# TEMPORAL SECOND-ORDER FINITE DIFFERENCE SCHEMES FOR VARIABLE-ORDER TIME-FRACTIONAL GENERALIZED OLDROYD-B FLUID MODEL

by

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*In this paper, we study the variable-order generalized time fractional Oldroyd-B fluid model, use the reduced order method and the L2-1σ method to establish the differential format with second-order accuracy, prove the stability and convergence of the format, and give numerical examples to illustrate the effectiveness of the differential format.*

Key words: *Caputo fractional derivative, fractional Oldroyd-B fluid model, fractional diffusion equation, finite difference method*

## Introduction

The Oldroyd-B fluid model is an important part of the non-Newtonian fluid model, many scholars have studied a series of results on the Oldroyd-B fluid model [1-5] obtained the analytical solutions of the fractional partial differential fluid model. In the recent works [6, 7], scholars used finite difference method, finite element method*, etc.* to establish the numerical solution format of Oldroyd-B fluid model. In [8], the numerical solution of the generalized Oldroyd-B fluid model have been discussed, two numerical difference formats with time accuracy less than second-order are proposed, the stability and convergence of the formats are proved using energy methods. In this paper, the following variable order generalized time-fractional Oldroyd-B fluid model are considered:

$$s_1 D_t^{\alpha(t)} u + s_2 \frac{\partial u}{\partial t} = s_3 \frac{\partial^2 u}{\partial x^2} + s_4 D_t^{\beta(t)} \frac{\partial^2 u}{\partial x^2} + f(x,t), \ \ (x,t) \in (0,L) \times (0,T] \tag{1}$$

$$u\left(x,0\right) = \phi_1(x), \ \frac{\partial u}{\partial t}\left(x,0\right) = \phi_2(x), \ \ x \in [0,L] \tag{2}$$

$$u\left(0,t\right) = u\left(L,t\right) = 0, \ \ t \in [0,T] \tag{3}$$

where $1 < \alpha(t) < 2$, $0 < \beta(t) < 1$, and both $D_t^{\alpha(t)}u$ and $D_t^{\beta(t)}u$ are defined as [9, 10]. Based on the aforementioned study, we will use the new difference method mentioned in [11] to establish the second-order accuracy difference format for the variable-order time fractional Oldroyd-B fluid model.

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Firstly, we reduce the model to a lower-order form using the order reduction method, then derive a difference format for the variable-order generalized time fractional Oldroyd-B fluid model, which has second-order accuracy in time and space, and prove the stability and convergence of the format finally use numerical example results to show that the difference format is valid.

## The finite difference scheme

Firstly, for the domain $[0, L] \times [0, T]$ we take the mesh points $x_i = ih$, $i = 0, 1,..., M$ and $t_n = n\tau$, $n = 0, 1,..., N$, where $h = L/M$ and $\tau = T/N$ are the step size.

Denote:

$$u_j^n = u(x_j, t_n),\ \gamma_n = \gamma(t_n),\ V_h^\tau = \{v_i^n = v(x_i, t_n) \big| v_0 = v_M = 0\},\ \text{and}$$

$$u^{n+\sigma_n} = \sigma_n u^{n+1} + (1-\sigma_n)u^n,\ u_i^{n-1/2} = \frac{u_i^n + u_i^{n-1}}{2},\ \delta_t u_i^n = \frac{u_i^n - u_i^{n-1}}{\tau},\ \delta_t u_i^n = \frac{u_i^n - u_i^{n-1}}{\tau}$$

$$\delta_x u_i^n = \frac{u_i^n - u_{i-1}^n}{h},\ \delta_x^2 u_i^n = \frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{h^2},\ (u,v) = h\sum_{i=1}^{M-1} u_i v_i, \langle \delta_x u, \delta_x v\rangle = h\sum_{i=1}^{M} \delta_x u_i \delta_x v_i$$

$$\|u\| = \sqrt{(u,u)},\ |u|_1 = \sqrt{\langle \delta_x u, \delta_x u\rangle},\ \text{we can obtain}\ \left(\delta_x^2 u, v\right) = -\langle \delta_x u, \delta_x v\rangle$$

