VARIATIONAL PRINCIPLE FOR AN INCOMPRESSIBLE FLOW

by

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This paper gives a general approach to the inverse problem of calculus of variations. The 2-D Euler equations of incompressible flow are used as an example to show how to derive a variational formulation. The paper begins with ideal Laplace equation for its potential flow without vorticity, which admits the Kelvin 1849 variational principle. The next step is to assume a small vorticity to obtain an approximate variational formulation, which is then amended by adding an additional unknown term for further determined, this process leads to the well-known semi-inverse method. Lagrange crisis is also introduced, and some methods to solve the crisis are discussed

Key words: *Kelvin's variational principle, semi-inverse method, Euler-Lagrange equation*

Introduction

The variational principle [1] plays an important role in practical applications, it considers a problem in an energy view, the most famous variational principle is the principle of least action or Hamilton principle. The variational principle is also the mathematical tool to economic analysis, we have Samuelson's variational principle and Evans variational principle in economics [2, 3]. Numerical simulation is also used for verification of the theoretical prediction for complex problems to guarantee the used analytical method is valid [4, 5], and the variational-based numerical methods have many advantages in physical compatibility and effective computation [6]. During the numerical simulation, the most iteration algorithm is the Newton's iteration method, it is extremely sensitive to the initial guess, some effective modifications were appeared, including Nadeem-Ali-He iteration method [7] and Chun-Hui He's iteration algorithm [8, 9].

Though the variational-based numerical methods have obvious merits, a critical hurdle in CFD is the lack of a variational formulation for a practical fluid, and Galerkin technology has to be used [10, 11]. Other advanced numerical methods were appeared recently, for examples, the block-pulse function method [12], He-Laplace method [13], the homotopy perturbation method [14], and Chebyshev pseudospectral technique [15]. There is not a universal method to search for a variational formulation from governing equations, though the semi-inverse method [16, 17] has been widely used in practical applications, it is to establish an en-

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ergy-like trial functional with an unknown function, many variational formulations were established for various complex problems, for examples, the reaction-diffusion problem [18], the fractal solitary waves [19-21], Sine-Gordon equation [22], Chen-Lee-Liu equation [23], and 1-D fluid [24].

This paper will show that the semi-inverse method is mathematically correct and physically relative. Additionally Lagrange crisis is also discussed and a new way to identify the multiplier is recommended. This paper gives a mathematical framework for establishment of a needed variational principle for 2-D incompressible Euler equations.

The 2-D Euler equations of incompressible flow

Advances in CFD has led to skyrocketing interest in the variational principle for various fluids [25, 26], because the variational-based numerical simulation has many advantages over the Galerkin technology [10, 11], the former requires less differentiability, conservation for discretization schemes, ability to deal with free boundaries and discontinuous shocks. One of the most prominent bottlenecks is the difficulty in establishing a needed variational formulation from the governing equations, this is why weak variational principles are widely used in computational physics [11].

The 2-D Euler equations of incompressible flow are [27-29]:

$$\omega_t + (u\omega)_x + (v\omega)_y = 0 \tag{1}$$

$$\varphi_{xx} + \varphi_{yy} + \omega = 0 \tag{2}$$

$$\varphi_x = -v \tag{3}$$

$$\varphi_y = u \tag{4}$$

$$v_x - u_y = \omega \tag{5}$$

where φ is the stream function, ω – the vorticity, and u and v – the velocities in x- and y-directions, respectively.

In this education process, the students will finally learn the semi-inverse method, which was first proposed by Chinese mathematician, Ji-Huan He, in 1997 [16] to establish a variational formulation for the mentioned system.

Kelvin 1849 variational principle and its modification

We begin with a simplified case, that is when $\omega = 0,$ we have the following Laplace equation:

$$\varphi_{xx} + \varphi_{yy} = 0 \tag{6}$$

Its variational formulation is known:

$$J(\varphi) = \iint \left\{ \frac{1}{2}\left[(\varphi_x)^2 + (\varphi_y)^2 \right] \right\} \mathrm{d}x\mathrm{d}y \tag{7}$$

If we consider eqs. (3) and (4) as constraints, the principle can be written:

$$J(\varphi) = \iint \left[\frac{1}{2}(u^2 + v^2) \right] \mathrm{d}x\mathrm{d}y \tag{8}$$

This is the well-known Kelvin 1849 variational principle of minimum potential energy [30].

