A NEW GENERAL FRACTIONAL DERIVATIVE RELAXATION PHENOMENON

by

Xiao-Jun YANG [a,b,c*], ***Abdulrahman Ali ALSOLAMI*** [b], ***and Sohail NADEEM*** [d]

[a] State Key Laboratory for Geo-Mechanics and Deep Underground Engineering, and School of Mathematics, China University of Mining and Technology, Xuzhou, China
[b] Department of Mathematics, Faculty of Science, King Abdulaziz University, Jeddah, Saudi Arabia
[c] Department of Mathematics, College of Science, Kyung Hee University, Seoul, Republic of Korea
[d] School of Mathematics and Statistics, Qujing Normal University, Qujing, Yunnan, China
[e] Department of Mathematics, Quaid-I-Azam University, Islamabad, Pakistan

Original scientific paper
https://doi.org/10.2298/TSCI2301721Y

This article addresses a novel anomalous relaxation model with the new general fractional derivative of the Sonine kernel. This operator is considered in the sense of general fractional derivative proposed in the work [Yang et al., General fractional derivatives with applications in viscoelasticity, Academic Press, New York, USA, 2020]. The solution of the mathematical model is obtained with the aid of Laplace transform. The comparison among the classical and anomalous relaxation models is discussed in detail. This result is proposed as a mathematical tool to model the anomalous relaxation behavior of the complex materials.

Key words: *anomalous relaxation, general fractional derivative, complex materials, Sonine kernel, Laplace transform*

Introduction

Recently, the general fractional calculus (GFC) has an increasing interesting from mathematics to engineering. History, theory and applications of the GFC were investigated in [1-5]. There are well-known results of the GFC, such as the Atangana-Baleanu fractional derivative [6], and Yang-Abdel-Aty-Cattani fractional derivative [7]. Omame *et al.* [8] applied the Atangana-Baleanu fractional derivative to give the mathematical models of the COVID-19 and tuberculosis co-infection. Jleli *et al.* [9] suggested the Yang-Abdel-Aty-Cattani fractional derivative to propose the general fractional wave model. There are more applications in GFC [10, 11].

There are many mathematical models for the anomalous relaxation phenomena, developed by different authors, such as Mainardi [12], Metzler and Nonnenmacher [13], Tofighi and Golestani [14], Chen [15], de Oliveira, *et al.* [16], Giona, M., *et al.* [17], and Yang *et al.* [18]. The target of this paper is to study a general fractional derivative relaxation equation with the new general fractional derivative of the Sonine kernel.

[*] Corresponding author; e-mail: dyangxiaojun@163.com

A new general fractional derivative of the Sonine kernel

In 1884, Nikolay Yakovlevich Sonin (Sonine), a Russian mathematician proposed well-known Sonine functions, defined as [3, 19]

$$\Theta\left(t^{\alpha}\right)=\sum_{i=0}^{\infty}\frac{t^{i-\alpha}}{\Gamma\left(i+1\right)\Gamma\left(i-\alpha+1\right)} \tag{1}$$

and

$$\Phi\left(t^{\alpha}\right)=\sum_{i=0}^{\infty}\frac{t^{i+\alpha-1}}{\Gamma\left(i+1\right)\Gamma\left(i+\alpha\right)}, \tag{2}$$

where

$$\Gamma\left(t\right)=\int_{0}^{\infty}\mathrm{e}^{-x}x^{t-1}\mathrm{d}x \tag{3}$$

is the gamma function [3], $t > 0$ and $0 < \alpha < 1$.

There are [3, 19]

$$\Theta\left(\kappa t^{\alpha}\right)=\sum_{i=0}^{\infty}\frac{\kappa^{i}t^{i-\alpha}}{\Gamma\left(i+1\right)\Gamma\left(i-\alpha+1\right)} \tag{4}$$

and

$$\Phi\left(\kappa t^{\alpha}\right)=\sum_{i=0}^{\infty}\frac{\kappa^{i}t^{i+\alpha-1}}{\Gamma\left(i+1\right)\Gamma\left(i+\alpha\right)}, \tag{5}$$

where $\kappa > 0$.

