AN EVEN ENTIRE FUNCTION OF ORDER ONE IS A SPECIAL SOLUTION FOR A CLASSICAL WAVE EQUATION IN ONE-DIMENSIONAL SPACE

by

Xiao-Jun YANG [a,b,c,*]**,** Abdulrahman Ali ***ALSOLAMI***[b], Ahmed Refaie ***ALI***[c]

[a] State Key Laboratory for Geo-Mechanics and Deep Underground Engineering, and School of Mathematics, China University of Mining and Technology, Xuzhou 221116, China
[b] Department of Mathematics, Faculty of Science, King Abdulaziz University P.O. Box 80257, Jeddah 21589, Saudi Arabia
[c] Department of Mathematics, College of Science, Kyung Hee University, 26 Kyungheedae-ro, Dongdaemun-gu, Seoul 02447, Republic of Korea
[d] Mathematics and Computer Science Department,Faculty of Science, Menoufia University, Egypt

In this article we consider an even entire function of order one as the solution of the classical wave equation in one-dimensional space. We suggest a conjecture that this function has only purely real zeros in the entire complex plane. This is given as a new prospective on a nice connection with number theory and wave equation.

Key words: *wave equation, solution, even entire function, conjecture, number theory*

Introduction

It is well known that de Bruijn [1] and Newman [2] have suggested a family of the function $\Xi(x,t):\mathbb{C}\to\mathbb{C}$ for $t\in\mathbb{R}$ by the Fourier cosine integral

$$\Xi(x,t)=\int_0^\infty e^{-t\tau^2}\Phi(\tau)\cos(x\tau)\,d\tau\,, \tag{1}$$

where

$$\Phi(\tau)=4\sum_{n=1}^{\infty}\left\{2\pi^2n^4\exp\left(\frac{9}{2}\tau\right)-3\pi n^2\exp\left(\frac{5}{2}\tau\right)\right\}\exp\left[-\pi n^2\exp(2\tau)\right]. \tag{2}$$

If we put $t=0$ into (1), we obtain [3]

$$\Xi(x,t=0):=\Xi(x)=\int_0^\infty \Phi(\tau)\cos(x\tau)\,d\tau\,, \tag{3}$$

which is equal to [4]

$$\Xi(x)=4\int_1^\infty \mathrm{H}(\chi)\cos\left(\frac{x}{2}\log\chi\right)d\chi\,, \tag{4}$$

* Corresponding author; e-mail: dyangxiaojun@163.com

in which [4]

$$\mathrm{H}(\chi)=\frac{d}{d\chi}\left(\chi^{\frac{3}{2}}\mathrm{M}^{(1)}(\chi)\right)\chi^{-\frac{1}{4}} \tag{5}$$

and [4]

$$\mathrm{M}(\chi)=\sum_{k=1}^{\infty}e^{-k^2\pi\chi} . \tag{6}$$

Here, (4) is the well-known Riemann Xi function [5].

Assume that the real and imaginary parts for the complex variable $s\in\mathbb{C}$ by $\Re(s)$ and $\Im(s)$, respectively. If we set for $i=\sqrt{-1}$,

$$s=\frac{1}{2}+ix , \tag{7}$$

we obtain the Riemann xi function [5]

$$\Xi\left[\frac{1}{i}\left(s-\frac{1}{2}\right)\right]:=\xi(s)=4\int_1^{\infty}\mathrm{H}(\chi)\cosh\left[\frac{1}{2}\left(s-\frac{1}{2}\right)\log\chi\right]d\chi , \tag{8}$$

which is equivalent to the functional equation [5]

$$\xi(s)=\frac{s}{2}(s-1)\pi^{-\frac{s}{2}}\Gamma(s)\zeta(s), \tag{9}$$

where

$$\Gamma(s)=\int_0^{\infty}e^{-r}r^{s-1}dr \tag{10}$$

represents the gamma function [6], and

$$\zeta(s)=\prod_p\frac{1}{1-p^{-s}} \tag{11}$$

is denoted as the Riemann zeta function with all primes p [4].