The next step is to consider the case when $\omega \ll 1$, so the previous variational principle (7) or (8) should be approximately valid. In order to include ω in the variational formulation, we give a modification of the Kelvin's principle:

$$J(\varphi) = \iint \left\{ \frac{1}{2}\left[(\varphi_x)^2 + (\varphi_y)^2 \right] + \omega\varphi \right\} \mathrm{d}x\mathrm{d}y \tag{9}$$

This is, however, an approximate variational formulation, because $\delta\omega \neq 0$. However, if ω is assumed to be a known function of *x* and *y*, so $\delta\omega(x, y) = 0$. This property is similar to that for differential of a constant, d*C*/d*x* = 0, where *C* is a constant.

In order to make it mathematically consistent, a new concept has to be introduced, that is the concept of a constrained function, and it is written like this $\tilde{\omega}$. So eq. (9) becomes:

$$J(\varphi) = \iint \left\{ \frac{1}{2}\left[(\varphi_x)^2 + (\varphi_y)^2 \right] + \tilde{\omega}\varphi \right\} \mathrm{d}x\mathrm{d}y \tag{10}$$

Before doing anything, the property of $\tilde{\omega}$ is given, that is $\delta\tilde{\omega} = 0$. That means the dependent variable ω is forcibly constrained to be a function of *x* and *y*. To understand this, we use a similar treatment in an algebraic equation:

$$ax^2 + bx + c = 0 \tag{11}$$

We have the root formulation:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \tag{12}$$

Now we propose a cubic equation in the form:

$$0.0001x^3 + ax^2 + bx + c = 0 \tag{13}$$

As the cubic term is small, and we assume that the cubic term is known. To do this, we introduce a constrained variable like that for $\tilde{\omega}$:

$$ax^2 + bx + (c + 0.0001\tilde{x}^3) = 0 \tag{14}$$

So we have:

$$x = \frac{-b \pm \sqrt{b^2 - 4a(c + 0.0001\tilde{x}^3)}}{2a} \tag{15}$$

Equation (15) leads to the following iteration algorithm:

$$x_{n+1} = \frac{-b \pm \sqrt{b^2 - 4a(c + 0.0001x_n^3)}}{2a} \tag{16}$$

Examples can be given to show its effectively by fixing the values of *a*, *b*, and *c*. The concept of the constrained function is also used in the variational iteration method for easy identification of the Lagrange multiplier involved in the variational iteration algorithm [31-33].

The approximate variational formulation of eq. (10) leaves much space to be further improved, and a genuine variational formulation should be derived. To this end, we change one interdepend function in eq. (10) to more than one as that for a generalized variational principle.

We re-write eq. (2) in the form:

$$\varphi_{xx} + \varphi_{yy} + v_x - u_y = 0 \tag{17}$$

and re-write the approximate variational formulation of eq. (10) in the form:

$$J(\varphi) = \iint \left\{ \frac{1}{2}\left[(\varphi_x)^2 + (\varphi_y)^2 \right] + \tilde{v}\varphi_x - \tilde{u}\varphi_y \right\} \mathrm{d}x\mathrm{d}y \tag{18}$$

Here we write down the Lagrange function:

$$L = \frac{1}{2}\left[(\varphi_x)^2 + (\varphi_y)^2 \right] + v(x,y)\varphi_x - u(x,y)\varphi_y \tag{19}$$

Its Euler-Lagrange equation is:

$$\frac{\partial L}{\partial \varphi} - \frac{\partial}{\partial x}\left(\frac{\partial L}{\partial \varphi_x} \right) - \frac{\partial}{\partial y}\left(\frac{\partial L}{\partial \varphi_y} \right) = 0 \tag{20}$$

It is easy to calculate the following components:

$$\frac{\partial L}{\partial \varphi} = 0$$

$$\frac{\partial L}{\partial \varphi_x} = \varphi_x + v(x,y)$$

$$\frac{\partial L}{\partial \varphi_y} = \varphi_y - u(x,y) \tag{21}$$

So eq. (20) leads to the following one:

$$0 - \frac{\partial}{\partial x}[\varphi_x + v(x,y)] - \frac{\partial}{\partial y}[\varphi_y - u(x,y)] = 0 \tag{22}$$

