Resonance in a Self-Sustained Bistable Noisy Oscillator, *Phys. Rev. E, 81* (2010), 1, 011106
[15] Wu, Z. Q., Hao, Y., Stochastic P-Bifurcations in Tri-Stable Van der Pol-Duffing Oscillator with Multiplicative Colored Noise (in Chinese), *Acta Phys. Sin., 64* (2015), 6, 060501
[16] Qian, J. M., Chen, L. C., Random Vibration of SDOF Vibro-Impact Oscillators with Restitution Factor Related to Velocity Under Wide-Band Noise Excitations, *Mech. Syst. Signal. Pr., 147* (2021), Jan., 107082

[17] Huang, Z. L., Jin, X. L., Response and Stability of a SDOF Strongly Non-linear Stochastic System with Light Damping Modeled by a Fractional Derivative, *J. Sound Vib.*, *319* (2009), 3-5, pp. 1121-1135
[18] Sun, Y. H., Yang, Y. G., Stochastic Averaging for the Piezoelectric Energy Harvesting System with Fractional Derivative Element, *IEEE Access, 8* (2020), Mar., pp. 59883-59890
[19] Li, W., *et al.,* Stochastic Bifurcations of Generalized Duffing-Van der Pol System with Fractional Derivative Under Colored Noise, *Chinese Phys. B 26* (2017), 9, pp. 62-69
[20] Li, Y. J., *et al.,* Stochastic Transition Behaviors in a Tri-Stable Van der Pol Oscillator with Fractional Delayed Element Subject to Gaussian White Noise, *Thermal Science, 26* (2022), 3B, pp. 2713- 2725
[21] Li, Y. J., *et al.,* Transition Behaviors of System Energy in a Bi-Stable Van der Pol Oscillator with Fractional Derivative Element Driven by Multiplicative Gaussian White Noise, *Thermal Science, 26* (2022), 3B, pp. 2727-2736
[22] Shen, Y., *et al*., A Periodic Solution of the Fractional Sine-Gordon Equation Arising in Architectural Engineering, *Journal of Low Frequency Noise, Vibration & Active Control, 40* (2021), 2, pp. 683-691
[23] He, J. H., Fractal Calculus and Its Geometrical Explanation, *Results in Physics, 10* (2018), Sept., pp. 272-276
[24] He, C. H., El-Dib, Y. O., A Heuristic Review on the Homotopy Perturbation Method for Non-Conservative Oscillators, *Journal of Low frequency Noise Vibration and Active Control, 41* (2022), 2, pp. 572-603
[25] He, C. H., *et al*., Controlling the Kinematics of a Spring-Pendulum System Using an Energy Harvesting Device, *Journal of Low frequency Noise Vibration and Active Control*, *41* (2022), 3, pp. 1234-1257
[26] He, C. H., *et al*., Hybrid Rayleigh-van der Pol-Duffing Oscillator: Stability Analysis and Controller, *Journal of Low frequency Noise Vibration and Active Control, 41* (2022), 1, pp. 244-268
[27] He, J. H., *et al*., Stability of Three Degrees-Of-Freedom Auto-Parametric System, *Alexandria Engineering Journal, 61* (2022), 11, pp. 8393-8415
[28] He, J. H., *et al*., Modelling of the Rotational Motion of 6-DOF Rigid Body According to the Bobylev-Steklov Conditions, *Results in Physics, 35* (2022), Apr., 105391
[29] Chen, L. C., *et al*., Stationary Response of Duffing Oscillator with Hardening Stiffness and Fractional Derivative, *Int. J. Non-lin. Mech., 48* (2013), Jan., pp. 44-50
[30] Chen, L. C., *et al*., First-Passage Failure of Single-Degree-Of-Freedom Non-linear Oscillators with Fractional Derivative, *J. Vib. Control., 19* (2013), 14, pp. 2154-2163
[31] Shen, Y. J., *et al*. Primary Resonance of Duffing Oscillator with Two Kinds of Fractional-Order Derivatives, *Int. J. Non-lin. Mech., 47* (2012), 9, pp. 975-983
[32] He, C. H., Liu, C., A Modified Frequency-Amplitude Formulation for Fractal Vibration Systems, *Fractals, 30* (2022), 3, 2250046
[33] Yang, Y. G., *et al*., Stochastic Response of van der Pol Oscillator with Two Kinds of Fractional Derivatives Under Gaussian White Noise Excitation, *Chinese Phys. B, 25* (2016), 2, pp. 13-21
[34] Chen, L. C., Zhu, W. Q., Stochastic Response of Fractional-Order Van der Pol Oscillator, *Theor. Appl. Mech. Lett., 4* (2014), 1, pp. 68-72
[35] Xu, C., Roberts, A. J., On the Low-Dimensional Modelling of Stratonovich Stochastic Differential Equations, *Physica, A, 225* (1996), 1, pp. 62-80
[36] Wu, Z. Q., Hao, Y., Three-pEak P-Bifurcations in Stochastically Excited Van der Pol-Duffing Oscillator (in Chinese), *Sci. Sin. Phys. Mech. Astron., 43* (2013), 4, pp. 524-529
[37] Zhu, W. Q., Random Vibration (in Chinese), Science Press, Beijing, China 1992
[38] Ling, F. H., Catastrophe Theory and its Applications (in Chinese), Shang Hai Jiao Tong University Press, Shangai, China, 1987

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