*Lemma 1*. [12] For any $u \in V_h^\tau$, we can get:

$$\|u\|_\infty \le \frac{\sqrt{L}}{2}|u|_1,\ \|u\| \le \frac{L}{\sqrt{6}}|u|_1,\ \|u^{n+\sigma_n}\| \le \|u^n\| + \|u^{n+1}\|$$

*Lemma 2*. [13] Assume $u \in C^3[0, T]$, we have:

$$\bar{\delta}_t u(t_n) \equiv \frac{1}{2\tau}\left[(2\sigma_n + 1)u^{n+1} - 4\sigma_n u^n + (2\sigma_n - 1)u^{n-1}\right] = \frac{\mathrm{d}u}{\mathrm{d}t}(t_{n+\sigma_n}) + \mathcal{O}(\tau^2)$$

Firstly we use the order reduction method to (1), let $\gamma(t) = \alpha(t) - 1$, $\partial u/\partial t = v$, then we have:

$$s_1 D_t^{\gamma(t)} v + s_2 v = s_3 \frac{\partial^2 u}{\partial x^2} + s_4\, {}^{RL}D_t^{-[1-\beta(t)]} \frac{\partial^2 v}{\partial x^2} + f(x,t),\ x \in (0,L),\ t \in (0,T] \tag{4}$$

Denote

$$y(\sigma) = \sigma - \left[1 - \frac{1}{2}\gamma(t_n + \sigma\tau)\right]$$

$\sigma_n$ satisfies $y(\sigma) = 0$, for $n = 1,..., N-1$, $\sigma_n$ can be generated by Newton's method, and $1/2 < \sigma_n < 1$. To discretize $D_t^{\gamma(t)} v(x, t)$ at $(x_i, t_{n+\sigma_n})$ by L1-2$\sigma$ formula, we have [9]:

$$s_1 D_t^{\gamma(t)} v(x,t) = A_n \sum_{k=0}^{n} c_{n-k}^{(n)} \left(v_i^{k+1} - v_i^k\right) + r_i^{n+\sigma} \tag{5}$$

*Lemma 4*. [13] If $0 < \gamma(t) < 1$, then $\{c_k^{(n)}\}$ satisfy the following relation:

$$(2\sigma_n - 1)c_0^{(n)} - \sigma_n c_1^{(n)} > 0, A_n c_0^{(n)} > A_n c_1^{(n)} > \cdots > A_n c_n^{(n)} > M_1 > 0$$

*Lemma 5*. [9] Suppose that $\alpha'(t) \le 0$ when $0 \le t \le T$, for any fixed $n (2 \le n \le N)$ we have:

$$A_n c_k^{(n)} \le (1 + M_2\tau) A_{n-1} c_k^{(n-1)},\ \tau\sum_{k=1}^{n} A_n \left(c_{k-1}^{(k)} - c_k^{(k)}\right) \le M_3 < \infty,\ \tau\sum_{k=1}^{n} A_n c_k^{(k)} \le M_4 < \infty$$

*Lemma 6.* [11] For any $v^1, v^2..., v^n \in V_h^\tau$, we have the following estimate:

$$\left[\sum_{k=0}^{n} c_k^{(n)}\left(v^{k+1}-v^k\right),\sigma_n v^{k+1}+\left(1-\sigma_n\right)v^k\right]\geq\frac{1}{2}\sum_{k=0}^{n}c_k^{(n)}\left(\left\|v^{k+1}\right\|^2-\left\|v^k\right\|^2\right)$$