The Laplace transforms of eqs. (1)-(4) are given as

$$\mathbb{L}\left\{\Theta\left(t^{\alpha}\right)\right\}=\mathbb{L}\left\{\sum_{i=0}^{\infty}\frac{t^{i-\alpha}}{\Gamma\left(i+1\right)\Gamma\left(i-\alpha+1\right)}\right\}=\sum_{i=0}^{\infty}\frac{s^{-(i-\alpha+1)}}{\Gamma\left(i+1\right)}=s^{\alpha-1}\mathrm{e}^{1/s}, \tag{6}$$

$$\mathbb{L}\left\{\Phi\left(t^{\alpha}\right)\right\}=\mathbb{L}\left\{\sum_{i=0}^{\infty}\frac{t^{i+\alpha-1}}{\Gamma\left(i+1\right)\Gamma\left(i+\alpha\right)}\right\}=s^{-\alpha}\mathrm{e}^{1/s}, \tag{7}$$

$$\mathbb{L}\left\{\Theta\left(\kappa t^{\alpha}\right)\right\}=\mathbb{L}\left\{\sum_{i=0}^{\infty}\frac{\kappa^{i}t^{i-\alpha}}{\Gamma\left(i+1\right)\Gamma\left(i-\alpha+1\right)}\right\}=\sum_{i=0}^{\infty}\frac{\kappa^{i}s^{-(i-\alpha+1)}}{\Gamma\left(i+1\right)}=s^{\alpha-1}\mathrm{e}^{\kappa/s} \tag{8}$$

and

$$\mathbb{L}\left\{\Phi\left(\kappa t^{\alpha}\right)\right\}=\mathbb{L}\left\{\sum_{i=0}^{\infty}\frac{\kappa^{i}t^{i+\alpha-1}}{\Gamma\left(i+1\right)\Gamma\left(i+\alpha\right)}\right\}=\sum_{i=0}^{\infty}\frac{\kappa^{i}s^{-(i+\alpha)}}{\Gamma\left(i+1\right)}=s^{-\alpha}\mathrm{e}^{\kappa/s}, \tag{9}$$

where

$$\mathbb{L}\left\{f\left(t\right)\right\}=f\left(s\right)=\int_{0}^{\infty}\mathrm{e}^{-st}f\left(t\right)\mathrm{d}t \tag{10}$$

$$\mathbb{L}\left\{\frac{t^{i-\alpha}}{\Gamma\left(i-\alpha+1\right)}\right\}=s^{-(i-\alpha+1)} \tag{11}$$

and

$$\mathbb{L}\left\{\frac{t^{i+\alpha-1}}{\Gamma(i+\alpha)}\right\}=s^{-(i+\alpha)}. \tag{12}$$

Let $L_p[a, b]$ be the space of the integrable functions such that $\phi \in L_p[a, b]$ and $\varphi \in L_p[a, b]$, where $0 \le a < b < \infty$.

A general fractional integral of the function $\varphi(t)$ is given as [2]

$$I^{k,(\alpha)}\varphi(t)=\int_0^t \Theta\left[\kappa(t-\tau)^\alpha\right]\varphi(\tau)\,\mathrm{d}\tau \tag{13}$$

A general fractional derivative of the function $\phi(t)$ is given as [2]

$$D^{k,(\alpha)}\phi(t)=\int_0^t \Phi\left[-\kappa(t-\tau)^\alpha\right]\phi^{(1)}(\tau)\,\mathrm{d}\tau \tag{14}$$

where $\Theta(t)$ and $\Phi(t)$ in eqs. (13) and (14) are called the Sonine kernels.

The Laplace transforms of eqs. (13) and (14) are given as

$$\mathbb{L}\left\{I^{k,(\alpha)}\varphi(t)\right\}=s^{\alpha-1}\mathrm{e}^{\kappa/s}\varphi(s) \tag{15}$$

and

$$\mathbb{L}\left\{D^{k,(\alpha)}\phi(t)\right\}=s^{-\alpha}\mathrm{e}^{-\kappa/s}\left[s\phi(s)-\phi(0)\right] \tag{16}$$

with the relation [2]

$$\mathbb{L}\left\{\int_0^t \Theta\left(\kappa\tau^\alpha\right)\Phi\left[-\kappa(t-\tau)^\alpha\right]\mathrm{d}\tau\right\}=\frac{1}{s}. \tag{17}$$

From eq. (17) we obtain the Sonine condition [2]

$$\int_0^t \Theta\left(\kappa t^\alpha\right)\Phi\left[-\kappa(t-\tau)^\alpha\right]\mathrm{d}\tau=1. \tag{18}$$