This can be simplified to eq. (17). Alternatively, the following approximate one is obtained:

$$J(\varphi) = \iint \left\{ \frac{1}{2}\left[(\varphi_x)^2 + (\varphi_y)^2 \right] - \tilde{v}_x\varphi + \tilde{u}_y\varphi \right\} \mathrm{d}x\mathrm{d}y \tag{23}$$

Its Lagrange function is:

$$L = \frac{1}{2}\left[(\varphi_x)^2 + (\varphi_y)^2 \right] - \tilde{v}_x\varphi + \tilde{u}_y\varphi \tag{24}$$

It is easy to obtain the following results:

$$\frac{\partial L}{\partial \varphi} = -v_x + u_y, \quad \frac{\partial L}{\partial \varphi_x} = \varphi_x, \quad \frac{\partial L}{\partial \varphi_y} = \varphi_y \tag{25}$$

So eq. (20) gives the following Euler-Lagrange equation:

$$-v_x + u_y - \frac{\partial}{\partial x}(\varphi_x) - \frac{\partial}{\partial y}(\varphi_y) = 0 \tag{26}$$

This is eq. (17).

So far we have limited to a single independent function of $\varphi,$ and multiple independent functions in a variational formulation should be considered. The next step is to use the Lagrange multiple method to establish a possible generalized variational formulation. Introducing two Lagrange multiples λ_1 and $\lambda_2,$ we have:

$$J(\varphi,u,v,\lambda_1,\lambda_2) = \iint \left\{ \frac{1}{2}\left[(\varphi_x)^2 + (\varphi_y)^2\right] + v(x,y)\varphi_x - u(x,y)\varphi_y + \right.$$
$$\left. + \lambda_1(\varphi_y - u) + \lambda_2(\varphi_x + v) \right\} \mathrm{d}x\mathrm{d}y \tag{27}$$

Here $\varphi,$ u and v are independent functions. The Euler-Lagrange equations are:

$$\begin{gathered} -\varphi_{xx} - \varphi_{yy} - v_x + u_x - \lambda_{1y} - \lambda_{2x} = 0 \\ -\varphi_y - \lambda_1 = 0 \\ \varphi_x + \lambda_2 = 0 \end{gathered} \tag{28}$$

After identification of the multipliers, we can not obtain the needed equations, so the try fails. The fail is due to the approximate one of eq. (18), while the Lagrange multipliers are valid only for a genuine variational principle, that might imply the terms in eq. (27) might be in an approximate opinion. If so, we introduce a function *F* defined:

$$F = \lambda_1(\varphi_y - u) + \lambda_2(\varphi_x + v) \tag{29}$$

So eq. (27) becomes:

$$J(\varphi,u,v) = \iint \left\{ \frac{1}{2}\left[(\varphi_x)^2 + (\varphi_y)^2\right] + v\varphi_x - u\varphi_y + F \right\} \mathrm{d}x\mathrm{d}y \tag{30}$$

Equation (30) is the idea from the semi-inverse method [16, 17].

Ji-Huan He's semi-inverse method

The semi-inverse method was proposed by Ji-Huan He in 1997 [16], and it is widely used to search for variational formulations for various practical problems.

Lagrange function of eq. (30) is:

$$L(\varphi,u,v) = \frac{1}{2}\left[(\varphi_x)^2 + (\varphi_y)^2\right] + v\varphi_x - u\varphi_y + F(u,v) \tag{31}$$

Hence, we have:

$$-\frac{\partial}{\partial x}(\varphi_x)-\frac{\partial}{\partial y}(\varphi_y)-\frac{\partial}{\partial x}(v)+\frac{\partial}{\partial y}(u)=0 \tag{32}$$

This is exactly equivalent to eq. (8).

Now the Euler-Lagrange equations with respect to u and v are given, respectively:

$$-\varphi_y+\frac{\delta F}{\delta u}=0 \tag{33}$$

$$\varphi_x+\frac{\delta F}{\delta v}=0 \tag{34}$$

Where $\delta F/\delta u$ is the variational derivative:

$$\frac{\delta F}{\delta u}=\frac{\partial F}{\partial u}-\frac{\partial}{\partial x}\left(\frac{\partial F}{\partial u_x}\right)-\frac{\partial}{\partial y}\left(\frac{\partial F}{\partial u_y}\right) \tag{35}$$

