ON THE SOLUTION FOR THE DIFFUSION EQUATION RELATED TO THE L-FUNCTIONS ATTACHED TO CUSP FORMS

by

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In this article we suggest the entire functions associated with the L-functions attached to cusp forms. The entire function defined by the Fourier cosine transform is the solution for the diffusion equation in one-dimensional case. We propose three conjectures for the zeros of three entire functions of order one via theory of entire functions.

Key words: *diffusion equation, entire function, L-functions, cusp forms, Fourier cosine transform*

Introduction

Assume that (see [1], p.15)

$$SL_2(\mathbb{Z})=\left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix}: a,b,c,d \in \mathbb{Z},\ ad-bc=1\right\} \tag{1}$$

is the modular group. Let $\mathbb{S}_k\left(SL_2(\mathbb{Z})\right)$ be the space of cusp forms of weight k and level 1. A holomorphic cusp form ϕ for $\phi \in \mathbb{S}_k\left(SL_2(\mathbb{Z})\right)$ and $\mathfrak{A}=\left\{u \in \mathbb{C} \middle| \Im(u)>0\right\}$ is defined by the Fourier series (see [1], p.65-66)

$$\phi(u)=\sum_{n=1}^{\infty}\alpha_{\phi}(n)e^{2\pi inu} . \tag{3}$$

The L-series attached to the cusp form ϕ (see [1], p.65-66)

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$$L(\phi,\ell)=\sum_{n=1}^{\infty}\alpha_{\phi}(n)n^{-\ell} \tag{4}$$

converges absolutely for $\Re(\ell)>k/2+1$.

It is showed that (4) has the following functional equation (see Theorem 5.3.7 in [1], p.66)

$$\Lambda(\phi,\ell)=(2\pi)^{-\ell}\,\Gamma(\ell)L(\phi,\ell)=i^{k}(2\pi)^{-(k-\ell)}\,\Gamma(k-\ell)L(\phi,k-\ell)=i^{k}\Lambda(\phi,k-\ell). \tag{6}$$

Making use of

$$\phi(iu)=i^{-k}u^{-k}\phi\left(\frac{i}{u}\right)=O\left(u^{-k}e^{-\frac{2\pi}{u}}\right) \tag{7}$$

as $u\to 0$, we have from (6) that (see [1], p.66)

$$\Lambda(\phi,\ell)=(2\pi)^{-s}\,\Gamma(s)L(\phi,\ell)=\int_{0}^{\infty}\phi(iu)u^{\ell-1}du. \tag{8}$$

From (7) and (8) we obtain (see [1], p.66)

$$\int_{1}^{\infty}\phi\left(\frac{i}{u}\right)u^{-\ell-1}du=i^{k}\int_{1}^{\infty}\phi(iu)u^{k-\ell-1}du, \tag{9}$$

which leads to (see [1], p.66)

$$\Lambda(\phi,\ell)=\int_{1}^{\infty}\phi(iu)\left(u^{\ell-1}+i^{k}u^{k-\ell-1}\right)du. \tag{10}$$

In fact, (10) was also considered and investigated by Berndt-Knopp [2] and Ogg [3]. From (10) we get

$$\Lambda(\phi,\ell)=\int_{1}^{\infty}\phi(iu)u^{\frac{k}{2}-1}\left(u^{\ell-\frac{k}{2}}+i^{k}u^{\frac{k}{2}-\ell}\right)du. \tag{11}$$

