A NEW COMBINED ZK-mZK DYNAMIC MODEL FOR ROSSBY SOLITARY WAVE

by

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In this article, using multi-scale and perturbation methods, a partial differential model of Rossby solitary waves with complete Coriolis force effect is obtained, which is called the combined ZK-mZK equation. This equation can reflect the propagation characteristics of Rossby waves in space, and is more suitable for real oceans and atmosphere than the (1+1)-dimensional model. According to the parameter composition of the new model, it can be seen that the effect of the complete Coriolis force affects not only the longitudinal structure of the model, but also the amplitude structure of the non-linear long wave.

Key words: *combined ZK-mZK equation, Rossby solitary wave, complete Coriolis force effect*

Introduction

In the past few decades, the construction of mathematical models to study the generation and evolution of Rossby solitary waves has attracted a lot of attention, and a series of mathematical models have been obtained. Many patterns from 1-D to multi-dimensional, for instance, KdV [1], mKdV [2] model, ZK-BO mode [3], ZK Burgers mode [4], local fractional Boussinesq mode [5], and non-linear time-fractional Heisenberg mode [6]. The mathematical models are used to explain some natural phenomena related to non-linear effects of Rossby waves. In the previous research work, we have paid attention the following problems, the equations of motion describing the ocean and atmosphere, including momentum equation and continuity equation are very complex. However, the 1-D PDE used to describe the evolution of non-linear Rossby waves has certain limitations, because the real motion of the ocean and atmosphere does not only occur in 1-D. So, the study of non-linear Rossby wave high dimensional motion needs to be solved urgently.

In addition, the horizontal component of Coriolis force parameters has received extensive attention from scholars [7-10]. White *et al.* [11] pointed out the necessity of preserving the horizontal component of the Coriolis force through the scale analysis of the zonal dynamic equation, it was also known as the *non-traditional approximation* mode with Coriolis cosine

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terms, which can more comprehensively and accurately analyzed atmospheric motion. Gerkema *et al.* [12] derived the near inertial wave from the original equation based on the vertical infinite water depth boundary. The exact solution shows that the near inertial wave can pass through the inertial latitude. Yin *et al.* [13] gave the non-linear KdV equation with exogenous forcing satisfied by the near-equatorial non-linear Rossby waves from the potential vortex equation containing the vertical and horizontal components of the Coriolis force. The aforementioned results show that the horizontal component of the complete Coriolis force yields the imbalance of motion, which is considered as an important factor leading to the development of the dynamical system.

In this paper, based on the quasi geostrophic potential vorticity mode, a new ZK-mZK model characterizing the (2+1)-dimensional combination of Rossby solitary waves is obtained by using multi-scale analysis and perturbation methods. Finally, we analyze the influence of each parameter qualitatively through the coefficients in the model.

The derivation of combined ZK-mZK model

Based on the geophysical fluid dynamics, we give the following quasi geostrophic potential vorticity equation:

$$\left(\frac{\partial}{\partial t}+\frac{\partial\Psi}{\partial x}\frac{\partial}{\partial y}-\frac{\partial\Psi}{\partial y}\frac{\partial}{\partial x}\right)\left[\nabla^2\Psi+\frac{\partial}{\partial z}\left(\frac{f_0^2}{N(z)^2}\frac{\partial\Psi}{\partial z}\right)+\beta y+\frac{f_0}{H}G(x,y)-f_H\frac{\partial G(x,y)}{\partial y}\right]=0 \tag{1}$$

