FRACTAL MODIFICATION OF SCHRÖDINGER EQUATION AND ITS FRACTAL VARIATIONAL PRINCIPLE

by

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With the help of a new fractal derivative, a fractal model for variable coefficients and highly non-linear Schrödinger equations on a non-smooth boundary are acquired. The variational principles of the fractal variable coefficients and highly non-linear Schrödinger equations are built successfully by coupling fractal semi-inverse and He's two-scale transformation methods, which are helpful to reveal the symmetry, to discover the conserved quantity, and the obtained variational principles have widespread applications in numerical simulation.

Key words: *variational principle, fractal semi-inverse method, fractal derivative, Schrödinger equations, He's two-scale transform method*

Introduction

In [1], the non-linear Schrödinger equations with high non-linearity and variable coefficients were studied, which are:

$$i\frac{\partial u}{\partial t}+\lambda_1 t^{\varepsilon_1}\frac{\partial^2 u}{\partial x^2}+\xi_1 t^{\varepsilon_2}\left|u\right|^2 u+\eta_1\left|u\right|^4 u=Kv \tag{1}$$

$$i\frac{\partial v}{\partial t}+\lambda_2 t^{\varepsilon_1}\frac{\partial^2 v}{\partial x^2}+\xi_2 t^{\varepsilon_2}\left|v\right|^2 v+\eta_2\left|v\right|^4 v=Ku \tag{2}$$

where $u=u(x,t)$, $v=v(x,t)$ are complex-valued functions of x and t, $\lambda_i,\ \xi_i,\eta_i$ (i = 1, 2) are the constant parameters, and K is the coupling constant.

There were many reports about the Schrödinger equations. Ain, *et al.* [2] studied time-fractional Schrödinger equation. He and El-Dib [3, 4] considered some modifications of the Schrödinger equation, and its periodic solutions were discussed. Yao and Chang [5], Ozis and Yildirim [6], and Zhou and Wang [1] established the variational formulations for various modifications of Schrödinger equations. But so far, the variational principle for variable coefficients and highly non-linear Schrödinger equation (VCHNSE) of fractal order has not been dealt with. Equations (1) and (2) describe the interaction between two waves of different frequencies or identical frequency, which pertain to two different polarities. In real application,

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the smooth space (*t*, *x*) should be replaced by the fractal space (t^α, x^β), with α and β are fractal dimensions in time and space, respectively, because the non-smooth boundary has a great influence on the two waves [7-9]. Fractal calculus provides a very effective manner to deal with discontinuous boundaries or porous media, and much achievement was obtained. For example, heat conduction in a porous concrete [10], vibration systems in a porous medium [10-14], mechanical and electrical properties of a composite [15], and fractal MEMS systems [16, 17]. Furthermore the discontinuous population growth [18] and the discontinuous economic development [19] can also be modelled by the fractal calculus. It was reported that the unsmooth boundary will greatly affect the solitary waves, though the wave shape is rarely affected [7-9].

When the Schrödinger equation with high non-linearity and variable coefficients is in a non-smooth or discontinuous boundary, the fractal derivative is used to describe the models:

$$i\frac{\partial u}{\partial t^\alpha} + \lambda_1 t^{\varepsilon_1}\frac{\partial u}{\partial x^{2\beta}} + \xi_1 t^{\varepsilon_2}|u|^2 u + \eta_1 |u|^4 u = Kv \tag{3}$$

$$i\frac{\partial v}{\partial t^\alpha} + \lambda_2 t^{\varepsilon_1}\frac{\partial v}{\partial x^{2\beta}} + \xi_2 t^{\varepsilon_2}|v|^2 v + \eta_2 |v|^4 v = Ku \tag{4}$$

where $\partial u/\partial t^\alpha$ and $\partial u/\partial x^\beta$ are He's partial fractal derivatives with respect to t^α and x^β, respectively, defined [20, 21]:

$$\frac{\partial u}{\partial t^\alpha}(t_0, x) = \Gamma(1+\alpha)\lim_{\substack{t-t_0=\Delta t\\ \Delta t\neq 0}}\frac{u(t,x)-u(t_0,x)}{(t-t_0)^\alpha} \tag{5}$$

$$\frac{\partial u}{\partial x^\beta}(t, x_0) = \Gamma(1+\beta)\lim_{\substack{x-x_0=\Delta x\\ \Delta x\neq 0}}\frac{u(t,x)-u(t,x_0)}{(x-x_0)^\beta} \tag{6}$$

