SPECIAL SOLUTIONS FOR THE LAPLACE AND DIFFUSION EQUATIONS ASSOCIATED WITH THE ALGEBRAIC NUMBER FIELD

by

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This article is devoted to the even entire functions, which are the exact solution for the Laplace and diffusion equations. These functions are considered in the algebraic number field. We guess that the functions have purely real zeros in the entire complex plane. These are proposed as new connections with algebraic number theory and mathematical physics.

Key words: *diffusion equation, Laplace equation, entire functions, algebraic number field, number theory*

Introduction

Let $\aleph_n$ denotes the number of ideals with the norm equal to n and the algebraic number field $\mathbb{G}$ with α real places and β complex places. Assume that $L_{\mathbb{G}}$ is the absolute value of the discriminant of $\mathbb{G}$. The Dedekind zeta function $\zeta_{\mathbb{G}}(s)$ of the algebraic number field $\mathbb{G}$ is expressed as [1]

$$\zeta_{\mathbb{G}}(r) = \sum_{n=1}^{\infty} \aleph_n n^{-r} \qquad (\mathrm{Re}(r) > 1), \tag{1}$$

which is connected with the Dedekind xi function $\xi_{\mathbb{G}}(r)$ by ([2])

$$\xi_{\mathbb{G}}(r) = \zeta_{\mathbb{G}}(r)\, r(r-1) L_{\mathbb{G}}^{2} \pi^{-\frac{\alpha r}{2}} (2\pi)^{-\beta r} \Gamma^{\alpha}(r), \tag{2}$$

with the functional equation [2]

$$\xi_{\mathbb{G}}(r) = \xi_{\mathbb{G}}(1-r). \tag{3}$$

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Let $\varpi_{\mathbb{G}}$ be the number of roots of unity of $\mathbb{G}$ and let $l_{\mathbb{G}}$ is the discriminant of $\mathbb{G}$ such that $\left|l_{\mathbb{G}}\right| = L_{\mathbb{G}}$.

If

$$\vartheta_2^2 - 4\vartheta_1\vartheta_3 = L_{\mathbb{G}} = \left|l_{\mathbb{G}}\right| \ \left(-\vartheta_1 < \vartheta_2 \le \vartheta_1 < \vartheta_3 \text{ or } 0 < \vartheta_2 \le \vartheta_1 = \vartheta_3\right),$$

we have the quadratic form [2,3]

$$\mathbb{F}(x,y) = \vartheta_1 x^2 + \vartheta_2 xy + \vartheta_3 y^2, \tag{4}$$

such that the Li's function $\mathrm{M}(\tau)$ reads [2,3]

$$\mathrm{M}_{\mathbb{F}}(\tau) = \sum_{\mathbb{F}} \sum_{m,k=-\infty}^{\infty} \frac{\pi\mathbb{F}(m,k)}{\sqrt{L_{\mathbb{G}}}} \left(\frac{\pi\tau\mathbb{F}(m,k)}{\sqrt{L_{\mathbb{G}}}} - 1\right) \exp\left(-\frac{2\pi\tau\mathbb{F}(m,k)}{\sqrt{L_{\mathbb{G}}}}\right) \tag{5}$$

where this runs over the inequivalent classes of the positive definite integral quadratic forms of $L_{\mathbb{G}}$ for all integers m,k with $k \ne 0$.

By (5), the Dedekind xi function (2) is represented by [4]

$$\xi_{\mathbb{G}}(r) = \frac{4}{\varpi_{\mathbb{G}}} \int_1^{\infty} \mathrm{M}_{\mathbb{F}}(\tau)\left(\tau^r + \tau^{1-r}\right) d\tau, \tag{6}$$

which is rewritten as [4]

$$\xi_{\mathbb{G}}(r) = \frac{8}{\varpi_{\mathbb{G}}} \int_1^{\infty} \mathrm{M}_{\mathbb{F}}(\tau)\tau^{\frac{1}{2}} \cosh\left[\left(r - \frac{1}{2}\right)\log\tau\right] d\tau. \tag{7}$$

