NON-DIFFERENTIABLE EXACT SOLUTIONS OF THE LOCAL FRACTIONAL KLEIN-FOCK-GORDON EQUATION ON CANTOR SETS

by

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Based on the local fractional derivative, a new local fractional Klein-Fock-Gordon equation is derived in this paper for the first time. A simple method namely Yang's special function method is used to seek for the non-differentiable exact solutions. The whole calculation process strongly shows that the proposed method is simple and effective, and can be applied to investigate the non-differentiable exact solutions of the other local fractional PDE.

Key words: *local fractional derivative, Mittag-Leffler function, Yang's special function method, Cantor sets*

Introduction

As we all know, a great deal of the complex phenomenon occurring in nature such as the optics [1-3], biology [4], vibration [5], thermal science [6, 7], and so on [8, 9] can be modeled by the PDE. It is of great significance to study the exact solution of the NLPDE since it can enable us to better understand and make use of natural phenomena. Recently, the fractal and fractional derivative have the wide attentions in different fields because they can model the complex problems that the integer derivative cannot [10-16]. In this work, we aim to derive a new fractional Klein-Fock-Gordon equation based on the local fractional derivative (LFD) [17, 18]:

$$\frac{\partial^{2\xi}\Xi_\xi}{\partial t^{2\xi}} + c\frac{\partial^{2\xi}\Xi_\xi}{\partial x^{2\xi}} + r\Xi_\xi + \delta\,\Xi_\xi^3 = 0 \tag{1}$$

where $\Xi_\xi = \Xi_\xi(x^\xi, t^\xi)$, ξ $(0 < \xi \le 1)$ is the fractional order, $\partial^\xi\Xi_\xi/\partial t^\xi$ and $\partial^\xi\Xi_\xi/\partial x^\xi$ are the local fractional derivatives and their definitions [19-22]. In this work, we aim to investigate the non-differentiable (ND) exact solutions of the local fractional Klein-Fock-Gordon equation by Yang's special function method.

Basic theory

In this section, we will introduce some basic theory.

Definition 1. The LFD for $\Xi_{(x)}$ of the fractional order ξ is defined [19]:

$$\left.\frac{\mathrm{d}^\xi\Xi(x)}{\mathrm{d}t^\xi}\right| x = x0 = \lim_{x\to x_0}\frac{\Delta\left[\Xi(x)-\Xi(x)\right]}{(x-x_0)^\xi} \tag{2}$$

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where $\Delta^{\xi}[\Xi(x) - \Xi(x)] \cong \Gamma(1 + \xi)[\Xi(x) - \Xi(x_0)]$. For the LFD, there is the rule chain [19]:

$$\frac{\mathrm{d}^{\xi k}\Xi(x)}{\mathrm{d}x^{k\xi}} = \overbrace{\frac{\mathrm{d}\xi}{\mathrm{d}x\xi}\cdots\frac{\mathrm{d}\xi}{\mathrm{d}x\xi}}^{k\ \text{times}}\Xi(x)$$

Definition 2. The local fractional integral (LFI) for $\Xi(x)$ ξ is defined [19]:

$$ {}_aI_b^{\xi}\Xi(x) = \frac{1}{\Gamma(1+\xi)}\int_a^b \Xi(x)(\mathrm{d}x)^{\xi} = \frac{1}{\Gamma(1+\xi)}\lim_{\Delta x_k \to 0}\sum_{k=0}^{N-1}\Xi(x_k)(\Delta x_k)^{\xi} \tag{3}$$

where $\Delta x_k = x_{k+1} - x_k$ and $x_0 = a < x_1 < ... < x_{N-1} < x_N = b$.

Property 1. The LFD admits the properties [22]:

$$\frac{\partial^{\xi}}{\partial t^{\xi}}\left[\phi(t) \pm \varphi(t)\right] = \frac{\partial^{\xi}}{\partial t^{\xi}}\phi(t) \pm \frac{\partial^{\xi}}{\partial t^{\xi}}\varphi(t)$$

$$\frac{\partial^{\xi}}{\partial t^{\xi}}\left[\phi(t)\varphi(t)\right] = \varphi(t)\frac{\partial^{\xi}}{\partial t^{\xi}}\phi(t) + \phi(t)\frac{\partial^{\xi}}{\partial t^{\xi}}\varphi(t)$$

$$\frac{\partial^{\xi}}{\partial t^{\xi}}\left[\frac{\phi(t)}{\varphi(t)}\right] = \frac{\left[\varphi(t)\dfrac{\partial^{\xi}}{\partial t^{\xi}}\phi(t) - \phi(t)\varphi(t)\right]}{\varphi(t)}$$

Definition 3. The definition of the Mittag-Leffler function (MLF) on the CS is [19]:

$$ML_{\xi}(\chi^{\xi}) = \sum_{\Im=0}^{\infty}\frac{\chi^{\Im\xi}}{\Gamma\left(1+\Im\xi\right)} \tag{4}$$