where Δx is mean pore size and Δt is the shortest time period of variable change at spacing distance of Δx. When the spatial scale is larger than Δx, the boundary is considered as smooth, and if not more, then the border is discontinuous and is considered a fractal curve. Likewise for time, when we observe the solitary wave on a scale exceeding Δt, a smooth wave morphology will appear. However, when we observe the wave on the scale not exceeding Δt, discontinuous wave morphology can be found. Detailed explanation of the fractal derivative is available in refs. [22-26]. For the fractal derivatives, we have the following chain rules [20, 21]:

$$\frac{\partial^2 u(x,t)}{\partial t^\alpha \partial x^\beta} = \frac{\partial u(x,t)}{\partial t^\alpha}\frac{\partial u(x,t)}{\partial x^\beta}, \quad \frac{\partial^2 u(x,t)}{\partial x^{2\beta}} = \frac{\partial u(x,t)}{\partial x^\beta}\frac{\partial u(x,t)}{\partial x^\beta} \tag{7}$$

Fractal derivative models were widely used in engineering, thermodynamics, electrospinning, electrochemical, biomechanism, tsunami travelling, thermal insulation, for examples, fractional Sine-Gordon equation [27], fractal Chen-Lee-Liu equation [28], fractal Langmuir model [29], fractal charge transport [30], fractal power law flow [31], fractal Harry Dym equation [32], fractal Klein-Gordon equation [33], fractal KdV-Burgers-Kuramoto equation [34], fractal KdV equation [35], fractional memristor model [36], fractal Burgers' equation [37, 38], and non-smooth initial value problem [39].

On substituting:

$$u(x^\alpha,t^\alpha)=\phi_1(x^\alpha,t^\alpha)+i\psi_1(x^\alpha,t^\alpha), v(x^\alpha,t^\alpha)=\phi_2(x^\alpha,t^\alpha)+i\psi_2(x^\alpha,t^\alpha)$$

where $\phi_i(x^\alpha,t^\alpha)$ and $\psi_i(x^\alpha,t^\alpha)(i = 1, 2)$ are real functions of x and t, we obtain the differential equations about $\phi_1,\psi_1,\phi_2,$ and ψ_2:

$$-\frac{\partial\psi_1}{\partial t^\alpha}+\lambda_1 t^{\varepsilon_1}\frac{\partial^2\phi_1}{\partial x^{2\beta}}+\xi_1 t^{\varepsilon_2}\phi_1(\phi_1^2+\psi_1^2)+\eta_1\phi_1(\phi_1^2+\psi_1^2)^2=K\phi_2 \tag{8}$$

$$\frac{\partial\phi_1}{\partial t^\alpha}+\lambda_1 t^{\varepsilon_1}\frac{\partial^2\psi_1}{\partial x^{2\beta}}+\xi_1 t^{\varepsilon_2}\psi_1(\phi_1^2+\psi_1^2)+\eta_1\psi_1(\phi_1^2+\psi_1^2)^2=K\psi_2 \tag{9}$$

$$-\frac{\partial\psi_2}{\partial t^\alpha}+\lambda_2 t^{\varepsilon_1}\frac{\partial^2\phi_2}{\partial x^{2\beta}}+\xi_2 t^{\varepsilon_2}\phi_2(\phi_2^2+\psi_2^2)+\eta_2\phi_2(\phi_2^2+\psi_2^2)=K\phi_1 \tag{10}$$

$$\frac{\partial\phi_2}{\partial t^\alpha}+\lambda_2 t^{\varepsilon_1}\frac{\partial^2\psi_2}{\partial x^{2\beta}}+\xi_2 t^{\varepsilon_2}\psi_2(\phi_2^2+\psi_2^2)+\eta_2\psi_2(\phi_2^2+\psi_2^2)^2=K\psi_1 \tag{11}$$

In this paper, the fractal variational formulas of general systems (8)-(11) are established with coupling fractal semi-inverse [40-42] and fractional complex transformation [43, 44], which was further developed into the two-scale transformation methods [2, 20, 21]. Although the highly non-linear or variable coefficient Schrödinger equation has been widely studied by many scientists for a long time, so far, the variational principle of fractal Schrödinger equation has not been studied in fractal spaces.

