EFFECT OF FRACTIONAL ORDER ON UNSTEADY MAGNETOHYDRODYNAMICS PULSATILE FLOW OF BLOOD INSIDE AN ARTERY

by

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This manuscript aims to investigate the velocity profile for the blood flow through an artery subject to magnetic field. It has been investigated how periodic acceleration of the body and slip conditions affect the irregular pulsatile blood flow across a porous media inside an artery if a magnetic field is present, under the assumption that blood is an incompressible electrically conducting fluid. A mathematical formulation involving Caputo fractional derivative serves as the basis of study. An analytical solution for fluid velocity is developed with the help of finite Hankel and Laplace transforms. The influence of fractional order on the fluid velocity is illustrated with the help of graphical simulations. The obtained results will be helpful in future research for the treatment of stenosis and other cardiovascular diseases.

Keyw ords: *MHD, blood flow, magnetic field, stenosis, Caputo time-fractional derivative*

Introduction

The MHD application brings down the rate of flow of blood in the human arterial (circulatory) system. This has been found very helpful to treat cardiovascular disorders particularly atherosclerosis (medically called stenosis). Stenosis is the most common arterial blood vessel condition which can cause death. From last few decades, researchers have been paying a lot of attention fluid dynamics to investigate the behavior of biological fluids when a magnetic field is applied on them. The reason is the significance of the studies on this topic in the field of medical sciences. Eldesoky [1] studied a mathematical formulation of blood flowing unsteadily across parallel plates when the magnetic field was applied on it. Latha and Kumar [2] proposed a biomagnetic fluid-flow in parallel plates on which radiation and heat source affect. Ali *et al*. [3] examined the problem of incompressible, electrically conducting, viscous, unsteady flow of blood and transfer of heat across a channel of plates with the lower plate kept stretched. Kumar and Diwakar [4] studied the nature of blood flowing inside an artery affected by stenosis. The blood under consideration exhibited fluid-like behavior in a homogeneous circular tube with a radially symmetric but axially non-symmetric stenosis. The governing model for boundary-constrained laminar, incompressible, and non-Newtonian fluid-flow (power-law fluid-flow)

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was numerically solved. Sharma *et al*. [5] examined the pulsatile MHD flow of blood inside an artery for double stenosis problem. Eldesoky [6] conducted research on time-dependent (injuction/suction) pulsatile MHD unsteady blood flow across porous medium in considering slip condition at permeable walls. Blood flow in capillaries with MHD was examined by Misra and Sinha [7], whose lumen was porous and wall permeable. The movement of Jeffery fluid subject to magnetic fields was examined by Nallapu and Radhakrishnamacharya [8] for the flow through tubes of very small diameters. The impact of permeability of a porous material on MHD blood flow through extremely small capillaries was researched by Agarwal and Varshney [9]. Verma et al. [10] investigated the pulsatile blood flow in moderate stenosis while accelerating the body. The pulsatile laminar flow of blood across and artery arterial was examined by Rabby *et al*. [11] for double stenoses. The axisymmetric flow of blood through s stenotic channel was examined by Kumar and Diwakar [12]. Some other interesting results available in literature are reported in [13-16].

Fractional order mathematical models have caught the attention of many researchers during the past few years. The fractional differential equations are found to be more beneficial in various theoretical investigation due to their flexibility and useful mathematical properties. Abro and Atangana [17] presented a mathematical fractional formulation of pulsatile MHD flow in the presence of porosity, considering two definitions of fractional derivatives. Anwar *et al*. [18] investigated a fractional order model for MHD flow of Oldroyd-B fluid using Caputo-Fabrizio derivative.