Vertical velocity:

$$W=\frac{f_0}{N(z)^2}\left(\frac{\partial}{\partial t}+\frac{\partial\Psi}{\partial x}\frac{\partial}{\partial y}-\frac{\partial\Psi}{\partial y}\frac{\partial}{\partial x}\right)\frac{\partial\Psi}{\partial z} \tag{2}$$

where Ψ is the stream function. The Coriolis force effect of the normal compnent is $f=f_0+\beta y$, where f_H is the Coriolis effect of the horizontal component, $G(x, y)$ is the topographical effect, and $N(z)$ is the buoyancy frequency. The Laplace operator is $\nabla^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2$. The side boundary condition reads:

$$\Psi(0)=\Psi(+\infty)=0 \tag{3}$$

the vertical boundary is given:

$$W(0)=W(+\infty)=0 \tag{4}$$

Dimensionless analysis is conducted for eqs. (1) and (2), and eqs. (1) and (2) are converted into:

$$\left(\frac{\partial}{\partial t}+\frac{\partial\Psi}{\partial x}\frac{\partial}{\partial y}-\frac{\partial\Psi}{\partial y}\frac{\partial}{\partial x}\right)\left[\nabla^2\Psi+\beta y+G(x,y)-\theta\frac{\partial G(x,y)}{\partial y}+\frac{\partial}{\partial z}\left(K^2\frac{\partial\Psi}{\partial z}\right)\right]=0 \tag{5}$$

$$W=R_0K^2\left(\frac{\partial}{\partial t}+\frac{\partial\Psi}{\partial x}\frac{\partial}{\partial y}-\frac{\partial\Psi}{\partial y}\frac{\partial}{\partial x}\right)\frac{\partial\Psi}{\partial z} \tag{6}$$

where

$$K^2=\frac{L_0^2f_0^2}{N^2H_0^2},\ R_0=\frac{U_0}{f_0L_0},\ \theta=\frac{f_HH}{f_0L_0}$$

the horizontal characteristic scale is L_0, the characteristic quantity of velocity is U_0, and the characteristic quantity of fluid height is H_0:

$$\Psi = -\int^{y}[U(s)-c_0]\mathrm{d}s + \varepsilon\psi \tag{7}$$

where $\varepsilon \ll 1$ ais the small parameter used to measure the non-linear characteristic scale. We assume that the main characteristic scale of topographical effect is in the y direction, $G(x, y) \to G_1(y)$, $(\theta = \varepsilon^{3/2})$.

Substituting the stream function into eq. (5) to get:

$$\frac{\partial}{\partial t}\left[\nabla^2\psi + \frac{\partial}{\partial z}\left(K^2\frac{\partial\psi}{\partial z}\right)\right] + (U-c_0)\frac{\partial}{\partial x}\left[\nabla^2\psi + \frac{\partial}{\partial z}\left(K^2\frac{\partial\psi}{\partial z}\right)\right] +$$
$$+\lambda\frac{\partial\psi}{\partial x} + \varepsilon^{\frac{3}{2}}J[\psi, \frac{\partial G_1(y)}{\partial y}] + \varepsilon J\left[\psi, \nabla^2\psi + \frac{\partial}{\partial z}\left(K^2\frac{\partial\psi}{\partial z}\right)\right] = 0 \tag{8}$$

$$W = \varepsilon R_0 K^2\left[\frac{\partial}{\partial t} + \left(U-c_0\right)\frac{\partial}{\partial x}\right]\frac{\partial\psi}{\partial z} + \varepsilon^2 J\left[\psi, \frac{\partial\psi}{\partial z}\right] \tag{9}$$

where

$$\lambda = \beta + G_1'(y) - U'',\ J[A,B] = \frac{\partial A}{\partial X}\frac{\partial B}{\partial Y} - \frac{\partial A}{\partial Y}\frac{\partial B}{\partial X}$$

Now we introduce multi-scale space-time transformation:

$$T = \epsilon^{3/2}t,\ Y = \epsilon^{3/4}y,\ z = z \tag{10}$$

Substituting eq. (10) into eq. (8):