Following the same method as [11] to discretize:

$${}_0^{RL}D_t^{-(1-\beta(t))}v(x,t),\ at\ (x_i,t_{n+\sigma_n})$$

we have:

$${}_0^{RL}D_t^{-[1-\beta(t)]}v(x_i,t_{n+\sigma_n})=B_n\sum_{k=0}^{n+1}d_{n-k+1}^{(n)}v^k+r_2 \tag{6}$$

*Lemma 7.* [14] For any $v^1, v^2..., v^n \in V_h^\tau$, it holds that:

$$\sum_{s=1}^{n}\left(\sum_{k=0}^{s+1}B_s d_{s-k+1}^{(s)}v^k,v^{s+\sigma_n}\right)\geq-\frac{Q_1}{2}\left\|v^0\right\|^2-3Q_2\left(5+6\sigma_n\right)\tau\sum_{s=1}^{n+1}\left\|v^s\right\|^2-20Q_2\sigma_n\tau\left\|v^1\right\|^2 \tag{7}$$

To discretise:

$$D_t^{\gamma(t)}v(x,t),\ \text{and}\ {}_0^{RL}D_t^{-(1-\beta(t))}v(x,t),\ \text{at}\ (x_i,t_{1/2})$$

we have:

$$s_1D_t^{\gamma}v(x_i,t_{\frac{1}{2}})=\hat{c}_0\left(v_i^1-v_i^0\right)+r_i^{1/2},\ s_4{}^{RL}D_t^{-[1-\beta(t)]}\frac{\partial^2 v}{\partial x^2}(x_i,t_{1/2})=B_0\left(d_1\delta_x^2v_i^0+d_0\delta_x^2v_i^1\right)+\hat{r}_i^{1/2} \tag{8}$$

where

$$\hat{c}_0=\frac{s_1\tau^{-\gamma_{1/2}}}{2^{1-\gamma_{1/2}}\Gamma(2-\gamma_{1/2})},\ B_0=\frac{s_4\tau^{1-\beta_{1/2}}}{\Gamma(2-\beta_{1/2})},\ d_1=\left[1-\frac{1}{2(2-\beta_{1/2})}\right]\left(\frac{1}{2}\right)^{1-\beta_{1/2}}$$

$$d_0=\left(\frac{1}{2}\right)^{2-\beta_{1/2}},\ r_i^{1/2}=\mathcal{O}\left(\tau^{2-\gamma_{1/2}}\right),\ \hat{r}_i^{1/2}=\mathcal{O}(\tau^2+h^2)$$

We propose finite difference scheme for the initial boundary value problem eqs. (2)-(4):

$$A_n\sum_{k=0}^{n}c_{n-k}^{(n)}\left(v_i^{k+1}-v_i^k\right)+s_2v_i^{n+\sigma_n}=s_3\delta_x^2u_i^{n+\sigma_n}+B_n\sum_{k=0}^{n+1}d_{n-k+1}^{(n)}\delta_x^2v_i^k+f_i^{n+\sigma_n}$$
$$1\leq i\leq M-1,\ 0\leq n\leq N-1 \tag{9}$$

$$\hat{c}_0\left(v_i^1-v_i^0\right)+s_2v_i^{1/2}=s_3\delta_x^2u_i^{1/2}+B_0\left(d_1\delta_x^2v_i^0+d_0\delta_x^2v_i^1\right)+f_i^{1/2},\ 1\leq i\leq M-1 \tag{10}$$

$$\delta_t\delta_x^2u_i^{1/2}=\delta_x^2v_i^{1/2}+g_i^{1/2},\ 1\leq i\leq M-1 \tag{11}$$

$$\overline{\delta}_t\delta_x^2u_i^n=\delta_x^2v_i^{n+\sigma_n}+g_i^{n+\sigma_n},\ 1\leq i\leq M-1,\ 1\leq n\leq N-1 \tag{12}$$

$$u_i^0=\phi_1(x_i),\ v_i^0=\phi_2(x_i),\ 1\leq i\leq M-1,\ u_0^n=u_M^n=0,\ 0\leq n\leq N \tag{13}$$