The classical wave equation in one dimension space can be expressed as [7]

$$\frac{\partial^2\Xi(x,t)}{\partial t^2}-\frac{\partial^2\Xi(x,t)}{\partial x^2}=0, \tag{12}$$

where $\Xi(x,t)$ represents the wave function.

In this article we consider a new family of the entire function $\aleph(x,t):\mathbb{C}\to\mathbb{C}$ for $t\in\mathbb{R}$ by the Fourier cosine integral

$$\aleph(x,t)=\int_0^{\infty}\exp(-t\tau)\Phi(\tau)\cos(x\tau)d\tau , \tag{13}$$

which has the functional equation

$$\aleph(x,t)=\aleph(-x,t) . \tag{14}$$

Putting $t=0$ into (12), we obtain

$$\aleph\left(x,t=0\right)=\Xi\left(x\right). \tag{15}$$

The main aim of this paper is to investigate the zeros of (13) and to consider the relationship between the classical wave equation (12) and a new family of the entire function (13). The structure is presented as follows. In Section 2 we investigate the properties and conjecture for the even entire function (13). In Section 3 we derive the classical wave equation from the even entire function (13). Finally, our conclusion is given in Section 4.

Properties and Conjecture

Let $t\in\mathbb{R}$ such that (13) becomes

$$\aleph\left(x,t\right)=\int_0^\infty \phi\left(\tau,t\right)\cos\left(x\tau\right)d\tau\,, \tag{16}$$

where

$$\phi\left(\tau,t\right)=\exp\left(-t\tau\right)\Phi\left(\tau\right). \tag{17}$$

If there exists [8]

$$\Phi\left(\tau\right)>0 \tag{18}$$

for $\tau>0$, we obtain

$$\phi\left(\tau,t\right)=\exp\left(-t\tau\right)\Phi\left(\tau\right)>0\,. \tag{19}$$

Since it is well known that

$$\Xi\left(x\right)=\int_0^\infty \Phi\left(\tau\right)\cos\left(x\tau\right)d\tau \tag{20}$$

is an even entire function of order one, we also find that

$$\aleph\left(x,t\right)=\int_0^\infty \exp\left(-t\tau\right)\Phi\left(\tau\right)\cos\left(x\tau\right)d\tau=\int_0^\infty \phi\left(\tau,t\right)\cos\left(x\tau\right)d\tau \tag{21}$$

is an even entire function of order one. Clearly, $\aleph\left(x,t\right)$ is of genus one. Thus, the convergence exponent of the zeros of $\aleph\left(x,t\right)$ is one (see [9], p.19). This implies that $\aleph\left(x,t\right)$ has infinity of zeros. Thus, we have the following conjecture as follows:

Conjecture I. $\aleph\left(x,t\right)$ has purely real zeros.

Remark. When $t=0$, the Jensen conjecture [3] reads that $\aleph\left(x,t=0\right)=\Xi\left(x\right)$ has purely real zeros.

An Application of (13) to Derive the Classical Wave Equation

The first derivative of (13) with respect to the variable t is considered as follows:

$$\frac{\partial\aleph\left(x,t\right)}{\partial t}=\int_0^\infty\left\{\frac{\partial\exp\left(-t\tau\right)}{\partial t}\right\}\Lambda\left(\tau\right)\cos\left(x\tau\right)d\tau\,. \tag{22}$$

To simplify (22), we obtain

$$\frac{\partial \aleph(x,t)}{\partial t} = -\int_0^\infty \tau \exp(-t\tau)\Lambda(\tau)\cos(x\tau)\,d\tau\,. \tag{23}$$