The Riemann-Liouville fractional integral of the function $\varphi(t)$ is defined as [1-3]

$$I^{(\alpha)}\varphi(t)=\frac{1}{\Gamma(\alpha)}\int_0^t \frac{\varphi(\tau)}{(t-\tau)^{1-\alpha}}\,\mathrm{d}\tau. \tag{19}$$

The Liouville-Sonine-Caputo fractional derivative of the function $\phi(t)$ is defined as [1-3]

$$D^{(\alpha)}\phi(t)=\frac{1}{\Gamma(1-\alpha)}\int_0^t \frac{\phi^{(1)}(\tau)}{(t-\tau)^{\alpha}}\,\mathrm{d}\tau \tag{20}$$

Equation (20) is also called the Caputo fractional derivative [1-3].

A new anomalous relaxation model

Let us suggest an anomalous relaxation model

$$D^{k,(\alpha)}\Lambda(t)+\mu\Lambda(t)=0, \tag{21}$$

subjected to the initial condition

$$\Lambda(t=0)=\Lambda(0), \tag{22}$$

where $\mu > 0$ is the relaxation constant.

The Laplace transform of eq. (21) reads [3]

$$s^{-\alpha}\mathrm{e}^{-\kappa/s}\left[s\Lambda(s)-\Lambda(0)\right]+\Lambda(s)=0, \tag{23}$$

where

$$\mathbb{L}\{\Lambda(t)\}=\Lambda(s)=\int_0^\infty \mathrm{e}^{-st}\Lambda(t)\mathrm{d}t. \tag{24}$$

By eq. (23), we have

$$\Lambda(s)=\Lambda(0)\frac{s^{-\alpha}\mathrm{e}^{-\kappa/2}}{s^{1-\alpha}\mathrm{e}^{-\kappa/s}+1}, \tag{25}$$

which can be written as

$$\Lambda(s)=\frac{\Lambda(0)}{s+s^{\alpha}\mathrm{e}^{\kappa/s}}. \tag{26}$$

Let us denote

$$\frac{1}{1+x^{\alpha-1}\mathrm{e}^{\kappa/x}}=\sum_{n=0}^{\infty}\aleph(\kappa,n,\alpha)\frac{x^{-n}}{n!}, \tag{27}$$

where

$$\aleph(\kappa,n,\alpha)=\lim_{x\to 0}\frac{\mathrm{d}^n}{\mathrm{d}x^n}\left[\frac{1}{1+x^{1-\alpha}\mathrm{e}^{\kappa x}}\right]$$

are called the hot numbers.

Thus eq. (26) becomes

$$\Lambda(s)=\frac{\Lambda(0)}{s+s^{\alpha}\mathrm{e}^{\kappa/s}}=\Lambda(0)\sum_{n=0}^{\infty}\aleph(\kappa,n,\alpha)\frac{s^{-(n+1)}}{n!}, \tag{28}$$

which leads to

$$\Lambda(t)=\Lambda(0)\sum_{n=0}^{\infty}\aleph(\kappa,n,\alpha)\frac{t^n}{(n!)^2}. \tag{29}$$

Compared with fractional and classical relaxation problems

The classical relaxation equation is given as [12]

$$\mu D^{(1)}X(t)=X(t), \tag{30}$$

subjected to the initial condition

$$X(t=0)=X(0), \tag{31}$$

where μ is the relaxation constant.

The solution of eq. (30) reads [12]

$$X(t)=X(0)\mathrm{e}^{-\mu t}. \tag{32}$$

The fractional relaxation equation is considered as [12]

$$\mu D^{(\alpha)}Y(t)=Y(t), \tag{33}$$

subjected to the initial condition

$$Y(t=0)=Y(0), \tag{34}$$

where μ is the relaxation constant.

The solution of eq. (33) is expressed as [12]

$$Y(t) = Y(0) E_\alpha\left(-\mu t^\alpha\right), \tag{35}$$

where

$$E_\alpha\left(t^\alpha\right) = \sum_{n=0}^{\infty} \frac{t^{n\alpha}}{\Gamma(n\alpha + 1)} \tag{36}$$

is the Mittag-Leffler function [3].