Variational principle for the fractal Schrödinger equation

The variational theory plays an important role in mathematics and physics because variational formulas show the structure of possible energy conservation laws and solutions. Wang *et al.* [45] set up a variational principle for anisotropic wave propagation in fractal space. Wang and He [46] extended Wang's variational principle [45] to fractal space/time. Now various fractal variational principles were appeared in literature for fractal KdV-Burgers equation [47], fractal plasma model [48], and fractal Telegraph equation [49].

He's semi-inverse method [40-42] is widely used to establish variational principles directly from governing equations. In this work, we will apply the He’s semi-inverse method [40-42] and the two-scale fractal theory [50, 51] to establish a variational principle for the fractal VCHNSE.

In order to find the variational formulation of the systems (8)-(11), we first set up a fractal trial functional:

$$J(\phi_1,\psi_1,\phi_2,\psi_2)=\iint L\mathrm{d}x^\beta\mathrm{d}t^\alpha \tag{15}$$

where L is the fractal trial-Lagrange functional.

There are many ways to establish a fractal trial functional [40-42]. Among them, we choose the following form:

$$L=\psi_1\frac{\partial\phi_1}{\partial t^\alpha}-\frac{\lambda_1}{2}t^{\varepsilon_1}\left(\frac{\partial\psi_1}{\partial x^\beta}\right)^2+\frac{\xi_1}{4}t^{\varepsilon_2}(\psi_1^4+2\phi_1^2\psi_1^2)+$$
$$+\frac{\eta_1}{6}(\psi_1^6+3\phi_1^4\psi_1^2+3\phi_1^2\psi_1^4)-K\psi_1\psi_2+F(\phi_1,\phi_2,\psi_2) \tag{16}$$

where F is an undetermined function with regard to ϕ_1, and/or ϕ_2, and/or ψ_2 derivatives. There exist various alternative approaches to construct the trial functional, illustrative examples can be found in [40-42]. The advantage of the above trial-functional lies on the fact that stationary condition with respect to ψ_1, and noting that F is absence of ψ_1 and its derivatives, is eq. (9).

At present, the variation of eq. (16) with respect to ϕ_1 gives the result Euler Lagrange equation:

$$-\frac{\partial\psi_1}{\partial t^{\alpha}} + \xi_1 t^{\varepsilon_2}\phi_1\psi_1^2 + 2\eta_1\phi_1^3\psi_1^2 + \eta_1\phi_1\psi_1^4 + \frac{\delta F}{\delta\phi_1} = 0 \tag{17}$$

where $\delta F/\delta\phi_1$ are called the fractal variational derivative about ϕ_1 [40-42] presented:

$$\frac{\delta F}{\delta\phi_1} = \frac{\partial F}{\partial\phi_1} - \frac{\partial}{\partial t}\left(\frac{\partial F}{\partial\phi_{1t}}\right) - \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial\phi_{1x}}\right) + \frac{\partial^2}{\partial t^2}\left(\frac{\partial F}{\partial\phi_{1tt}}\right) + \frac{\partial^2}{\partial x^2}\left(\frac{\partial F}{\partial\phi_{1xx}}\right) + \cdots \tag{18}$$

We look for such an F so that eq. (17) leads to eq. (8). So we set:

$$\frac{\delta F}{\delta\phi_1} = \frac{\partial\psi_1}{\partial t^{\alpha}} - \xi_1 t^{\varepsilon_1}\phi_1\psi_1^2 - 2\eta_1\phi_1^3\psi_1^2 - \eta_1\phi_1\psi_1^4 = \lambda_1 t^{\varepsilon_1}\frac{\partial^2\phi_1}{\partial x^{2\beta}} + \xi_1 t^{\varepsilon_2}\phi_1^3 + \eta_1\phi_1^5 - K\phi_2 \tag{19}$$

from which the undetermined F can be defined:

$$F = -\frac{1}{2}\lambda_1 t^{\varepsilon_1}\left(\frac{\partial\phi_1}{\partial x^{\beta}}\right)^2 + \frac{1}{4}\xi_1 t^{\varepsilon_2}\phi_1^4 + \frac{1}{6}\phi_1^6 - K\phi_1\phi_2 + F_1(\phi_2,\psi_2) \tag{20}$$

where F_1 is a newly undetermined function that was introduced in regard to ϕ_2 or/and ψ_2 derivatives. The fractal trial-Lagrange, eq. (16), is updated:

$$L = \psi_1\frac{\partial\phi_1}{\partial t^{\alpha}} - \frac{\lambda_1}{2}t^{\varepsilon_1}\left[\left(\frac{\partial\phi_1}{\partial x^{\beta}}\right)^2 + \left(\frac{\partial\psi_1}{\partial x^{\beta}}\right)^2\right] + \frac{\xi_1}{4}t^{\varepsilon_2}(\phi_1^2+\psi_1^2)^2 +$$
$$+\frac{\eta_1}{6}(\phi_1^2+\psi_1^2)^3 - K(\phi_1\phi_2+\psi_1\psi_2) + F_1 \tag{21}$$