By using (11), we show

$$\Lambda(\phi,\ell)=\vartheta(\phi,\ell)+i^{k}\theta(\phi,\ell), \tag{12}$$

where

$$\vartheta(\phi,\ell)=\int_{1}^{\infty}\phi(iu)u^{\ell-1}du \tag{13}$$

and

$$\theta(\phi,\ell)=\int_{1}^{\infty}\phi(iu)u^{k-\ell-1}du. \tag{14}$$

Let us denote

$$\wp(\phi,\ell)=\vartheta(\phi,\ell)+\theta(\phi,\ell), \tag{15}$$

and

$$\Phi(\phi,\ell)=\vartheta(\phi,\ell)-\theta(\phi,\ell) \tag{16}$$

such that

$$\begin{aligned}\wp(\phi,\ell)&=\int_1^\infty \phi(iu)u^{\frac{k}{2}-1}\left(u^{\ell-\frac{k}{2}}+u^{\frac{k}{2}-\ell}\right)du\\&=\int_1^\infty \phi(iu)u^{\frac{k}{2}-1}\cosh\left[\left(\ell-\frac{k}{2}\right)\log u\right]du\end{aligned} \tag{17}$$

and

$$\begin{aligned}\Phi(\phi,\ell)&=\int_1^\infty \phi(iu)u^{\frac{k}{2}-1}\left(u^{\ell-\frac{k}{2}}-u^{\frac{k}{2}-\ell}\right)du\\&=\int_1^\infty \phi(iu)u^{\frac{k}{2}-1}\sinh\left[\left(\ell-\frac{k}{2}\right)\log u\right]du.\end{aligned} \tag{18}$$

With (17) and (18), we obtain the functional equations

$$\wp(\phi,\ell)=\wp(\phi,k-\ell)$$

and

$$\Phi(\phi,\ell)=-\Phi(\phi,k-\ell).$$

The above cases are equal to (6) when k is even number.

It is easy to know that (17) and (18) are the entire functions of order 1 and infinite type [4]. Taking $k=1$ and $\tau=\ell-\frac{1}{2}$ into (17) and (18), we get

$$\widehat{\wp}(\phi,\tau)=\int_1^\infty \phi(iu)u^{-\frac{1}{2}}\cosh(\tau\log u)\,du \tag{19}$$

and

$$\widehat{\Phi}(\phi,\tau)=\int_1^\infty \phi(iu)u^{-\frac{1}{2}}\sinh(\tau\log u)\,du \tag{20}$$

It is easy to know that (19) and (20) are the entire functions of order 1 and infinite type [4]. Let us substitute

$$u=e^{\hbar} \tag{21}$$

into (19) and (20), we get

$$\widehat{\wp}(\phi,\tau)=\int_0^\infty \Theta(\hbar)e^{\frac{\hbar}{2}}\cosh(\tau\hbar)\,d\hbar \tag{22}$$

and

$$\hat{\theta}(\phi,\tau)=\int_0^\infty \Theta(\hbar)e^{\frac{\hbar}{2}}\sinh(\tau\hbar)\,d\hbar \tag{23}$$

where

$$\Theta(\hbar)=\phi\left(ie^{\hbar}\right)=\sum_{n=1}^{\infty}\alpha_{\phi}(n)e^{-2\pi ne^{\hbar}} . \tag{24}$$

It is easy to see the following functional equations

$$\widehat{\wp}(\phi,\tau)=\widehat{\wp}(\phi,-\tau) \tag{25}$$

and

$$\widehat{\Phi}(\phi,-\tau)=-\widehat{\Phi}(\phi,\tau) . \tag{26}$$

Let us consider

$$\widehat{\mathrm{H}_{\nu}}(\phi,\tau)=\int_{0}^{\infty}e^{-\nu\hbar^{2}}\Theta(\hbar)e^{\frac{\hbar}{2}}\cos(\tau\hbar)\,d\hbar \tag{27}$$

and

$$\widehat{\theta_{\nu}}(\phi,\tau)=\int_{0}^{\infty}e^{-\nu\hbar^{2}}\Theta(\hbar)e^{\frac{\hbar}{2}}\sinh(\tau\hbar)\,d\hbar . \tag{28}$$

Here, (27) is analogous of the result considered by de Bruijn [5] and Newman [6].

The aim of this paper is to investigate the diffusion equation [7] by using (27). The structure of the paper is given as follows. In Section 2 we study the properties and four conjectures of (22), (23), (27) and (28). In Section 3 we consider to deduce the diffusion equation. Finally, we give our conclusions in Section 4.