Formulation of problem

The unsteady pulsatile blood flow through an axisymmetric, cylindrical artery with a diameter $2R$ through some porous structure with body acceleration as shown in the fig. 1. The cylindrical co-ordinates (r, θ, z) are induced with z-axis taken along the central axis of the artery. The flow is considered under the assumption that the blood is an electrically conducting, incompressible, Newtonian fluid subject to a constant magnetic field $\mathcal{B}(0, B_0, 0)$ which is orthogonal to the artery. The blood's viscosity is constant. The magnetic Renolds number is considered to be sufficiently small for the flow so that the induced electric and magnetic fields can be ignored. The following terms used in the model are taken:

- For steady flow A_0 is the pressure gradient.
- Amplitude A_1 is for oscillatory part.
- Amplitude a_0 is for body acceleration.
- The $\omega_p = 2\prod f_p$ where f_p is pulse rate.
- The $\omega_b = 2\prod f_b$ where f_b is frequency of body acceleration.
- The $u(r, t)$ is the velocity at time, t.
- The k denotes the permeability parameter for porous medium.
- The ρ, σ, and ϖ denote the density, electric conductivity and dynamic viscosity of blood, respectively.

The expressions for Hartmann number, Ha, Womersley parameter, β, and Knudsen number, Kn, are given:

$$\text{Ha} = B_0 R\sqrt{\frac{\sigma}{\varpi}}, \quad \beta = R\sqrt{\frac{\rho\omega}{\varpi}}, \quad \text{Kn} = \frac{A}{R}$$

The body acceleration and pressure gradient can be expressed:

$$G = a_0 \cos(\omega_b t) \tag{1}$$

$$-\frac{\partial p}{\partial z} = \Lambda_0 + \Lambda_1 \cos(\omega_p t) \tag{2}$$

The flow equation in cylindrical polar co-ordinates can be written:

$$\rho \frac{\partial u}{\partial t} = -\frac{\partial p}{\partial z} + \varpi \nabla 2u + \rho G - \left(\frac{\varpi}{k}\right) u + \mathcal{J} \times \mathcal{B} \tag{3}$$

The Maxwell's equations state:

$$\overline{\nabla}\mathcal{B} = 0, \ \ \overline{\nabla} \times \mathcal{B} = \varpi_0 \mathcal{J}, \ \ \overline{\nabla} \times \overline{\overline{E}} = -\frac{\partial \mathcal{B}}{\partial t} \tag{4}$$

The Ohm's law implies:

$$\mathcal{J} = \sigma(\overline{E} + \mathcal{V} \times \mathcal{B}) \tag{5}$$

where the velocity distribution is denoted by $\mathcal{V}$ = (0, 0, u), ϖ_0 is the magnetic permeability, $\mathcal{J}$ is the current density, and E is the electric field. Since the magnetic Reynolds number is small, therefore, the linearized magnetohydrodynamic force is expressible:

$$\mathcal{J} \times \mathcal{B} = -\sigma B_0^2 u \tag{6}$$

whereas the shear stress τ is [19]:

$$\tau = -\varpi \frac{\partial u}{\partial r} \tag{7}$$

The equation of motion using the given assumptions can be written:

$$\rho \frac{\partial u}{\partial t} = \Lambda_0 + \Lambda_1 \cos(\omega_p t) + \varpi \left(\frac{\partial 2u}{\partial r2} + \frac{1}{r}\frac{\partial u}{\partial r}\right) + \rho\left[a_0 \cos(\omega_b t)\right] - \left(\frac{\varpi}{k}\right) u - \sigma B_0^2 u \tag{8}$$

Using the assumptions:

$$u^* = \frac{u}{\omega R'}, \ r^* = \frac{r}{R'}, \ t^* = t\omega, \ \Lambda_0^* = \frac{R}{\varpi\omega}\Lambda_0, \ \Lambda_1^* = \frac{R}{\varpi\omega}\Lambda_1$$

$$a_0^* = \frac{\rho R}{\varpi\omega} a_0, \ z^* = \frac{z}{R'}, \ k^* = \frac{k}{R^2}, \ b^* = \frac{\omega_b}{\omega_p}$$

The non-dimensional form of eq. (8) can be written:

$$\beta^2 \frac{\partial u}{\partial t} = \Lambda_0 + \Lambda_1 \cos(t) + a_0 \cos(bt) + \left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r}\right) - \left(\text{Ha}^2 + \frac{1}{k}\right) u$$

where the stars are dropped for sake of convenience.