Putting

$$r = \frac{1}{2} + i\ell \tag{8}$$

into (7), we have [4]

$$\Xi_{\mathbb{G}}(\ell) = \frac{8}{\varpi_{\mathbb{G}}} \int_1^{\infty} \mathrm{M}_{\mathbb{F}}(\tau)\tau^{\frac{1}{2}} \cos(\ell\log\tau) d\tau \tag{9}$$

if

$$\Xi_{\mathbb{G}}(\ell) = \xi_{\mathbb{G}}\left(\frac{1}{2} + i\ell\right). \tag{10}$$

Putting $\tau = e^{2\hbar}$ into (9) gives

$$\Xi_{\mathbb{G}}(\ell) = \frac{8}{\varpi_{\mathbb{G}}} \int_0^{\infty} \mathrm{M}_{\mathbb{F}}\left(e^{2\hbar}\right) e^{2\hbar} \cos(\ell\hbar) d\hbar. \tag{11}$$

If (12) becomes

$$\psi_{\mathbb{F}}(\hbar) = \frac{8e^{2\hbar}}{\varpi_{\mathbb{G}}} \sum_{\mathbb{F}} \sum_{m,k=-\infty}^{\infty} \frac{\pi\mathbb{F}(m,k)}{\sqrt{L_{\mathbb{G}}}} \left(\frac{\pi e^{2\hbar}\mathbb{F}(m,k)}{\sqrt{L_{\mathbb{G}}}} - 1\right) \exp\left(-\frac{2\pi e^{2\hbar}\mathbb{F}(m,k)}{\sqrt{L_{\mathbb{G}}}}\right), \tag{12}$$

for all integers m,k with $k \neq 0$, then (11) can be rewritten as

$$\Xi_{\mathbb{G}}\left(\ell\right)=\int_{0}^{\infty}\psi_{\mathbb{F}}\left(\hbar\right)\cos\left(\ell\hbar\right)d\hbar\,. \tag{13}$$

The aims of this paper are to suggest an entire function

$$\Xi_{\mathbb{G}}\left(\ell,\eta\right)=\int_{0}^{\infty}e^{\eta\hbar}\psi_{\mathbb{F}}\left(\hbar\right)\sin\left(\ell\hbar\right)d\hbar\,, \tag{14}$$

to deduce the Laplace equation [5]

$$\frac{\partial^{2}\Xi_{\mathbb{G}}\left(\ell,\eta\right)}{\partial\ell^{2}}+\frac{\partial^{2}\Xi_{\mathbb{G}}\left(\ell,\eta\right)}{\partial\eta^{2}}=0\,, \tag{15}$$

and to present

$$\Phi_{\mathbb{G}}\left(\ell,t\right)=\int_{0}^{\infty}e^{-t\hbar^{2}}\psi_{\mathbb{F}}\left(\hbar\right)\sin\left(\ell\hbar\right)d\hbar\,, \tag{16}$$

to reduce to the diffusion equation [6]

$$\frac{\partial\Xi_{\mathbb{G}}\left(\ell,t\right)}{\partial t}=\frac{\partial\Xi_{\mathbb{G}}\left(\ell,t\right)}{\partial\ell^{2}}\,. \tag{17}$$

An Odd Entire Function in Algebraic Number Field is a Special Solution for the Laplace Equation

We now define a family of the entire function $\Xi_{\mathbb{G}}\left(\ell,\eta\right):\mathbb{C}\to\mathbb{C}$ for $\eta\in\mathbb{R}$ by the Fourier sine integral

$$\Xi_{\mathbb{G}}\left(\ell,\eta\right)=\int_{0}^{\infty}e^{\eta\hbar}\psi_{\mathbb{F}}\left(\hbar\right)\sin\left(\ell\hbar\right)d\hbar\,. \tag{18}$$

Because of $\Xi_{\mathbb{G}}\left(-\ell,\eta\right)=-\Xi_{\mathbb{G}}\left(\ell,\eta\right)$, (18) is an odd entire function of order one and of genus one. Since the convergence exponent of the zeros of (18) is one, we have the result:

Conjecture A. (18) has purely real zeros in the entire complex plane $\ell\in\mathbb{C}$.