$$\varepsilon^{1/2}\left\{\frac{\partial}{\partial X}\left[\frac{\partial^2\psi}{\partial y^2} + \frac{\partial}{\partial z}\left(K^2\frac{\partial\psi}{\partial z}\right)\right] + \lambda\frac{\partial\psi}{\partial X}\right\} + \varepsilon^{5/4}\left\{2\left(U-c_0\right)\frac{\partial^3\psi}{\partial X\partial y\partial Y}\right\} +$$
$$+\varepsilon^{6/4}\left\{\frac{\partial^3\psi}{\partial T\partial y^2} + \frac{\partial^2}{\partial T\partial z}\left(K^2\frac{\partial\psi}{\partial z}\right) + \left(U-c_0\right)\frac{\partial^3}{\partial X^3} +\right.$$
$$\left.+\frac{\partial\psi}{\partial X}\frac{\partial^3\psi}{\partial y^3} + \frac{\partial\psi}{\partial X}\frac{\partial^2}{\partial y\partial z}\left(K^2\frac{\partial\psi}{\partial z}\right) - \frac{\partial\psi}{\partial y}\frac{\partial^3\psi}{\partial X\partial y^2} - \frac{\partial\psi}{\partial y}\frac{\partial^2}{\partial X\partial z}\left(K^2\frac{\partial\psi}{\partial z}\right)\right\} + \varepsilon^2\left\{\left(U-c_0\right)\frac{\partial^3\psi}{\partial X\partial Y^2}\right\} +$$
$$+\varepsilon^2\frac{\partial\psi}{\partial X}\frac{\partial^2 G_1(y)}{\partial y^2} + \varepsilon^{9/4}\left\{2\frac{\partial^3\psi}{\partial T\partial y\partial Y} + 3\frac{\partial\psi}{\partial X}\frac{\partial^3\psi}{\partial y^2\partial Y} + \frac{\partial\psi}{\partial X}\frac{\partial^2}{\partial z\partial Y}\left(K^2\frac{\partial\psi}{\partial z}\right) - \frac{\partial\psi}{\partial y}\frac{\partial^3\psi}{\partial X\partial y\partial Y}\right.$$
$$\left.-\frac{\partial\psi}{\partial Y}\frac{\partial^3\psi}{\partial X\partial y^2} - \frac{\partial\psi}{\partial Y}\frac{\partial^2}{\partial z\partial X}\left(K^2\frac{\partial\psi}{\partial z}\right)\right\} + \varepsilon^{5/2}\left\{\frac{\partial^3\psi}{\partial T\partial X^2} + \frac{\partial\psi}{\partial X}\frac{\partial^3\psi}{\partial y\partial X^2} - \frac{\partial\psi}{\partial y}\frac{\partial^3\psi}{\partial X^3}\right\}$$
$$+\varepsilon^3\left\{\frac{\partial^3\psi}{\partial T\partial Y^2} + 3\frac{\partial\psi}{\partial X}\frac{\partial^3\psi}{\partial y\partial Y^2} - \frac{\partial\psi}{\partial y}\frac{\partial^3\psi}{\partial X\partial Y^2} - 2\frac{\partial\psi}{\partial Y}\frac{\partial^3\psi}{\partial X\partial y\partial Y}\right\} + o(\varepsilon^3) = 0 \tag{11}$$

The side boundary condition:

$$\psi(0) = \psi(+\infty) = 0 \tag{12}$$

Therefore, the vertical boundary condition can be written:

$$\frac{\partial\psi}{\partial z} = 0,\ z \to +\infty \tag{13}$$

Assuming that the perturbed stream function is expanded by the perturbations about ε:

$$\psi = \psi_0 + \varepsilon^{1/4}\psi_1 + \varepsilon^{2/4}\psi_2 + \varepsilon^{3/4}\psi_3 + \varepsilon^{4/4}\psi_4 + \varepsilon^{5/4}\psi_5 + \varepsilon^{6/4}\psi_6 + \cdots \tag{14}$$