Definition 4. By the MLF, the special functions can be constructed [19]:

$$SH_{\xi}(\wp^{\xi}) = \frac{2}{ML_{\xi}(\chi^{\xi}) + ML_{\xi}(-\chi^{\xi})}$$

$$CH_{\xi}(\chi^{\xi}) = \frac{2}{ML_{\xi}(\chi^{\xi}) + ML_{\xi}(-\chi^{\xi})}$$

$$SE_{\xi}(\chi^{\xi}) = \frac{2}{ML_{\vartheta}(i^{\xi}\chi^{\xi}) + ML_{\xi}(-i^{\xi}\chi^{\xi})}$$

$$CS_{\xi}(\chi^{\xi}) = \frac{2i^{\xi}}{ML_{\xi}(i^{\xi}\chi^{\xi}) - ML_{\xi}(-i^{\xi}\chi^{\xi})}$$

Property 2. The MLF owns the following properties [19]:

$$D^{(\xi)}ML_{\xi}(\Lambda\chi^{\xi}) = \Lambda ML_{\xi}(\chi^{\xi})$$

$$ML_{\xi}(\chi^{\xi})ML_{\xi}(\lambda^{\xi}) = ML_{\xi}(\chi^{\xi} + \lambda^{\xi})$$

$$ML_{\xi}(\chi^{\xi})ML_{\xi}(-\lambda^{\xi}) = ML_{\xi}(\chi^{\xi} - \lambda^{\xi})$$

$$ML_{\xi}(\chi^{\xi})ML_{\xi}(i^{\xi}\lambda^{\xi}) = ML_{\xi}(\chi^{\xi} + i^{\xi}\lambda^{\xi})$$

$$ML_{\xi}(i^{\xi}\lambda^{\xi})ML_{\xi}(i^{\xi}\lambda^{\xi}) = ML_{\xi}(i^{\xi}\chi^{\xi} + i^{\xi}\lambda^{\xi})$$

The ND exact solutions

The aim of this section is to apply Yang's special function method to construct the ND exact solutions. To this end, we introduce the ND transformation [20]:

$$\Xi_{\xi}(x^{\xi},t^{\xi})=\psi_{\xi}(\chi^{\xi}),\ \ \chi^{\xi}=\rho^{\xi}x^{\xi}-\varpi^{\xi}t^{\xi} \tag{5}$$

Taking eq. (5) into eq. (1) yields:

$$\left(\omega^{2\xi}+c\rho^{2\xi}\right)\frac{\mathrm{d}^{2\xi}\psi_{\xi}}{\mathrm{d}\chi^{2\xi}}+\gamma\psi_{\xi}+\delta\psi_{\xi}^{3}=0 \tag{6}$$

In the view of Yang's special function method [21, 22], there are the results:

$$\left[D^{(\xi)}\Theta_{\xi}(\chi^{\xi})\right]^{2}=p\varepsilon_{2}^{2}\Theta_{\xi}^{2}(\chi^{\xi})+q\frac{\varepsilon_{2}^{2}}{\varepsilon_{1}^{2}}\Theta_{\xi}^{4}(\chi^{\xi}) \tag{7}$$

Its ND exact solutions are given as:

$$\Theta_{\xi}(\chi^{\xi})=\begin{cases}\varepsilon_{1}SH_{\xi}(\varepsilon_{2}\chi^{\xi})\ \text{ for }\ p=1,\ q=-1\\ \varepsilon_{1}CH_{\xi}(\varepsilon_{2}\chi^{\xi})\ \text{ for }\ p=1,\ q=1\\ \varepsilon_{1}SE_{\xi}(\varepsilon_{2}\chi^{\xi})\ \text{ for }\ p=-1,\ q=1\\ \varepsilon_{1}CS_{\xi}(\varepsilon_{2}\chi^{\xi})\ \text{ for }\ p=-1,\ q=1\end{cases} \tag{8}$$

To solve eq. (6), we multiply both sides of eq. (6) by $\mathrm{d}^{\xi}\psi_{\xi}/\mathrm{d}\chi^{\xi}$ as:

$$\left(\omega^{2\xi}+c\rho^{2\xi}\right)\frac{\mathrm{d}^{2\xi}\psi_{\xi}}{d\chi^{2\xi}}\frac{\mathrm{d}^{\xi}\psi_{\xi}}{d\chi^{\xi}}+\gamma\psi_{\xi}\frac{\mathrm{d}^{\xi}\psi_{\xi}}{d\chi^{\xi}}+\delta\psi_{\xi}^{3}\frac{\mathrm{d}^{\xi}\psi_{\xi}}{d\chi^{\xi}}=0 \tag{9}$$