A Family of the Entire Functions

A family of the entire function $\widehat{\wp_{\nu}}(\phi,\tau):\mathbb{C}\to\mathbb{C}$ for $\nu\in\mathbb{R}$ is defined as

$$\widehat{\wp_{\nu}}(\phi,\tau)=\int_{0}^{\infty}e^{-\nu\hbar^{2}}\Theta(\hbar)e^{\frac{\hbar}{2}}\cosh(\tau\hbar)\,d\hbar . \tag{29}$$

There is the functional equation

$$\widehat{\wp_{\nu}}(\phi,\tau)=\widehat{\wp_{\nu}}(\phi,-\tau) . \tag{30}$$

Since

$$\widehat{\wp}(\phi,\tau)=\int_{1}^{\infty}\phi(iu)u^{-\frac{1}{2}}\cosh(\tau\log u)\,du \tag{31}$$

is an entire function of order 1 and infinite type [4], we see that (31) is of genus 1, and that the convergence exponent of its zeros is 1. Thus, (31) has infinity of zero. Similarly, (29) is an entire function of order 1 and of genus 1, and the convergence exponent of zeros of (29) is 1 [8]. This implies that we have the following result:

Conjecture A. $\widehat{\wp_{\nu}}(\phi,\tau)$ has purely imaginary zeros in the entire complex plane $\tau\in\mathbb{C}$.

A family of the entire function $\widehat{\theta_{\nu}}(\phi,\tau):\mathbb{C}\to\mathbb{C}$ for $\nu\in\mathbb{R}$ is defined as

$$\widehat{\theta_\nu}(\phi,\tau)=\int_0^\infty e^{-\nu\hbar^2}\Theta(\hbar)e^{\frac{\hbar}{2}}\sinh(\tau\hbar)\,d\hbar\,. \tag{32}$$

Here, (32) has the functional equation

$$\widehat{\theta_\nu}(\phi,-\tau)=-\widehat{\theta_\nu}(\phi,\tau). \tag{33}$$

Since

$$\hat{\theta}(\phi,\tau)=\int_0^\infty \Theta(\hbar)e^{\frac{\hbar}{2}}\sinh(\tau\hbar)\,d\hbar \tag{34}$$

is an entire function of order 1 and infinite type [4], we see that (34) is of genus 1, and that the convergence exponent of its zeros is 1. Thus, (34) has infinity of zero. Similarly, (32) is an entire function of order 1 and of genus 1, and the convergence exponent of zeros of (32) is 1 [8]. Since (32) has a real zero $\tau=0$, this reduces the following result:

Conjecture B. $\widehat{\theta_\nu}(\phi,\tau)$ has purely imaginary zeros in the complex plane $\tau\in\mathbb{C}\setminus\{0\}$.

A family of the entire function $\widehat{\mathrm{H}_\nu}(\phi,\tau):\mathbb{C}\to\mathbb{C}$ for $\nu\in\mathbb{R}$ is defined by the Fourier cosine transform

$$\widehat{\mathrm{H}_\nu}(\phi,\tau)=\int_0^\infty e^{-\nu\hbar^2}\Theta(\hbar)e^{\frac{\hbar}{2}}\cos(\tau\hbar)\,d\hbar\,. \tag{35}$$

There is the functional equation

$$\widehat{\mathrm{H}_\nu}(\phi,\tau)=\widehat{\mathrm{H}_\nu}(\phi,-\tau). \tag{36}$$

Making use of (29) and (35), we obtain

$$\widehat{\wp_\nu}(\phi,i\tau)=\widehat{\mathrm{H}_\nu}(\phi,\tau) \tag{37}$$

and

$$\widehat{\mathrm{H}_\nu}(\phi,i\tau)=\widehat{\wp_\nu}(\phi,\tau). \tag{38}$$

It is the fact that (35) is an entire function of order 1 and of genus 1, and the convergence exponent of zeros of (35) is 1 [8].