Substituting eq. (14) into eq. (11) to obtain the equation of each order about ε. To facilitate the introduction of L operator:

$$L = \left(U - c_0\right)\frac{\partial}{\partial X}\left[\frac{\partial^2}{\partial y^2} + \frac{\partial}{\partial z}\left(K^2\frac{\partial}{\partial z}\right) + \frac{\lambda}{U - c_0}\right] \tag{15}$$

$$\varepsilon^{1/2} : L(\psi_0) = 0 \tag{16}$$

Lateral boundary condition:

$$\psi_0(0) = \psi_0(+\infty) = 0 \tag{17}$$

Vertical boundary:

$$\frac{\partial \psi_0}{\partial z} = 0,\ \ z \to +\infty \tag{18}$$

Solving eq. (16) is a linear problem, and using the method of separating variables, the assumption is ψ_0:

$$\psi_0 = A(X, Y, T)\phi_0(y)W_0(z) \tag{19}$$

Substituting eq. (19) into eqs. (16) and (17) to obtain the eigenvalue problem abou ϕ_0, W_0:

$$\begin{aligned} &\phi_0'' + k_0^2\phi_0 + \frac{\lambda}{U - c_0}\phi_0 = 0 \\ &\left(K^2 W_0'\right)' + k_0^2 W_0 = 0 \end{aligned} \tag{20}$$

Similarly, we continue to solve ε higher order problems:

$$\varepsilon^{3/4} : L(\psi_1) = 0 \tag{21}$$

The variable ψ_1 has the expressions:

$$\psi_1 = A^2(X, Y, T)\phi_1(y)W_1(z) \tag{22}$$

further

$$\varepsilon^1 : L(\psi_2) = 0 \tag{23}$$

the same method assuming ψ_1:

$$\psi_2 = A(X, Y, T)\phi_2(y)W_2(z) \tag{24}$$

Substituting eqs. (22) and (24) into eqs. (21) and (23), respectively, and the characteristic equation of ψ_1, ψ_2, W_1, W_2 can be converted into the form:

$$\phi_1'' + k_1^2\phi_1 + \frac{\lambda}{U-c_0}\phi_1 = 0$$
$$\left(K^2W_1'\right)' + k_1^2W_1 = 0$$
$$\phi_2'' + k_2^2\phi_2 + \frac{\lambda}{U-c_0}\phi_2 = 0 \tag{25}$$
$$\left(K^2W_1'\right)' + k_2^2W_2 = 0$$

Similarly, side boundary conditions can be written:

$$\psi_1(0) = \psi_1(+\infty) = 0$$
$$\psi_2(0) = \psi_2(+\infty) = 0 \tag{26}$$

The vertical boundary:

$$\frac{\partial\psi_1}{\partial z} = 0 \quad \frac{\partial\psi_1}{\partial z} = 0, \ z \to +\infty \tag{27}$$

In order to obtain the condition of eliminating singularity:

$$\varepsilon^{5/4} : L(\psi_3) = -2(U-c_0)\frac{\partial^3\psi_0}{\partial X\partial y\partial Y} \tag{28}$$

we separate the variables ψ_3:

$$\psi_3 = A_Y(X,Y,T)\phi_3(y)W_3(z) \tag{29}$$

Substituting eqs. (19) and (29) into eq. (28):

$$\phi_3'' + k_3^2\phi_3 + \frac{\lambda}{U-c_0}\phi_3 = -2\phi_0'$$
$$(K^2W_3')' + k_3^2W_3 = -2W_0 \tag{30}$$

$$\varepsilon^{3/2} : L(\psi_4) = -\left\{2(U-c_0)\frac{\partial^3\psi_1}{\partial X\partial y\partial Y} + \frac{\partial^3\psi_0}{\partial T\partial y^2} + \frac{\partial^2}{\partial T\partial z}\left(K^2\frac{\partial\psi_0}{\partial z}\right) + (U-c_0)\frac{\partial^3\psi_0}{\partial X^3} + \right.$$
$$+(U-c_0)\frac{\partial}{\partial x}\left(\frac{\partial G_1}{\partial x} - \theta\frac{\partial G_1}{\partial y}\right) + \frac{\partial\psi_0}{\partial X}\frac{\partial^3\psi_0}{\partial y^3} + \frac{\partial\psi_0}{\partial X}\frac{\partial^2}{\partial y\partial z}\left(K^2\frac{\partial\psi_0}{\partial z}\right) - \frac{\partial\psi_0}{\partial y}\frac{\partial^3\psi_0}{\partial X\partial y^2} -$$
$$\left. -\frac{\partial\psi_0}{\partial y}\frac{\partial^2}{\partial X\partial z}\left(K^2\frac{\partial\psi_0}{\partial z}\right)\right\}$$
$$= -\left\{A_T\left[\phi_0''W_0 + k_0^2\phi_0\left(K^2W_0'\right)'\right] + (U-c_0)A_{XXX}\phi_0W_0 + \right.$$
$$\left. + AA_X\left(\phi_0\phi_0'''W_0^2 - \phi_0'\phi_0''W_0^2\right) + 4(U-c_0)(A_YA_X + AA_{XY})\phi_1'W_1\right\} \tag{31}$$