Conclusion

In this work we have proposed a novel general fractional derivative relaxation model for the first time. We have considered the hot numbers to represent its solution. The result is compared with the solutions for the classical and fractional relaxation equations. It is showed that this has more complex behaviors for the materials.

Acknowledgment

This work is supported by the Yue-Qi Scholar of the China University of Mining and Technology (No. 102504180004).

Nomenclature

t – time co-ordinate, [second]

References

[1] Yang, X. J., *et al.*, *General Fractional Derivatives with Applications in Viscoelasticity*, Academic Press, New York, USA, 2020
[2] Yang, X. J., *General Fractional Derivatives: Theory, Methods and Applications*, Chapman and Hall/CRC, London, UK, 2019
[3] Yang, X. J., *Theory and Applications of Special Functions for Scientists and Engineers,* Springer Singapore, New York, USA, 2021
[4] Tarasov, V. E., General Fractional Dynamics, *Mathematics*, *9* (2021), 13, ID1464
[5] Luchko, Y., General Fractional Integrals and Derivatives with the Sonine Kernels, *Mathematics*, *9* (2021), 6, 594
[6] Atangana, A., Baleanu, D., New Fractional Derivatives with Non-Local and Non-Singular Kernel: Theory and Application Heat Transfer Model, *Thermal Science*, *20* (2016), 2, pp. 763-769
[7] Yang, X. J., *et al.*, A New General Fractional-Order Derivative with Rabotnov Fractional-Exponential Kernel Applied to Model the Anomalous Heat Transfer, *Thermal Science*, *23* (2019), 3A, pp. 1677-1681
[8] Omame, A., *et al*., A Fractional-Order Model for COVID-19 and Tuberculosis Co-Infection Using Atangana–Baleanu Derivative, *Chaos, Solitons and Fractals*, *153* (2021), Dec., ID111486
[9] Jleli, M., *et al*., Analytical Approach for Time Fractional Wave Equations in the Sense of Yang-Abdel-Aty-Cattani via the Homotopy Perturbation Transform Method, *Alexandria Engineering Journal*, *59* (2020), 5, pp. 2859-2863
[10] Yang, X. J., *et al*., Fundamental Solutions of Anomalous Diffusion Equations with the Decay Exponential Kernel, *Mathematical Methods in the Applied Sciences*, *42* (2019), 11, pp. 4054-4060
[11] Yang, X. J., *et al*., A New General Fractional-Order Derivative with Rabotnov Fractional-Exponential Kernel, *Thermal Science*, *23* (2019), 6B, pp. 3711-3718
[12] F. Mainardi, Fractional Relaxation-Oscillation and Fractional Diffusion-Wave Phenomena, *Chaos, Solitons and Fractals*, *7* (1996), 9, pp. 1461-1477
[13] Metzler, R., Nonnenmacher, T. F., Fractional Relaxation Processes and Fractional Rheological Models for the Description of a Class of Viscoelastic Materials, *International Journal of Plasticity*, *19* (2003), 7, pp. 941-959

[14] Tofighi, A., Golestani, A., A Perturbative Study of Fractional Relaxation Phenomena, *Physica A: Statistical Mechanics and its Applications*, *387* (2008), 8-9, pp. 1807-1817
[15] Chen, W., *et al*., Investigation on Fractional and Fractal Derivative Relaxation-Oscillation Models, *International Journal of Non-Linear Sciences and Numerical Simulation*, *11* (2010), 1, pp. 3-10
[16] De Oliveira, *et al*., Fractional Models of Anomalous Relaxation Based on the Kilbas and Saigo Function, *Meccanica*, *49* (2014), 9, pp. 2049-2060
[17] Giona, M., *et al.*, Fractional Diffusion Equation and Relaxation in Complex Viscoelastic Materials, *Physica A: Statistical Mechanics and its Applications*, *191* (1992), 1-4, pp. 449-453
[18] Yang, X. J., Fractional Derivatives of Constant and Variable Orders Applied to Anomalous Relaxation Models in Heat Transfer Problems, *Thermal Science*, *21* (2017), 3, pp. 1161-1171
[19] Sonine, N., Sur la Généralisation d'une Formule d'Abel (in French), *Acta Mathematica*, *4* (1884), 1, pp. 171-176

Paper submitted: November 7, 2022
Paper revised: November 21, 2022
Paper accepted: November 29, 2022