At the moment, the stationary condition about ϕ_2 is:

$$-K\phi_1 + \frac{\delta F_1}{\delta\phi_2} = 0 \tag{22}$$

In eq. (22), we set:

$$\frac{\delta F_1}{\delta\phi_2} = K\phi_1 = -\frac{\partial\psi_2}{\partial t} + \lambda_2 t^{\varepsilon_1}\frac{\partial^2\phi_2}{\partial x^2} + \xi_2 t^{\varepsilon_2}\phi_2(\phi_2^2+\psi_2^2) + \eta_2\phi_2(\phi_2^2+\psi_2^2)^2 \tag{23}$$

From eq. (23), the unknown F_1 can be distinguished:

$$F_1 = -\phi_2\frac{\partial\psi_2}{\partial t^{\alpha}} - \frac{\lambda_2}{2}t^{\varepsilon_1}\left(\frac{\partial\phi_2}{\partial x^{\beta}}\right)^2 + \frac{\xi_2}{4}t^{\varepsilon_2}(\phi_2^4 + 2\phi_2^2\psi_2^2) + \frac{\eta_2}{6}\phi_2(\phi_2^6 + 3\phi_2^4\psi_2^2 + 3\phi_2^2\psi_2^4) + F_2(\psi_2) \tag{24}$$

where F_2 is an undetermined function of ψ_2, or/and its derivatives.

The fractal Lagrangian can go further updated as:

$$L=\psi_1\frac{\partial\phi_1}{\partial t^\alpha}-\frac{a_1}{2}t^m\left[\left(\frac{\partial\phi_1}{\partial x^\beta}\right)^2+\left(\frac{\partial\psi_1}{\partial x^\beta}\right)^2\right]+\frac{b_1}{4}t^n(\phi_1^2+\psi_1^2)^2+$$
$$+\frac{c_1}{6}(\phi_1^2+\psi_1^2)^3-K(\phi_1\phi_2+\psi_1\psi_2)-\phi_2\frac{\partial\psi_2}{\partial t^\alpha}-\frac{\lambda_2}{2}t^{\varepsilon_1}\left(\frac{\partial\phi_2}{\partial x}\right)^2+$$
$$+\frac{\xi_2}{4}t^{\varepsilon_2}(\phi_2^4+2\phi_2^2\psi_2^2)+\frac{\eta_2}{6}\phi_2(\phi_2^6+3\phi_2^4\psi_2^2+3\phi_2^2\psi_2^4)+F_2(\psi_2) \tag{25}$$

The fractal Euler equation on ψ_2 is:

$$-K\psi_1+\frac{\partial\phi_2}{\partial t^\alpha}+\xi_2t^{\varepsilon_2}\phi_2^2\psi_2+\eta_2(\phi_2^4\psi_2+2\phi_2^2\psi_2^3)+\frac{\delta F_2}{\delta\psi_2}=0 \tag{26}$$

Equation (26) equates to eq. (11), as a result, we presume:

$$\frac{\delta F_2}{\delta\psi_2}=K\psi_1-\frac{\partial\phi_2}{\partial t^\alpha}-\xi_2t^{\varepsilon_2}\phi_2^2\psi_2-\eta_2(\phi_2^4\psi_2+2\phi_2^2\psi_2^3)=\lambda_2t^{\varepsilon_1}\frac{\partial^2\psi_2}{\partial x^{2\beta}}+\xi_2t^{\varepsilon_2}\psi_2^3+\eta_2\psi_2^5 \tag{27}$$

So we can identify F_2 without difficulty:

$$F_2=-\lambda_2t^{\varepsilon_1}\left(\frac{\partial\psi_2}{\partial x^\alpha}\right)^2+\frac{\xi_2}{4}t^{\varepsilon_2}\psi_2^4+\frac{1}{6}\eta_2\psi_2^6 \tag{28}$$