By the Taylor expansion and (5)-(8), the truncation error of (9)-(12) is in the following

$$R_i^n=\mathcal{O}(\tau^2+h^2),\ R_i^{1/2}=\mathcal{O}\left(\tau^{2-\gamma_{1/2}}+h^2\right),\ r_i^{1/2}=\mathcal{O}\left(\tau^2+h^2\right),\ r_i^n=\mathcal{O}\left(\tau^2+h^2\right)$$

## Convergence and stability

*Lemma 8.* [13] For any $w^1, w^2..., w^n \in V_h^\tau$, it holds that:

$$\left(\overline{\delta}_t w^n,w^{n+\sigma_n}\right)\geq\frac{1}{4\tau}\left(E^{n+1}-E^n\right),\ 1\leq n\leq N-1 \tag{14}$$

where

$$E^{n+1}=(2\sigma_n+1)\left\|w^{n+1}\right\|^2-(2\sigma_n-1)\left\|w^n\right\|^2+(2\sigma_n^2+\sigma_n-1)\left\|w^{n+1}-w^n\right\|^2,\ E^{n+1}\ge\frac{1}{\sigma_n}\left\|w^{n+1}\right\|^2$$

*Lemma 9***.** [15] Let $u$, $v$, $g^n \in V_h^\tau$ be non-negative temporal grid functions, $g^n$ be non-decreasing, and $A$, $B$ be non-negative constants.

(I) If

$$v^n\le(1+\tau B)v^{n-1}+\tau w^{n-1}\ \text{ then }\ v^n\le\exp\left(Bn\tau\right)\left(v^0+\tau\sum_{p=0}^{n-1}w^p\right)$$

(II) If

$$v^n\le g^n+B\tau\sum_{p=1}^{n-1}v^p,\ \text{ then }\ v^n\le g^n\exp\left(Bn\tau\right)$$

When $n=0$, substituting (11) to (9) and taking product with $-\delta_x^2v^{1/2}$, we have:

$$\frac{A_0c_0^{(0)}}{2}\left(\left|v^1\right|_1^2-\left|v^0\right|_1^2\right)+\frac{s_2}{2}\left[\sigma_0\left|v^1\right|_1^2+\left(1-\sigma_0\right)\left|v^0\right|_1^2+\left\langle\delta_xv^0,\delta_xv^1\right\rangle\right]=$$
$$=s_3\left(\delta_x^2u^0,-\delta_x^2v^{1/2}\right)-s_3\sigma_0\tau\left\|\delta_x^2v^{1/2}\right\|^2-\frac{B_0d_0^{(0)}}{2}\left(\left\|\delta_xv^1\right\|^2+\left\langle\delta_x^2v^1,\delta_x^2v^0\right\rangle\right)-$$
$$-\frac{B_0d_1^{(0)}}{2}\left(\left\|\delta_xv^0\right\|^2+\left\langle\delta_x^2v^1,\delta_x^2v^0\right\rangle\right)+\left\langle\delta_xf^{\sigma_0},\delta_xv^{1/2}\right\rangle+s_3\sigma_0\tau\left\langle\delta_xg^{1/2},\delta_xv^{1/2}\right\rangle$$

Using Cauchy-Schwarz inequality, we have:

$$\left|v^1\right|_1^2\le\left[5+\frac{2s_2^2}{(A_0c_0^{(0)})^2}\right]\left|v^0\right|_1^2+\frac{\left(B_0d_1^{(0)}+B_0d_0^{(0)}\right)^2}{8B_0d_0^{(0)}}\left\|\delta_x^2v^0\right\|^2+\frac{24s_3^2}{\left(A_0c_0^{(0)}\right)^2}\left|\delta_x^2u^0\right|_1^2+$$
$$+\frac{24s_3}{\left(A_0c_0^{(0)}\right)^2}\left(s_3\sigma_0^2\tau^2\left|\delta_xg^{\frac{1}{2}}\right|_1^2+\frac{1}{s_3}\left|\delta_xf^{\sigma_0}\right|_1^2\right)\tag{15}$$