In view of eqs. (3) and (4), we have:

$$\frac{\delta F}{\delta u}=\varphi_y=u \tag{36}$$

$$\frac{\delta F}{\delta v}=-\varphi_x=v \tag{37}$$

From eqs. (36) and (37), we have:

$$F=\frac{1}{2}(u^2+v^2) \tag{38}$$

Finally we obtain the following variational formulation:

$$J(\varphi,u,v)=\iint\left\{\frac{1}{2}\left[(\varphi_x)^2+(\varphi_y)^2\right]+v\varphi_x-u\varphi_y+\frac{1}{2}(u^2+v^2)\right\}\mathrm{d}x\mathrm{d}y \tag{39}$$

Lagrange multiplier

The variational principle of eq. (39) is subject to the constraint of eq. (5). The general approach to elimination of the constraint is the Lagrange multiplier method, that is:

$$J(\varphi,u,v,\omega,\lambda)=\iint\left\{\frac{1}{2}\left[(\varphi_x)^2+(\varphi_y)^2\right]+v\varphi_x-u\varphi_y+\frac{1}{2}(u^2+v^2)+\lambda(v_x-u_y-\omega)\right\}\mathrm{d}x\mathrm{d}y \tag{40}$$

where λ is the Lagrange multiplier. Identification of the multiplier reads:

$$\lambda=0 \tag{41}$$

This is called Lagrange crisis [34], and there are many methods to overcome the crisis. Here we recommend two methods. The identification of the Lagrange multiplier plays also an important role in the variational-based analytical methods [35, 36], numerical methods [37-40], and establishment of a generalized variational formulation [41, 42], and Hamiltonian-based frequency-amplitude formulation [43, 44].

We re-construct eq. (39) in the form:

$$J(\varphi,u,v)=\iint\left\{\frac{1}{2}\left[(\varphi_x)^2+(\varphi_y)^2\right]-(v_x-u_y)\varphi+\frac{1}{2}(u^2+v^2)\right\}\mathrm{d}x\mathrm{d}y \tag{42}$$

Now introducing the Lagrange multiplier, we have:

$$J(\varphi,u,v,\omega,\lambda)=\iint\left\{\frac{1}{2}\left[(\varphi_x)^2+(\varphi_y)^2\right]-\omega\varphi+\frac{1}{2}(u^2+v^2)+\lambda(v_x-u_y-\omega)\right\}\mathrm{d}x\mathrm{d}y \tag{43}$$

The Lagrange multiplier can be now identified:

$$\lambda=-\varphi \tag{44}$$

Hence, we obtain the following generalized variational principle:

$$J(\varphi,u,v,\omega)=\iint\left\{\frac{1}{2}\left[(\varphi_x)^2+(\varphi_y)^2\right]-\omega\varphi+\frac{1}{2}(u^2+v^2)-\varphi(v_x-u_y-\omega)\right\}\mathrm{d}x\mathrm{d}y=$$

$$=\iint\left\{\frac{1}{2}\left[(\varphi_x)^2+(\varphi_y)^2\right]+\frac{1}{2}(u^2+v^2)-\varphi(v_x-u_y)\right\}\mathrm{d}x\mathrm{d}y \tag{45}$$

The multiplier of eq. (41) can be also identified in the form:

$$\lambda=\mu(v_x-u_y-\omega) \tag{46}$$

where μ is a non-zero constant. We obtain the following generalized variational principle:

$$J(\varphi,u,v,\omega)=\iint\left\{\frac{1}{2}\left[(\varphi_x)^2+(\varphi_y)^2\right]+v\varphi_x-u\varphi_y+\frac{1}{2}(u^2+v^2)+\mu(v_x-u_y-\omega)^2\right\}\mathrm{d}x\mathrm{d}y \tag{47}$$

Proof. The Euler-Lagrange equations with respect to $\varphi,u,v,\omega,$ are, respectively:

$$-\varphi_{xx}-\varphi_{yy}-v_x+u_y=0 \tag{48}$$

$$-\varphi_y+u+2\mu(v_x-u_y-\omega)_y=0 \tag{49}$$

$$\varphi_x+v-2\mu(v_x-u_y-\omega)_x=0 \tag{50}$$

$$-2\mu(v_x-u_y-\omega)=0 \tag{51}$$

It is easy to find that the previous equations turn out to the governing equations, eqs. (2)-(5).

Conclusion

This paper elucidates a step-to-step solving process to establish a needed variational principle from the governing equations, and it reveals that semi-inverse method is mathematically reliable and physically relative.

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