$$\varepsilon^{7/4}: L(\psi_5) = -\left\{ 2(U-c_0)\frac{\partial^3\psi_2}{\partial X\partial y\partial Y} + \frac{\partial^3\psi_1}{\partial T\partial y^2} + \frac{\partial^2}{\partial T\partial z}\left(K^2\frac{\partial\psi_1}{\partial z}\right) + (U-c_0)\frac{\partial^3\psi_1}{\partial X^3} + \right.$$
$$+\frac{\partial\psi_0}{\partial X}\frac{\partial^3\psi_1}{\partial y^3} + \frac{\partial\psi_0}{\partial X}\frac{\partial^2}{\partial y\partial z}\left(K^2\frac{\partial\psi_1}{\partial z}\right) - \frac{\partial\psi_0}{\partial y}\frac{\partial^3\psi_1}{\partial X\partial y^2} - \frac{\partial\psi_0}{\partial y}\frac{\partial^2}{\partial X\partial z}\left(K^2\frac{\partial\psi_1}{\partial z}\right) +$$
$$\left. +\frac{\partial\psi_1}{\partial X}\frac{\partial^3\psi_0}{\partial y^3} + \frac{\partial\psi_1}{\partial X}\frac{\partial^2}{\partial y\partial z}\left(K^2\frac{\partial\psi_0}{\partial z}\right) - \frac{\partial\psi_1}{\partial y}\frac{\partial^3\psi_0}{\partial X\partial y^2} - \frac{\partial\psi_1}{\partial y}\frac{\partial^2}{\partial X\partial z}\left(K^2\frac{\partial\psi_0}{\partial z}\right) \right\} \tag{32}$$

$$\varepsilon^2: L(\psi_6) = -\left\{ \left[\phi_2'' W_2 + k_2^2\phi_2\left(K^2W_2'\right)' \right] A_T + \phi_0'' G_1''(y) W_0 A_X + (U-c_0)A_{XXX}\phi_2 W_2 + \right.$$
$$+(U-c_0)A_{XYY}\left(2\phi_3' W_3 + \phi_0 W_0\right) + A\left\{ A_X \left[\phi_0\phi_2''' W_0 W_2 + k_2^2\phi_0\phi_2'\left(K^2W_2'\right)' - \phi_0'\phi_2'' W_0 W_2 - \right.\right.$$
$$\left. -k_2^2\phi_0'\phi_2\left(K^2W_2'\right)' + \phi_2\phi_0''' W_0 W_2 + k_0^2\phi_2\phi_0'\left(K^2W_0'\right)' - \phi_2'\phi_0'' W_0 W_2 - k_0^2\phi_2'\phi_0\left(K^2W_0'\right)' \right] +$$
$$\left.\left. +2A^2 A_X\left[\phi_1\phi_1''' W_1^2 - \phi_1'\phi_1'' W_1^2 \right] \right\}\right\} \tag{33}$$

Equation (33) is multiplied ϕ_0 by the left and right, and then integrating it x and y the integration interval is [0, +∞). According to the boundary conditions, we have the identical equation:

$$\phi_0\frac{\partial^2\psi_6}{\partial y^2} = \frac{\partial}{\partial y}\left(\phi_0\frac{\partial\psi_6}{\partial y}\right) - \frac{\partial}{\partial y}\left(\psi_6\frac{\partial\psi_0}{\partial y}\right) + \psi_6\frac{\partial^2\psi_0}{\partial y^2} \tag{34}$$