The second derivative of (13) with respect to the variable t reads

$$\frac{\partial^2 \aleph(x,t)}{\partial t^2} = \int_0^\infty \left\{\frac{\partial^2 \exp(-t\tau)}{\partial t^2}\right\}\Lambda(\tau)\cos(x\tau)\,d\tau\,, \tag{24}$$

which leads to

$$\frac{\partial^2 \aleph(x,t)}{\partial t^2} = \int_0^\infty \tau^2 \exp(-t\tau)\Lambda(\tau)\cos(x\tau)\,d\tau\,. \tag{25}$$

The first derivative of (13) with respect to the variable x is given as follows:

$$\frac{\partial \aleph(x,t)}{\partial x} = -\int_0^\infty \tau \exp(-t\tau)\Lambda(\tau)\sin(x\tau)\,d\tau. \tag{26}$$

Also, the second derivative of (13) with respect to the variable x can be expressed as

$$\frac{\partial^2 \aleph(x,t)}{\partial x^2} = -\int_0^\infty \tau^2 \exp(-t\tau)\Lambda(\tau)\cos(x\tau)\,d\tau. \tag{27}$$

It follows from (24) and (27) that the classical wave equation [7]

$$\frac{\partial^2 \aleph(x,t)}{\partial t^2} - \frac{\partial^2 \aleph(x,t)}{\partial x^2} = 0\,(t \ge 0, a \le x \le b). \tag{28}$$

From (13) and (23) we have the initial value conditions

$$\aleph(x,t=0) = \int_0^\infty \Phi(\tau)\cos(x\tau)\,d\tau \tag{29}$$

and

$$\frac{\partial \aleph(x,t=0)}{\partial t} = -\int_0^\infty \tau\Lambda(\tau)\cos(x\tau)\,d\tau\,. \tag{30}$$

From (13) we obtain the boundary value conditions

$$\aleph(x=a,t) = \int_0^\infty \exp(-t\tau)\Phi(\tau)\cos(a\tau)\,d\tau \tag{31}$$

and

$$\aleph(x=b,t) = \int_0^\infty \exp(-t\tau)\Phi(\tau)\cos(b\tau)\,d\tau\,. \tag{32}$$

Here, (29) can be written as

$$\aleph(x,t=0) = \Xi(x) = \int_0^\infty \Phi(\tau)\cos(x\tau)\,d\tau\,. \tag{34}$$

Conclusion

In this work we have proposed an even entire function of order one to structure a classical wave equation in one-dimensional space. We conjectured that this function has purely real zeros. This has the connection with the wave equation and number theory.

Acknowledgements:

This work is supported by the Yue-Qi Scholar of the China University of Mining and Technology (No. 102504180004).

Nomenclature

$\mathbb{R}$ -set of real numbers, [−] $\mathbb{C}$ -set of complex numbers, [−]
t -time coordinate, [s] x -space coordinate, [m]

References

[1] De Bruijn, N. G., The roots of trigonometric integrals, *Duke Mathematical Journal*, 17(1950), 3, pp.197-226
[2] Newman, C. M., Fourier transforms with only real zeros, *Proceedings of the American Mathematical Society*, 61 (1976), 2, pp.245-251
[3] Jensen, J. L., Recherches sur la théorie des équations, *Acta Mathematica*, 36(1913), 1, pp.181-195
[4] Riemann, G. F. B., Über die Anzahl der Primzahlen unter einer gegebenen Grösse, *Monatsberichte der Berliner Akademie*, 2 (1859), November, pp.671-680
[5] Edwards, H. M., *Riemanns'Zeta Function*, Academic press, New York, 1974
[6] Yang, X. J., *Theory and Applications of Special Functions for Scientists and Engineers*, Springer Singapore, 2021
[7] Balazs, N. L., On the solution of the wave equation with moving boundaries, *Journal of Mathematical Analysis and Applications*, 3(1961), 3, pp.472-484
[8] Wintner, A., A note on the Riemann ξ-function, *Journal of the London Mathematical Society*, 1(1935), 2, pp.82-83
[9] Boas, R. P., *Entire Functions*, Academic Press, New York, USA, 1954

Paper submitted: November 11, 2022
Paper revised: December 15, 2022
Paper accepted: December 17, 2022