Taking the LFI of previous equation and ignoring the integral constant yields:

$$\left(\frac{\mathrm{d}^{\xi}\psi_{\xi}}{\mathrm{d}\chi^{\xi}}\right)=-\frac{\gamma}{\omega^{2\xi}+c\rho^{2\xi}}\psi_{\xi}^{2}-\frac{\delta}{2\left(\omega^{2\xi}+c\rho^{2\xi}\right)}\psi_{\xi}^{4} \tag{10}$$

By comparing eq. (10) and eq. (7), we can get the solutions in just one step as:
Set 1.

$$-\frac{\gamma}{\omega^{2\xi}+c\rho^{2\xi}}=\varepsilon_{2}^{2},\ -\frac{\delta}{2\left(\omega^{2\xi}+c\rho^{2\xi}\right)}=-\frac{\varepsilon_{2}^{2}}{\varepsilon_{1}^{2}}\ \left(\text{for }\ p=1,\ q=1\right)$$

Thus we have:

$$\varepsilon_{1}=\pm\sqrt{-\frac{2\gamma}{\delta}},\ \ \varepsilon_{2}=\pm\sqrt{-\frac{\gamma}{\omega^{2\xi}+c\rho^{2\xi}}} \tag{11}$$

Then the ND exact solution of eq. (1) on the CS is obtained as:

$$\Xi_{\xi}(x^{\xi},t^{\xi})=\pm\sqrt{-\frac{2\gamma}{\delta}}SH_{\xi}\left[\sqrt{-\frac{\gamma}{\omega^{2\xi}+c\rho^{2\xi}}}(\rho^{\xi}x^{\xi}-\varpi^{\xi}t^{\xi})\right] \tag{12}$$

Set 2.

$$-\frac{\gamma}{\omega^{2\xi}+c\rho^{2\xi}}=\varepsilon_{2}^{2},\ -\frac{\delta}{2\left(\omega^{2\xi}+c\rho^{2\xi}\right)}=\frac{\varepsilon_{2}^{2}}{\varepsilon_{1}^{2}}\ \left(\text{for }\ p=1,\ q=1\right)$$

So there are:

$$\varepsilon_1 = \pm\sqrt{\frac{2\gamma}{\delta}},\ \varepsilon_2 = \pm\sqrt{\frac{\gamma}{\omega^{2\xi} + c\rho^{2\xi}}} \tag{13}$$

Thus the ND exact solution of eq. (1) on the CS can be got as:

$$\Xi_\xi(x^\xi, t^\xi) = \pm\sqrt{\frac{2\gamma}{\delta}} CH_\xi\left[\sqrt{-\frac{\gamma}{\omega^{2\xi} + c\rho^{2\xi}}}(\rho^\xi x^\xi - \varpi^\xi t^\xi)\right] \tag{14}$$

Set 3.

$$-\frac{\gamma}{\omega^{2\xi} + c\rho^{2\xi}} = -\varepsilon_2^2,\ -\frac{\delta}{2\left(\omega^{2\xi} + c\rho^{2\xi}\right)} = \frac{\varepsilon_2^2}{\varepsilon_1^2}\ \left(\text{for}\ \ p = -1,\ q = 1\right)$$

Thus we have:

$$\varepsilon_1 = \pm\sqrt{-\frac{2\gamma}{\delta}},\ \varepsilon_2 = \pm\sqrt{\frac{\gamma}{\omega^{2\xi} + c\rho^{2\xi}}} \tag{15}$$

Then the ND exact solution of eq. (1) on the CS is obtained:

$$\Xi_\xi(x^\xi, t^\xi) = \pm\sqrt{-\frac{2\gamma}{\delta}} SE_\xi\left[\sqrt{\frac{\gamma}{\omega^{2\xi} + c\rho^{2\xi}}}(\rho^\xi x^\xi - \varpi^\xi t^\xi)\right] \tag{16}$$

Set 4.

$$-\frac{\gamma}{\omega^{2\xi} + c\rho^{2\xi}} = -\varepsilon_2^2,\ -\frac{\delta}{2\left(\omega^{2\xi} + c\rho^{2\xi}\right)} = \frac{\varepsilon_2^2}{\varepsilon_1^2}\ \left(\text{for}\ \ p = -1,\ q = 1\right)$$

Thus we have:

$$\varepsilon_1 = \pm\sqrt{-\frac{2\gamma}{\delta}},\ \varepsilon_2 = \pm\sqrt{\frac{\gamma}{\omega^{2\xi} + c\rho^{2\xi}}} \tag{17}$$

Then the ND exact solution of eq. (1) on the CS is obtained:

$$\Xi_\xi(x^\xi, t^\xi) = \pm\sqrt{-\frac{2\gamma}{\delta}} CS_\xi\left[\sqrt{\frac{\gamma}{\omega^{2\xi} + c\rho^{2\xi}}}(\rho^\xi x^\xi - \varpi^\xi t^\xi)\right] \tag{18}$$

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Nomenclature

x – space co-ordinate, [m]

t – time co-ordinate, [second]

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