Obviously, we see that $\ell=0$ is a zero of (18). Let us consider that

$$\frac{\partial\Xi_{\mathbb{G}}\left(\ell,\eta\right)}{\partial\ell}=\frac{\partial}{\partial\ell}\left\{\int_{0}^{\infty}e^{\eta\hbar}\psi_{\mathbb{F}}\left(\hbar\right)\sin\left(\ell\hbar\right)d\hbar\right\}=\int_{0}^{\infty}\hbar e^{\eta\hbar}\psi_{\mathbb{F}}\left(\hbar\right)\cos\left(\ell\hbar\right)d\hbar\,, \tag{19}$$

$$\frac{\partial^{2}\Xi_{\mathbb{G}}\left(\ell,\eta\right)}{\partial\ell^{2}}=\frac{\partial^{2}}{\partial\ell^{2}}\left\{\int_{0}^{\infty}e^{\eta\hbar}\psi_{\mathbb{F}}\left(\hbar\right)\sin\left(\ell\hbar\right)d\hbar\right\}=-\int_{0}^{\infty}\hbar^{2}e^{\eta\hbar}\psi_{\mathbb{F}}\left(\hbar\right)\sin\left(\ell\hbar\right)d\hbar\,, \tag{20}$$

$$\frac{\partial\Xi_{\mathbb{G}}\left(\ell,\eta\right)}{\partial\eta}=\frac{\partial}{\partial\eta}\left\{\int_{0}^{\infty}e^{\eta\hbar}\psi_{\mathbb{F}}\left(\hbar\right)\sin\left(\ell\hbar\right)d\hbar\right\}=\int_{0}^{\infty}\hbar e^{\eta\hbar}\psi_{\mathbb{F}}\left(\hbar\right)\sin\left(\ell\hbar\right)d\hbar\,, \tag{21}$$

and

$$\frac{\partial^{2}\Xi_{\mathbb{G}}\left(\ell,\eta\right)}{\partial\eta^{2}}=\frac{\partial^{2}}{\partial\eta^{2}}\left\{\int_{0}^{\infty}e^{\eta\hbar}\psi_{\mathbb{F}}\left(\hbar\right)\sin\left(\ell\hbar\right)d\hbar\right\}=\int_{0}^{\infty}\hbar^{2}e^{\eta\hbar}\psi_{\mathbb{F}}\left(\hbar\right)\sin\left(\ell\hbar\right)d\hbar\,. \tag{22}$$

By (20) and (22), this leads to the Laplace equation [5]

$$\frac{\partial^2 \Xi_{\mathbb{G}}(\ell,\eta)}{\partial \ell^2}+\frac{\partial^2 \Xi_{\mathbb{G}}(\ell,\eta)}{\partial \eta^2}=0\,. \tag{23}$$

Here, the initial and boundary value conditions are considered as

$$\Xi_{\mathbb{G}}(\ell=a,\eta)=\int_0^\infty e^{\eta\hbar}\psi_{\mathbb{F}}(\hbar)\sin(a\hbar)\,d\hbar\,, \tag{24}$$

$$\frac{\partial \Xi_{\mathbb{G}}(\ell=b,\eta)}{\partial \ell}=\int_0^\infty \hbar e^{\eta\hbar}\psi_{\mathbb{F}}(\hbar)\cos(b\hbar)\,d\hbar\,, \tag{25}$$

$$\Xi_{\mathbb{G}}(\ell,\eta=c)=\int_0^\infty e^{c\hbar}\psi_{\mathbb{F}}(\hbar)\sin(\ell\hbar)\,d\hbar\,, \tag{26}$$

and

$$\frac{\partial \Xi_{\mathbb{G}}(\ell,\eta=d)}{\partial \ell}=\int_0^\infty \hbar e^{d\hbar}\psi_{\mathbb{F}}(\hbar)\cos(\ell\hbar)\,d\hbar\,. \tag{27}$$

An Odd Entire Function in Algebraic Number Field is a Special Solution for the Diffusion Equation

We now define a family of the function $\Phi_{\mathbb{G}}(\ell,t):\mathbb{C}\to\mathbb{C}$ for $t\in\mathbb{R}$ by the Fourier cosine integral

$$\Phi_{\mathbb{G}}(\ell,t)=\int_0^\infty e^{-t\hbar^2}\psi_{\mathbb{F}}(\hbar)\sin(\ell\hbar)\,d\hbar\,. \tag{28}$$

In view of $\Phi_{\mathbb{G}}(-\ell,t)=-\Phi_{\mathbb{G}}(\ell,t)$, (28) is an odd entire function of order one and of genus one. Since the convergence exponent of the zeros of (28) is one, we also obtain the result:

Conjecture B. (28) has purely real zeros in the entire complex plane $\ell\in\mathbb{C}$.