The initial condition:

$$u(r,0) = 1 \ \text{at} \ t = 0$$

whereas the boundary conditions are:

$$u(0,t) \ \text{is finite (axis of the pipe)}$$

$$u(1,t) = 0 \ \text{at} \ r = 1, \ \text{(no slip)}$$

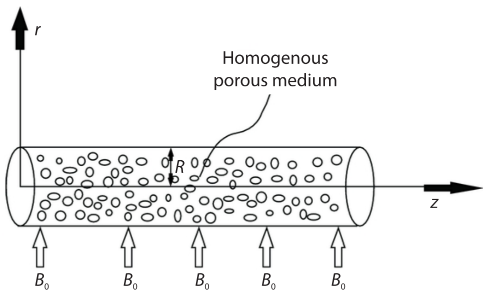

Figure 1. Schematic diagram of the problem

Extraction of solution for fractional order model

Consider the governing equation:

$$\beta^2 \frac{\partial u}{\partial t} = \Lambda_0 + \Lambda_1 \cos(t) + a_0 \cos(bt) + \left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r}\right) - \left(\mathrm{Ha}^2 + \frac{1}{k}\right)u \tag{9}$$

Applying the Caputo fractional derivative, the following equation can be obtained:

$$\beta^2 D_t^\alpha u = \Lambda_0 + \Lambda_1 \cos(t) + a_0 \cos(bt) + \left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r}\right) - \left(\mathrm{Ha}^2 + \frac{1}{k}\right)u \tag{10}$$

where

$$D_t^\alpha u = \begin{cases} \dfrac{1}{\Gamma(1-\alpha)} \displaystyle\int_0^t \dfrac{1}{(1-\tau)^\alpha} \dfrac{\partial f(r,t)}{\partial \tau} \mathrm{d}r & 0 < \alpha < 1 \\ \dfrac{\partial f(r,t)}{\partial t} & \alpha = 1 \end{cases} \tag{11}$$

is the Caputo fractional derivative operator of order α [20, 21].

Applying Laplace transform corresponding to time, t, on eq. (10), the result can be obtained:

$$s^\alpha \overline{u}(r,s) - s^{\alpha-1} u(r,0) = \frac{\Lambda_0}{\beta^2 s} + \frac{\Lambda 1}{\beta^2}\frac{s}{s^2+1} + \frac{a_0}{\beta^2}\frac{s}{s^2+b^2}\frac{1}{\beta^2}\left[\frac{\partial^2 \overline{u}(r,s)}{\partial r^2} + \frac{1}{r}\frac{\partial \overline{u}(r,s)}{\partial r}\right] - \\ -\frac{1}{\beta^2}\left(\mathrm{Ha}^2 + \frac{1}{k}\right)\overline{u}(r,s) \tag{12}$$

where

$$\overline{u}(r,s) = \int_0^\infty u(r,t) \exp^{-st} \mathrm{d}t, \ \ (s > 0)$$

Using the initial condition, the resulting equation:

$$\overline{u}(r,s) = \frac{1}{s} + \frac{\Lambda_0}{\beta^2 s^{\alpha+1}} + \frac{\Lambda 1}{\beta^2 s^\alpha}\frac{s}{s^2+1} + \frac{a_0}{\beta^2 s^\alpha}\frac{s}{s^2+b^2} + \frac{1}{\beta^2 s^\alpha}\left[\frac{\partial^2 \overline{u}(r,s)}{\partial r^2} + \frac{1}{r}\frac{\partial \overline{u}(r,s)}{\partial r}\right] - \\ -\frac{1}{\beta^2 s^\alpha}\left(\mathrm{Ha}^2 + \frac{1}{k}\right)\overline{u}(r,s) \tag{13}$$

Using the finite Hankel transformation [22, 23] of order zero, the resulting equation is obtained:

$$\overline{u}_H(r_n,s) = \left[\frac{1}{s} + \frac{\Lambda_0}{\beta^2 s^{\alpha+1}} + \frac{\Lambda_1}{\beta^2 s^\alpha}\frac{s}{s^2+1} + \frac{a_0}{\beta^2 s^\alpha}\frac{s}{s^2+b^2}\right]\frac{dJ_1(dr_n)}{r_n} + \frac{1}{\beta s^\alpha}\left(-r_n^2 \overline{u}_H(r_n s)\right) - \\ -\frac{1}{\beta s^\alpha}\left(\mathrm{Ha}^2 + \frac{1}{k}\right)\overline{u}_H(r_n s) \tag{14}$$

where

$$\overline{u}_H(r_n,s) = \int_0^1 r\overline{u}(r,s) J_0(rr_n) \mathrm{d}r$$

is the finite Hankel transformation of $\bar{u}(r, s)$. Moreover, $r_n > 0$ where $n = 1,2...$ are the roots of $J_0(x)$, J_0 being the zeroth order Bessel function of first kind. Further simplification yields the following relation.