By eq. (11) taking inner product with $v^{1/2}$ and $-s_3u^{1/2}$, respectively and using Cauchy-Schwarz inequality and noticing $\hat{c}_0$, we have:

$$s_3\left|u^1\right|_1^2\le\frac{B_0(d_0+d_1)^2}{2d_0}\tau\left|v^0\right|_1^2+\frac{3}{\hat{c}_0}\tau\left\|v^0\right\|^2+2s_3\left|u^0\right|_1^2+\frac{6s_3}{L^2}\left\|u^0\right\|^2+$$
$$+\frac{L^2}{3s_3}\tau^2\left\|g^{1/2}\right\|^2+s_1\Gamma(2-\gamma_{1/2})2^{2-\gamma_{1/2}}\tau^{1+\gamma_{1/2}}\left\|f^{1/2}\right\|^2\tag{16}$$

When $n\le1$, eq. (9) taking inner product with $v^{n+\sigma n}$ and using *Lemma 6*, we have:

$$\frac{1}{2}\left[A_nc_0^{(n,\gamma)}\left\|v^{n+1}\right\|^2-\sum_{k=1}^{n}A_n\left(c_{n-k}^{(n)}-c_{n-k+1}^{(n)}\right)\left\|v^k\right\|^2-A_nc_n^{(n)}\left\|v^0\right\|^2\right]+s_2\left\|v^{n+\sigma_n}\right\|^2\le$$
$$\le-s_3\left\langle\delta_xu^{n+\sigma_n},\delta_xv^{n+\sigma_n}\right\rangle-\sum_{k=0}^{n+1}B_nd_{n-k+1}^{(n)}\left\langle\delta_xv^k,\delta_xv^{n+\sigma_n}\right\rangle+\left(f^{n+\sigma_n},v^{n+\sigma_n}\right)\tag{17}$$

By (12) taking inner product with $-s_3u^{n+\sigma n}$, adding with eq. (17) and using *Lemma 8*, we have:

$$\frac{s_3}{4\tau}\left(F^{n+1}-F^n\right)+s_2\left\|v^{n+\sigma_n}\right\|^2+\frac{1}{2}\left[A_nc_0^{(n)}\left\|v^{n+1}\right\|^2-\sum_{k=1}^{n}A_n\left(c_{n-k}^{(n)}-c_{n-k+1}^{(n)}\right)\left\|v^k\right\|^2-A_nc_n^{(n)}\left\|v^0\right\|^2\right]\le$$
$$\le-\sum_{k=0}^{n+1}B_nd_{n-k+1}^{(n)}\left\langle\delta_xv^k,\delta_xv^{n+\sigma_n}\right\rangle+\left(f^{n+\sigma_n},v^{n+\sigma_n}\right)-s_3\left(g^{n+\sigma_n},u^{n+\sigma_n}\right)\tag{18}$$

Denoting by

$$G^{n+1} = s_3 F^{n+1} + 2\tau \sum_{k=2}^{n+1} A_n c_{n-k+1}^{(n)} \left\| v^k \right\|^2$$

and using *Lemma 5* we have:

$$s_3 F^n + 2\tau \sum_{k=2}^{n} A_n c_{n-k}^{(n)} \left\| v^k \right\|^2 \le \left(1 + M_2 \tau\right) \left( s_3 F^n + 2\tau \sum_{k=2}^{n} A_{n-1} c_{n-k}^{(n-1)} \left\| v^k \right\|^2 \right)$$