It is easy to obtain that $\ell=0$ is a zero of (28).

Making use of (28), we present

$$\frac{\partial \Phi_{\mathbb{G}}(\ell,t)}{\partial \ell}=\frac{\partial}{\partial \ell}\left\{\int_0^\infty e^{-t\hbar^2}\psi_{\mathbb{F}}(\hbar)\sin(\ell\hbar)\,d\hbar\right\}=\int_0^\infty \hbar e^{-t\hbar^2}\psi_{\mathbb{F}}(\hbar)\cos(\ell\hbar)\,d\hbar\,, \tag{29}$$

$$\frac{\partial^2 \Phi_{\mathbb{G}}(\ell,t)}{\partial \ell^2}=\frac{\partial^2}{\partial \ell^2}\left\{\int_0^\infty e^{-t\hbar^2}\psi_{\mathbb{F}}(\hbar)\sin(\ell\hbar)\,d\hbar\right\}=-\int_0^\infty \hbar^2 e^{-t\hbar^2}\psi_{\mathbb{F}}(\hbar)\sin(\ell\hbar)\,d\hbar\,, \tag{30}$$

and

$$\frac{\partial \Phi_{\mathbb{G}}(\ell,t)}{\partial t}=\frac{\partial}{\partial t}\left\{\int_0^\infty e^{-t\hbar^2}\psi_{\mathbb{F}}(\hbar)\sin(\ell\hbar)\,d\hbar\right\}=-\int_0^\infty \hbar^2 e^{-t\hbar^2}\psi_{\mathbb{F}}(\hbar)\cos(\ell\hbar)\,d\hbar\,, \tag{31}$$

which lead to

$$\frac{\partial \Phi_{\mathbb{G}}(\ell,t)}{\partial t}=\frac{\partial^2 \Phi_{\mathbb{G}}(\ell,t)}{\partial \ell^2}\,, \tag{32}$$

which is the diffusion equation in the one-dimensional space [6].

We now consider the initial value conditions

$$\Phi_{\mathbb{G}}\left(\ell, t=0\right)=\int_{0}^{\infty}\psi_{\mathbb{F}}\left(\hbar\right)\sin\left(\ell\hbar\right)d\hbar \tag{33}$$

and

$$\frac{\partial\Phi_{\mathbb{G}}\left(\ell, t=0\right)}{\partial t}=-\int_{0}^{\infty}\hbar^{2}\psi_{\mathbb{F}}\left(\hbar\right)\cos\left(\ell\hbar\right)d\hbar \tag{34}$$

and the boundary value conditions

$$\Phi_{\mathbb{G}}\left(\ell=\alpha, t\right)=\int_{0}^{\infty}e^{-t\hbar^{2}}\psi_{\mathbb{F}}\left(\hbar\right)\sin\left(\alpha\hbar\right)d\hbar \tag{35}$$

and

$$\Phi_{\mathbb{G}}\left(\ell=\beta, t\right)=\int_{0}^{\infty}e^{-t\hbar^{2}}\psi_{\mathbb{F}}\left(\hbar\right)\sin\left(\beta\hbar\right)d\hbar . \tag{36}$$

Conclusion

In this work we have proposed the even entire functions in the algebraic number field. They have been considered as the exact solution for the Laplace and diffusion equations. They have purely real zeros in the entire complex plane. We have suggested the initial and boundary problems for the Laplace and diffusion equations.

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Nomenclature

$\mathbb{R}$ -set of real numbers, [$-$]	$\mathbb{C}$ -set of complex numbers, [$-$]
t -time coordinate, [s]	η -space coordinate, [m]
$\mathrm{Re}\left(r\right)$ -real part, [-]	ℓ -space coordinate, [m]

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