At the moment, we get the fractal Lagrangian function:

$$L=\psi_1\frac{\partial\phi_1}{\partial t^\alpha}-\frac{\lambda_1}{2}t^{\varepsilon_1}\left[\left(\frac{\partial\phi_1}{\partial x^\beta}\right)^2+\left(\frac{\partial\psi_1}{\partial x^\beta}\right)^2\right]+\frac{\xi_1}{4}t^{\varepsilon_2}(\phi_1^2+\psi_1^2)^2+$$
$$+\frac{\eta_1}{6}(\phi_1^2+\psi_1^2)^3-K(\phi_1\phi_2+\psi_1\psi_2)-\phi_2\frac{\partial\psi_2}{\partial t^\alpha}-$$
$$-\frac{\lambda_2}{2}t^{\varepsilon_1}\left[\left(\frac{\partial\phi_2}{\partial x^\beta}\right)^2+\left(\frac{\partial\psi_2}{\partial x^\beta}\right)^2\right]+\frac{\xi_2}{4}t^{\varepsilon_2}(\phi_2^2+\psi_2^2)^2+\frac{\eta_2}{6}(\phi_2^2+\psi_2^2)^3 \tag{29}$$

Superseding:

$$\phi_1=\frac{u+u^*}{2},\quad \psi_1=i\frac{u^*-u}{2},\quad \phi_2=\frac{v+v^*}{2},\quad \psi_2=i\frac{v^*-v}{2}$$

here $u^*=\phi_1-i\psi_1$, $v^*=\phi_2-i\psi_2$, we can obtain the Lagrangian eq. (29) on the basis of u and v:

$$L=\frac{i}{4}\left[(u^*-u)\left(\frac{\partial u}{\partial t^\alpha}+\frac{\partial u^*}{\partial t^\alpha}\right)+(v^*+v)\left(\frac{\partial v}{\partial t^\alpha}-\frac{\partial v^*}{\partial t^\alpha}\right)\right]-$$
$$-\frac{1}{2}t^{\varepsilon_1}\left(a_1\left|\frac{\partial u}{\partial x^\beta}\right|^2+a_2\left|\frac{\partial v}{\partial x^\beta}\right|^2\right)+\frac{1}{6}(\eta_1|u|^6+\eta_2|v|^6)+$$
$$+\frac{1}{4}t^{\varepsilon_2}(\xi_1|u|^4+\xi_2|v|^4)-\frac{K}{2}(uv^*+\mathrm{u}^*v) \tag{30}$$

Finally, the variational principle is obtained about u and v, displaying:

$$J(u,v)=\iint\left\{\frac{i}{4}\left[(u^*-u)\left(\frac{\partial u}{\partial t^\alpha}+\frac{\partial u^*}{\partial t^\alpha}\right)+(v^*+v)\left(\frac{\partial v}{\partial t^\alpha}-\frac{\partial v^*}{\partial t^\alpha}\right)\right]-\right.$$
$$-\frac{1}{2}t^{\varepsilon_1}\left(\lambda_1\left|\frac{\partial u}{\partial x^\beta}\right|^2+\lambda_2\left|\frac{\partial v}{\partial x^\beta}\right|^2\right)+\frac{1}{4}t^{\varepsilon_2}(\xi_1|u|^4+\xi_2|v|^4)+$$
$$\left.+\frac{1}{6}(\eta_1|u|^6+\eta_2|v|^6)-\frac{K}{2}(uv^*+\mathrm{u}^*v)\right\}\mathrm{d}x^\beta\mathrm{d}t^\alpha \tag{31}$$

Discussion and conclusion

In this article, the fractal non-linear Schrödinger equation with variable coefficients and high non-linearity in fractal space is studied. With the help of He's two-scale transformation method [2, 20, 21], the fractal non-linear Schrödinger equation can be changed to its partner.

We assume:

$$T=t^\alpha,\quad X=x^\beta \tag{32}$$

Therefore, systems of eqs. (8)-(11) can be transformed into the undermentioned:

$$-\frac{\partial\psi_1}{\partial T}+\lambda_1T^{\varepsilon_1}\frac{\partial^2\phi_1}{\partial X^2}+\xi_1T^{\varepsilon_2}\phi_1(\phi_1^2+\psi_1^2)+\eta_1\phi_1(\phi_1^2+\psi_1^2)^2=K\phi_2 \tag{33}$$