In fact, eq. (18) can be converted:

$$G^{n+1} \le (1+M_2\tau) G^n - 4\tau \sum_{k=0}^{n+1} B_n d_{n-k+1}^{(n)} \left\langle \delta_x v^k, \delta_x v^{n+\sigma_n} \right\rangle + 2\tau A_n c_n^{(n)} \left\| v^0 \right\|^2 - 4 s_2 \tau \left\| v^{n+\sigma_n} \right\|^2 +$$
$$+ 2\tau A_n \left( c_{n-1}^{(n)} - c_n^{(n)} \right) \left\| v^1 \right\|^2 + 4\tau \left( f^{n+\sigma_n}, v^{n+\sigma_n} \right) - 4 s_3 \tau \left( g^{n+\sigma_n}, u^{n+\sigma_n} \right)$$

Using *Lemmas 7* and *9(I)*, we have:

$$G^{n+1} \le \mathrm{e}^{M_2 n\tau} \Bigg[ G^1 + 2\tau \left\| v^0 \right\|^2 \sum_{p=1}^{n} A_p c_p^{(p)} + 2\tau \left\| v^1 \right\|^2 \sum_{p=1}^{n} A_p \left( c_{p-1}^{(p)} - c_p^{(p)} \right) + 80 Q_2 \sigma_n \tau \left| v^1 \right|_1^2 + 2 Q_1 \tau \left| v^0 \right|_1^2 +$$
$$+ 12\left(5 + 6\sigma_n\right) Q_2 \tau^2 \sum_{p=1}^{n+1} \left| v^p \right|_1^2 + 4\tau \sum_{p=1}^{n} \left( f^{p+\sigma_p}, v^{p+\sigma_p} \right) - 4 s_3 \tau \sum_{p=1}^{n} \left( g^{p+\sigma_p}, u^{p+\sigma_p} \right) - 4 s_2 \tau \sum_{p=1}^{n} \left\| v^{p+\sigma_p} \right\|^2 \Bigg]$$

Using *Lemma 5* and *Lemma 8* and

$$|\cdot|_1^2 \le \frac{4}{h^2} \|\cdot\|^2$$

we have:

$$G^{n+1} \ge s_3 \left| u^{n+1} \right|_1^2 + M_1 \frac{\tau h^2}{4} \sum_{P=2}^{N+1} \left| v^k \right|_1^2, \quad G^1 \le \left( 4\sigma_0^2 + 4\sigma_0 - 1 \right) \left| u^1 \right|_1^2 + \left( 4\sigma_0^2 - 1 \right) \left| u^0 \right|_1^2$$

Combining previous two equations, using *Lemma 5* and Cauchy-Schwarz inequality, we have:

$$s_3 \left| u^{n+1} \right|_1^2 + \frac{M_1 \tau h^2}{4} \sum_{p+2}^{n+1} \left| v^k \right|_1^2 \le \mathrm{e}^{M_2 T} \Big\{ \left( 4\sigma_0^2 + 4\sigma_0 - 1 \right) \left| u^1 \right|_1^2 + 2 M_3 \left\| v^1 \right\|^2 +$$
$$+ \left[ 12\left(5 + 6\sigma_n\right) + 80\sigma_n \right] Q_2 \tau^2 \left| v^1 \right|_1^2 + \left( 4\sigma_0^2 - 1 \right) \left| u^0 \right|_1^2 + 2 M_4 \left\| v^0 \right\|^2 +$$
$$+ 2 Q_1 \tau \left| v^0 \right|_1^2 + 12\left(5 + 6\sigma_n\right) Q_2 \tau^2 \sum_{p+2}^{n+1} \left| v^p \right|_1^2 + 4\tau s_2 \sum_{p=1}^{n} \left\| v^{p+\sigma_p} \right\|^2 +$$
$$+ \frac{\tau}{s_2} \sum_{p=1}^{n} \left\| f^{p+\sigma_p} \right\|^2 + \frac{2L^2}{3 s_3} \tau \sum_{p=1}^{n} \left\| g^{p+\sigma_p} \right\|^2 + \frac{6 s_3}{L^2} \tau \sum_{p=1}^{n} \left\| u^{p+\sigma_p} \right\|^2 - 4 s_2 \tau \sum_{p=1}^{n} \left\| v^{p+\sigma_p} \right\|^2 \Big\}$$