Finally, we get:

$$A_T + a_0 A_X + a_1 A A_X + a_2 A^3 A_X + a_3 A_{XXX} + a_4 A_{XYY} = 0 \tag{35}$$

where

$$a = \int_0^{+\infty}\int_0^{+\infty}\frac{1}{U-c_0}\left[\phi_2'' W_2 + k_2^2\phi_2\left(K^2W_2'\right)' \right]\mathrm{d}y\mathrm{d}z$$

$$a_0 = \int_0^{+\infty}\int_0^{+\infty}\phi_0'' G_1''(y) W_0\,\mathrm{d}y\mathrm{d}z$$

$$a_1 = \int_0^{+\infty}\int_0^{+\infty}\frac{1}{U-c_0}\left[\phi_0\phi_2''' W_0 W_2 + k_2^2\phi_0\phi_2'\left(K^2W_2'\right)' - \phi_0'\phi_2'' W_0 W_2 - k_2^2\phi_0'\phi_2\left(K^2W_2'\right)' + \right.$$
$$\left. +\phi_2\phi_0''' W_0 W_2 + k_0^2\phi_2\phi_0'\left(K^2W_0'\right)' - \phi_2'\phi_0'' W_0 W_2 - k_0^2\phi_2'\phi_0\left(K^2W_0'\right)' \right]\mathrm{d}y\mathrm{d}z / a \tag{36}$$

$$a_2 = \int_0^{+\infty}\int_0^{+\infty}\frac{2}{U-c_0}\left[\phi_1\phi_1''' W_1^2 - \phi_1'\phi_1'' W_1^2 \right]\mathrm{d}y\mathrm{d}z / a$$

$$a_3 = \int_0^{+\infty}\int_0^{+\infty}\phi_2 W_2\,\mathrm{d}y\mathrm{d}z / a$$

$$a_4 = \int_0^{+\infty}\int_0^{+\infty}(2\phi_3' W_3 + \phi_0 W_0)\,\mathrm{d}y\mathrm{d}z / a$$

Equation (35) is a new (2+1)-dimensional mode. The coefficient a_1 is related to the topographical effect with the complete Coriolis effect. It can be seen that a_1AA_X and a_2AA_X these two items are the combined ZK equation and mZK equation, so the new model is named as the combined ZK-mZK equation. Compared to other models, the combined ZK-mZK equation is more suitable to describe the influencing factors of Rossby solitary wave. If the forcing term are ignored, eq. (35) is simplified to ZK equation:

$$A_T + a_1 AA_X + a_3 A_{XXX} + a_4 A_{XYY} = 0 \tag{37}$$

Conclusions

Based on the quasi-geostrophic potential vorticity equation, combined with multi-scale analysis and perturbation method, we obtain a new (2+1)-dimensional ZK-mZK model to characterize Rossby solitary waves. On the basis of the new model, we draw the following conclusions.

- The ZK-mZK model including the complete Coriolis force effect is derived, which is a new (2+1)-dimensional model including Rossby solitary wave characteristics. The influence of the Coriolis force is an important factor to generate solitary waves, while the horizontal component of rotation in the Coriolis force and the topographic effect affect the coefficient of the model. In the actual ocean and atmosphere, the (2+1)-dimensional model can better reflect the propagation laws of waves in space.
- According to the combined ZK-mZK equation, a_0, a_1, a_2, a_3, a_4 include topographic effect and complete Coriolis force action. In addition, the effect of the complete Coriolis force affects not only the longitudinal structure of the model, but also the amplitude structure of the non-linear long wave. Finally, the coefficients of the new mode contain soliton solutions when they satisfy certain algebraic relations. In the future work, we will study the solution structure of the model and wave propagation characteristics through theoretical analysis [14, 15] and numerical methods.

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Nomenclature

t – time, [second]

x, y, z – co-ordinates, [m]

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