Conjecture A is equivalent to the following result:

Conjecture C. $\widehat{\mathrm{H}_\nu}(\phi,\tau)$ has purely real zeros in the entire complex plane $\tau\in\mathbb{C}$.

From the Entire Function (35) to the Diffusion Equations

By (35), we find that

$$\frac{\partial\widehat{\mathrm{H}_\nu}(\phi,\tau)}{\partial\nu}=-\int_0^\infty \hbar^2 e^{-\nu\hbar^2}\Theta(\hbar)e^{\frac{\hbar}{2}}\cos(\tau\hbar)\,d\hbar\,, \tag{39}$$

$$\frac{\partial\widehat{\mathrm{H}_\nu}(\phi,\tau)}{\partial\tau}=-\int_0^\infty \hbar e^{-\nu\hbar^2}\Theta(\hbar)e^{\frac{\hbar}{2}}\sin(\tau\hbar)\,d\hbar\,, \tag{40}$$

and

$$\frac{\partial^2 \widehat{\mathrm{H}_\nu}(\phi,\tau)}{\partial\tau^2} = -\int_0^\infty \hbar^2 e^{-\nu\hbar^2}\Theta(\hbar)e^{\frac{\hbar}{2}}\cos(\tau\hbar)\,d\hbar\,. \quad (41)$$

From (39) and (41) we present the diffusion equation [7]

$$\frac{\partial \widehat{\mathrm{H}_\nu}(\phi,\tau)}{\partial\nu} = \frac{\partial^2 \widehat{\mathrm{H}_\nu}(\phi,\tau)}{\partial\tau^2}, \quad (42)$$

where $\nu > 0$ and $\alpha < \tau < \beta$.

The initial value condition of (42) reads

$$\widehat{\mathrm{H}_{\nu=0}}(\phi,\tau) = \int_0^\infty \Theta(\hbar)e^{\frac{\hbar}{2}}\cos(\tau\hbar)\,d\hbar\,. \quad (43)$$

We now consider the boundary value conditions of two types as follows.

By (35), we give the Dirichlet-type valule conditions for (42) as follows:

$$\widehat{\mathrm{H}_\nu}(\phi,\tau=\alpha) = \int_0^\infty e^{-\nu\hbar^2}\Theta(\hbar)e^{\frac{\hbar}{2}}\cos(\beta\hbar)\,d\hbar \quad (44)$$

and

$$\widehat{\mathrm{H}_\nu}(\phi,\tau=\beta) = \int_0^\infty e^{-\nu\hbar^2}\Theta(\hbar)e^{\frac{\hbar}{2}}\cos(\beta\hbar)\,d\hbar\,. \quad (45)$$

With (40), we have the Neumann-type valule conditions for (42) as follows:

$$\frac{\partial \widehat{\mathrm{H}_\nu}(\phi,\tau=\alpha)}{\partial\tau} = -\int_0^\infty \hbar e^{-\nu\hbar^2}\Theta(\hbar)e^{\frac{\hbar}{2}}\sin(\alpha\hbar)\,d\hbar \quad (46)$$

and

$$\frac{\partial \widehat{\mathrm{H}_\nu}(\phi,\tau=\beta)}{\partial\tau} = -\int_0^\infty \hbar e^{-\nu\hbar^2}\Theta(\hbar)e^{\frac{\hbar}{2}}\sin(\beta\hbar)\,d\hbar\,. \quad (47)$$

Conclusion

In our work we have proposed the entire functions associated with the L-functions attached to cusp forms. The conjectures for them have been suggested based on the theory of the entire functions. We have presented that the solution for the one-dimensional diffusion equation is the entire function associated with the L-functions attached to cusp forms.

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Nomenclature

$\mathbb{R}$ -set of real numbers, [$-$]

$\mathbb{C}$ -set of complex numbers, [$-$]

ν -time coordinate, [s]

τ -space coordinate, [m]

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