Let:

$$\frac{\tau}{h^2} < \frac{M_1}{\left[ 48 e^{MT} \left(5 + 6\sigma_n\right) Q_2 \right]}$$

Combining eqs. (15) and (16) and using *Lemma 1*, there exists constant $\kappa$ such that:

$$\left| u^{n+1} \right|_1^2 \le \kappa H^{n+1} + \kappa\tau \sum_{p+1}^{n} \left| u^p \right|_1^2$$

where

$$H^{n+1}=\left|v^0\right|_1^2+\left\|\delta_x^2v^0\right\|^2+\left|u^0\right|_1^2+\left|\delta_x^2u^0\right|_1^2+\tau^{1+\gamma 1/2}\left\|f^{1/2}\right\|^2+\left|\delta_xf^{\sigma_0}\right|_1^2+\left\|g^{1/2}\right\|^2+\left|\delta_xg^{1/2}\right|_1^2+$$
$$+\tau\sum_{p=1}^{n}\left\|f^{p+\sigma_p}\right\|^2+\tau\sum_{p=1}^{n}\left\|g^{p+\sigma_p}\right\|^2$$

Using *Lemma 9(II)*, we get:

$$|u^{n+1}|_1^2\leq\kappa \mathrm{e}^{\kappa T}H^{n+1} \tag{19}$$

The stability of the finite difference scheme (9) can deduce from (19).

Denote:

$$e_i^n=u_i^n-U_i^n,\ \overline{e}_i^n=v_i^n-V_i^n,\ 0\leq i\leq M,\ 0\leq n\leq N$$

We have:

$$\sum_{k=0}^{n}\hat{c}_{n-k}^{(n,\gamma)}\left(\overline{e}_i^{k+1}-\overline{e}_i^k\right)+s_2\overline{e}_i^{n+\sigma}=s_3\delta_x^2e_i^{n+\sigma}+\sum_{k=0}^{n+1}B_nd_{n-k+1}^{(n,\beta)}\delta_x^2\overline{e}_i^k+R_i^n,\ 1\leq i\leq M-1,\ 0\leq n\leq N-1$$

$$\hat{c}_0(\overline{e}_i^1-\overline{e}_i^0)+s_2\overline{e}_i^{1/2}=s_3\delta_x^2e_i^{1/2}+B_0\left(d_1\delta_x^2\overline{e}_i^0+d_0\delta_x^2\overline{e}_i^1\right)+\hat{R}_i^{1/2},\ 1\leq i\leq M-1$$

$$\delta_t\delta_x^2e_i^{1/2}=\delta_x^2\overline{e}_i^{1/2}+r_i^0,\ 1\leq i\leq M-1$$

$$e_i^0=0,\ \overline{e}_i^0=0,\ 1\leq i\leq M-1,\ e_0^n=e_M^n=0,\ \overline{e}_0^n=\overline{e}_M^n=0,\ 0\leq n\leq N$$

where

$$R_i^n=\mathcal{O}\left(\tau^2+h^2\right),\ \hat{R}_i^{1/2}=\mathcal{O}\left(\tau^{2-\gamma_{1/2}}+h^2\right),\ r_i^0=\mathcal{O}\left(\tau^2+h^2\right)\tau^2+h^2,\ \text{and}\ r_i^n=\mathcal{O}\left(\tau^2+h^2\right)$$