$$\bar{u}_H(r_n,s)=\frac{1}{\beta^2 Q}\left[\frac{\beta^2 s^\alpha}{s}+\frac{\Lambda_0}{s}+\frac{\Lambda_1 s}{s^2+1}+\frac{a_0 s}{s^2+b^2}\right]\frac{dJ_1(dr_n)}{r_n} \tag{15}$$

where

$$Q=\left[s^\alpha+\left(\frac{r_n^2}{\beta^2}+\frac{\mathrm{Ha}^2}{\beta^2}+\frac{1}{K\beta^2}\right)\right] \tag{16}$$

Applying Laplace inverse transformation on eq. (15), gives:

$$u_H(r_n,s)=\frac{1}{\beta^2}F_\alpha\left[-\left(\frac{r_n^2}{\beta^2}+\frac{\mathrm{Ha}^2}{\beta^2}+\frac{1}{K\beta^2}\right),t\right]*\left(\frac{\beta^2 t^{-\alpha}}{\Gamma(1-\alpha)}+\Lambda_0+\Lambda_1\cos(t)+a_0\cos(bt)\right)\frac{dJ_1(dr_n)}{r_n} \tag{17}$$

where $f*g$ is the convolution of f and g. After applying the inverse Hankel transform the result is obtained:

$$u(r,t)=\frac{2}{d^2}\sum_{n=1}^{\infty}\frac{J_0(rr_n)}{J_1^2(dr_n)}u_H(r_n,s),\ \ 0\le r\le d \tag{18}$$

After simplification, the result is acquired:

$$u(r,t)=\frac{2}{\beta^2 d}\sum_{n=1}^{\infty}\frac{J_0(rr_n)}{r_n J_1(dr_n)}\left(F_\alpha\left[-\left(\frac{r_n^2}{\beta^2}+\frac{\mathrm{Ha}^2}{\beta^2}+\frac{1}{K\beta^2}\right),t\right]\right)*\left(\frac{\beta^2 t^{-\alpha}}{\Gamma(1-\alpha)}\Lambda_0+\Lambda_1\cos(t)+a_0\cos(bt)\right)$$

Numerical simulations and results

The study is based on pulsatile blood flow across a porous medium inside an artery. The unsteady flow is considered with the periodic body acceleration when the magnetic field is applied. The corresponding fractional order mathematical model is considered taking the fractional derivative in Caputo's sense. The influence of fractional order α on fluid velocity is investigated. The graph of the solution is obtained using the MAPLE software. Graphically, fig. 2 demonstrates that fluid velocity reduces when the fractional order value α is increased. Figures 3 and 4 illustrate the effect of different fractional order α on velocity profile.