$$\frac{\partial\phi_1}{\partial T}+\lambda_1T^{\varepsilon_1}\frac{\partial^2\psi_1}{\partial X^2}+\xi_1T^{\varepsilon_2}\psi_1(\phi_1^2+\psi_1^2)+\eta_1\psi_1(\phi_1^2+\psi_1^2)^2=K\psi_2 \tag{34}$$

$$-\frac{\partial\psi_2}{\partial T}+\lambda_2T^{\varepsilon_1}\frac{\partial^2\phi_2}{\partial X^2}+\xi_2T^{\varepsilon_2}\phi_2(\phi_2^2+\psi_2^2)+\eta_2\phi_2(\phi_2^2+\psi_2^2)=K\phi_1 \tag{35}$$

$$\frac{\partial\phi_2}{\partial T}+\lambda_2T^{\varepsilon_1}\frac{\partial^2\psi_2}{\partial X^2}+\xi_2T^{\varepsilon_2}\psi_2(\phi_2^2+\psi_2^2)+\eta_2\psi_2(\phi_2^2+\psi_2^2)^2=K\psi_1 \tag{36}$$

Employing the previous approach, we acquire the variational formulation set of eqs. (33)-(36) in the light of u and v:

$$J(u,v)=\iint\left\{\frac{i}{4}\left[(u^*-u)\left(\frac{\partial u}{\partial T}+\frac{\partial u^*}{\partial T}\right)+(v^*+v)\left(\frac{\partial v}{\partial T}-\frac{\partial v^*}{\partial T}\right)\right]-\frac{1}{2}T^{\varepsilon_1}\left(\lambda_1\left|\frac{\partial u}{\partial X}\right|^2+\lambda_2\left|\frac{\partial v}{\partial X}\right|^2\right)+\right.$$
$$\left.+\frac{1}{4}T^{\varepsilon_2}(\xi_1|u|^4+\xi_2|v|^4)+\frac{1}{6}(\eta_1|u|^6+\eta_2|v|^6)-\frac{K}{2}(uv^*+\mathrm{u}^*v)\right\}\mathrm{d}X\mathrm{d}T \tag{37}$$

For the unique fractal Schrödinger equation:

$$i\frac{\partial u}{\partial t^\alpha}+\lambda_1t^{\varepsilon_1}\frac{\partial u}{\partial x^{2\beta}}+\xi_1t^{\varepsilon_2}|u|^2u+\eta_1|u|^4u=0 \tag{38}$$

can be changed into under mentioned:

$$i\frac{\partial u}{\partial T}+\lambda_1 T^{\varepsilon_1}\frac{\partial u}{\partial X^2}+\xi_1 T^{\varepsilon_2}\left|u\right|^2 u+\eta_1\left|u\right|^4 u=0 \tag{39}$$

Using a resemblance procedure, the variational principle of eq. (39) can be received:

$$J(u,v)=\iint\left\{\frac{i}{4}\left[(u^*-u)\left(\frac{\partial u}{\partial T}+\frac{\partial u^*}{\partial T}\right)\right]-\frac{\lambda_1}{2}T^{\varepsilon_1}\left|\frac{\partial u}{\partial X}\right|^2+\frac{\xi_1}{4}T^{\varepsilon_2}\left|u\right|^4+\frac{\eta_1}{6}\left|u\right|^6\right\}\mathrm{d}X\mathrm{d}T \tag{40}$$

With the help of eq. (40), we obtain a few special samples:

– $\varepsilon_1 = \varepsilon_2 = 0$, eq. (40) reduces to the results obtained by Yao and Chang [5],
– $\eta_1 = 0$, eq. (40) is equivalent to the fruit obtained by He [40],
– $\varepsilon_1 = \varepsilon_2 = 0$, $\lambda_1 = 0$, $\eta_1 = 0$, eq. (40) reverts to:

$$i\frac{\partial u}{\partial T}+\frac{\partial u}{\partial X^2}+\xi_1\left|u\right|^2 u=0$$

which is expounded by Ozis and Yildirim [6].

Zhou and Wang [1] have found a variational formulation for VCHNSE. In this work, we successfully extended VCHNSE to fractal VCHNSE by using He's fractal derivative via semi-inverse and He's two-scale transformation methods. The fractal models of coupled VCHNSE on non-smooth boundary are obtained and the fractal variational principle of coupled VCHNSE is successfully constructed. The correctness of the obtained variational principle is verified. It shows that the semi-inverse method is concise and effective. Based on the obtained variational principle, we can further study the motion law of solitary waves, and the problems discussed can be solved numerically with the help of variational method

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