So, we can get

$$\left|\mathrm{e}^{n+1}\right|_1^2\leq K\mathrm{e}^{2\sigma T}\hat{H}^{n+1}$$

where

$$\hat{H}^{n+1}=\left|\overline{e}^0\right|_1^2+\left\|\delta_x^2\overline{e}^0\right\|^2+\left|e^0\right|_1^2+\left|\delta_x^2e^0\right|_1^2+\tau^{1+\gamma 1/2}\left\|\hat{R}_i^{1/2}\right\|^2+\left|\delta_xR_i^n\right|_1^2+\left[r_i^0\right]^2+\left|\delta_xr_i^0\right|_1^2+\tau\sum_{p=1}^{n}\left\|R_i^p\right\|^2+\tau\sum_{p=1}^{n}\left\|r_i^p\right\|^2$$

Noticing:

$$R_i^n,\hat{R}_i^{1/2},r_i^0,r_i^n$$

there exists a constant $K$ s.t. $|e^{n+1}|1\leq K(\tau^2+h^2)$.

## Numerical experiments

Let us consider:

$$D_t^{\alpha(t)}u+\frac{\partial u}{\partial t}=\frac{\partial^2u}{\partial x^2}+D_t^{\beta(t)}\frac{\partial^2u}{\partial x^2}+f(x,t),\quad(x,t)\in(0,L)\times(0,T]$$

$$u(x,0)=0,\ \frac{\partial u}{\partial t}(x,0)=0,\ x\in[0,L],\ u(0,t)=0,\ u(L,t)=0,\ t\in[0,T]$$

The exact solution is $u(x,t)$ and the source term is:

$$f(x,t)=\left[\frac{6t^{3-\alpha(t)}}{\Gamma\left(4-\alpha(t)\right)}+3t^2+t^3+\frac{6t^{3-\beta(t)}}{\Gamma\left(4-\beta(t)\right)}\right]\sin x$$

Denote

$$E(h,\tau)=\max_{0\le n\le N}|U^n-u^n|_1,\ \text{Order}=\log_2\frac{E(h,2\tau)}{E(h,\tau)}$$

We take $L = \pi$, $T = 1$, and $h = \pi/500$, for different $\tau$, tab. 1 shows that the maximum $H_1$-norm error and convergence rates for $\alpha(t) = 2 - t^2$, $\beta(t) = t^2$, and $\alpha(t) = (6 + \sin t)/4$, $\beta(t) = t^2$, we can see that numerical results are in perfect agreement with the exact solution and has the theoretical accuracy of the second order in time.

**Table 1. The $H_1$-norm error and convergence rates**

| $\alpha(t) = 2 - t^2$, $\beta(t) = t^2$ | $E(h, \tau)$ | Order | $\alpha(t) = (6 + \sin t)/4$, $\beta(t) = t^2$ | $E(h, \tau)$ | Order |
|---|---|---|---|---|---|
| $\tau = 1/10$ | $7.871 \cdot 10^{-3}$ | | $\tau = 1/10$ | $5.8500 \cdot 10^{-3}$ | |
| $\tau = 1/20$ | $1.9442 \cdot 10^{-3}$ | 2.00 | $\tau = 1/20$ | $1.5535 \cdot 10^{-3}$ | 1.91 |
| $\tau = 1/30$ | $4.8490 \cdot 10^{-4}$ | 2.00 | $\tau = 1/30$ | $4.0070 \cdot 10^{-4}$ | 1.95 |
| $\tau = 1/40$ | $1.2179 \cdot 10^{-4}$ | 1.99 | $\tau = 1/40$ | $1.0227\text{E} \cdot 10^{-4}$ | 1.97 |

## Conclusion

In this paper a novel numerical scheme for the initial-boundary value problem of variable-order fractional Oldroyd-B fluid model have been considered. A difference scheme is proposed with theoretical accuracy $\mathcal{O}(\tau^2 + h^2)$. We also established the stability and convergence for the finite difference scheme. Numerical example can effectively verify the reliability of our method.

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## Nomenclature

$t$ – time co-ordinate, [second]

$x$ – space co-ordinate, [m]

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