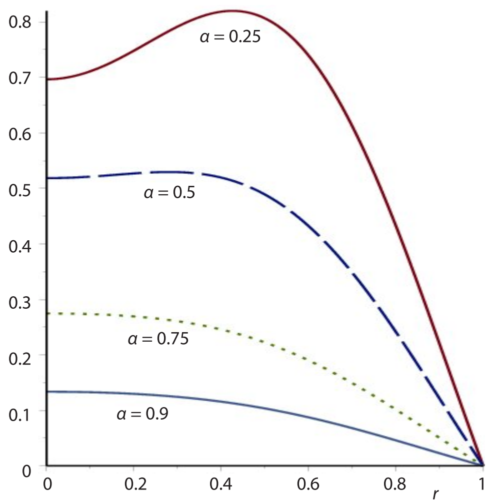

Figure 2. Velocity $u(r, t)$ for different values of fractional order α

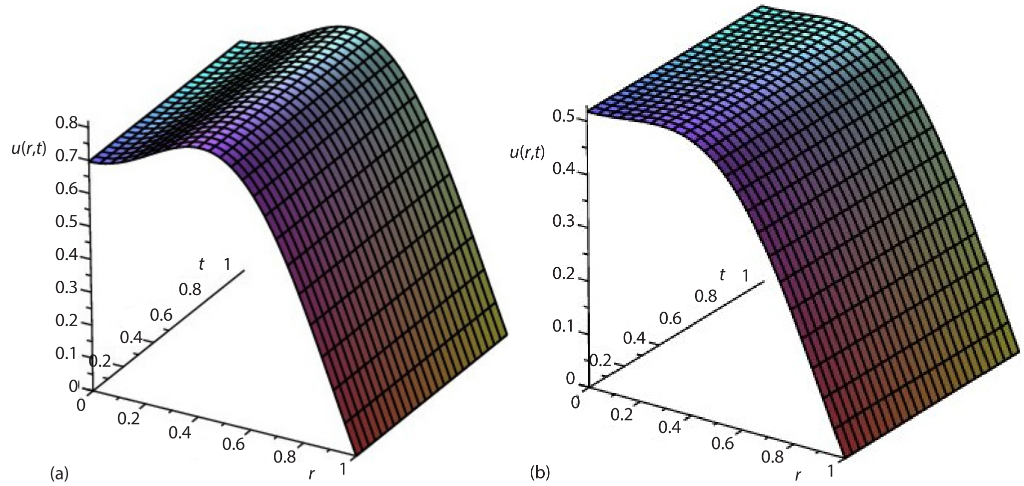

Figure 3. The 3D effect of fractional order α on plot of velocity profile $u(r, t)$ (a) when $\alpha = 0.75$ and (b) when $\alpha = 0.90$

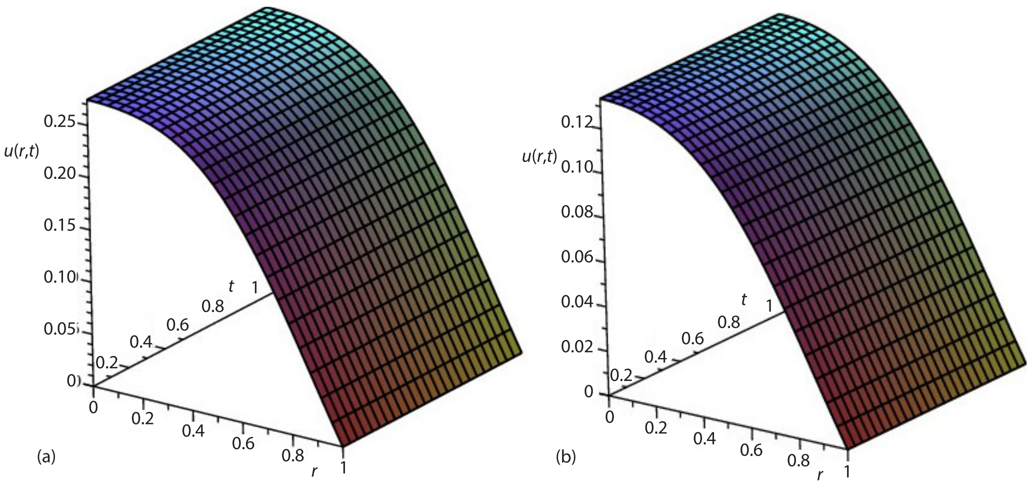

Figure 4. The effect of fractional order α on plot of velocity profile $u(r, t)$ (a) when $\alpha = 0.75$ and (b) when $\alpha = 0.90$

Conclusion

Considering a MHD approach, the impact of Caputo's fractional derivative on blood flow characteristics in a cylindrical domain has been investigated. Using the finite Hankel and Laplace transforms, the exact solutions of the governing fractional order evolution equation are retrieved. The calculated results are illustrated through few graphical representations. It is hoped that this study will be useful for future research in the medical area and the use of magnetic fields to treat some cardiovascular diseases. The findings of this research will be helpful to understand the pathological condition of flow of blood in arterial system when blood clots or fatty cholesterol plaques develop in the artery lumen.

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Competing interests

The authors declare that they have no conflicts of interest to report regarding the present study.

Authors contributions

All authors equally contributed to this work. All authors read and approved the final manuscript.

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