A NEW ODD ENTIRE FUNCTION OF ORDER ONE ARISING IN THE HEAT PROBLEM

by

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In this article we suggest a new odd entire function of order one. We discover that it is the solution of the heat equation. We propose a new conjecture that this function has only real zeros. The obtained result gives a connection with the heat problem and number theory.

Key words: *heat equation, odd entire function, heat equation, real zeros, number theory*

Introduction

Riemann [1] introduced the Xi function $\Xi(x)$, defined as

$$\Xi(x) = 4\int_1^\infty \mathrm{A}(\tau)\cos\left(\frac{x}{2}\log\tau\right)d\tau\ , \tag{1}$$

where

$$O(\tau) = \sum_{k=1}^\infty e^{-k^2\pi\tau} \tag{2}$$

and

$$\mathrm{A}(\tau) = \frac{d}{d\tau}\left(\tau^{\frac{3}{2}}O^{(1)}(\tau)\right)\tau^{-\frac{1}{4}}. \tag{3}$$

Here, (1) is called the Riemann Xi function [2].

Jensen [3] proposed that (2) can be expressed by the Fourier cosine integral as follows (see [4], p.255):

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$$\Xi(x) = 2\int_0^{\infty} \Phi(z)\cos(xz)\,dz\,, \tag{4}$$

where

$$\Phi(z) = 2\sum_{n=1}^{\infty}\left(2\pi^2 n^4 e^{\frac{9}{2}z} - 3\pi n^2 e^{\frac{5}{2}z}\right)e^{-\pi n^2 e^{2z}}\,. \tag{5}$$

In fact, de Bruijn [5] and Newman [6] suggested a family of the function $\Xi(x,t):\mathbb{C}\to\mathbb{C}$ for $t\in\mathbb{R}$ by the Fourier cosine integral

$$\Xi(x,t) = 2\int_0^{\infty} e^{-ty^2}\Phi(y)\cos(xy)\,dy\,, \tag{6}$$

where $x\in\mathbb{C}$. Here, $t\in\mathbb{R}$ is the de Bruijn-Newman constant.

Csordas, Smith and Varga [7] suggested the backward heat equation

$$\frac{\partial^2\Xi(x,t)}{\partial x^2} + \frac{\partial\Xi(x,t)}{\partial t} = 0\,. \tag{7}$$

Based on (2) and (3), author [8] introduced the Lambda function $\Lambda(x)$, i.e.,

$$\Lambda(x) = 4\int_1^{\infty} \mathrm{A}(\tau)\sin\left(\frac{x}{2}\log\tau\right)d\tau\,, \tag{8}$$

with the functional equation

$$\Lambda(-x) = -\Lambda(x)\,. \tag{9}$$

Applying (5), the target of this paper is to study a set of the functional family $\Lambda(x,t):\mathbb{C}\to\mathbb{C}$ for $t\in\mathbb{R}$ by the Fourier sine integral

$$\Lambda(x,t) = 2\int_0^{\infty} e^{-ty^2}\Phi(y)\sin(xy)\,dy\,, \tag{10}$$

where $x\in\mathbb{C}$, and to investigate the heat equation [9]

$$\frac{\partial\Lambda(x,t)}{\partial t} - \frac{\partial^2\Lambda(x,t)}{\partial x^2} = 0\,, \tag{11}$$

with the solution expressed by (10) for $x\in\mathbb{R}$ and $t\in\mathbb{R}$.

The structure of this paper is given as follows. In Section 2 we introduce the Fourier sine integral representation of the Lambda function. In Section 3 we suggest a new odd entire function of order one. In Section 4 we propose the conjectures associated with this function. In Section 4 we consider this function as the solution for the heat equation. In Section 5 we draw the conclusion.

The Fourier Sine Integral Representation of the Lambda Function

Putting $\tau = e^{2z}$ in (1), we also obtain (see [4], p.260)

$$\Xi(x) = 4\int_1^{\infty}\frac{d}{d\tau}\left(\tau^{\frac{3}{2}}O^{(1)}(\tau)\right)\tau^{-\frac{1}{4}}\cos\left(\frac{x}{2}\log\tau\right)dz = 4\int_0^{\infty}\frac{d}{dz}\left(e^{3z}O^{(1)}\left(e^{2z}\right)\right)e^{-\frac{1}{2}z}\cos(xz)\,dz. \tag{12}$$

From (12) one shows that

$$\Phi(z) = 2\sum_{n=1}^{\infty}\left(2\pi^2 n^4 e^{\frac{9}{2}z} - 3\pi n^2 e^{\frac{5}{2}z}\right)e^{-\pi n^2 e^{2z}} , \tag{13}$$

which leads to

$$\Phi(z) = 2\left\{\frac{d}{dz}\left(e^{3z}O^{(1)}\left(e^{2z}\right)\right)e^{-\frac{1}{2}z}\right\} = 2\sum_{n=1}^{\infty}\left(2\pi^2 n^4 e^{\frac{9}{2}z} - 3\pi n^2 e^{\frac{5}{2}z}\right)e^{-\pi n^2 e^{2z}} \tag{14}$$

and

$$\Xi(x) = 2\int_0^{\infty}\Phi(z)\cos(x\mathrm{z})\,dz . \tag{15}$$

Putting $\tau = e^{2z}$ in (8), we also have

$$\Lambda(x) = 4\int_1^{\infty}\mathrm{A}(\tau)\sin\left(\frac{x}{2}\log\tau\right)d\tau , \tag{13}$$

such that

$$\begin{aligned}\Lambda(x) &= 4\int_1^{\infty}\frac{d}{d\tau}\left(\tau^{\frac{3}{2}}O^{(1)}(\tau)\right)\tau^{-\frac{1}{4}}\sin\left(\frac{x}{2}\log\tau\right)d\tau \\ &= 4\int_0^{\infty}\frac{d}{dz}\left(e^{3z}O^{(1)}\left(e^{2z}\right)\right)e^{-\frac{1}{2}z}\sin(xz)\,dz \\ &= 2\int_0^{\infty}\Phi(z)\sin(xz)\,dz.\end{aligned} \tag{14}$$

Let us denote by

$$\begin{aligned}P(x) &= 4\int_1^{\infty}\frac{d}{d\tau}\left(\tau^{\frac{3}{2}}O^{(1)}(\tau)\right)\tau^{-\frac{1}{4}}\,\mathrm{e}^{i\frac{x}{2}\log\tau}\,d\tau \\ &= 4\int_0^{\infty}\frac{d}{dz}\left(e^{3z}O^{(1)}\left(e^{2z}\right)\right)e^{-\frac{1}{2}z}e^{ixz}dz \\ &= 2\int_0^{\infty}\Phi(z)e^{ixz}dz\end{aligned} \tag{15}$$

and

$$\begin{aligned}Q(x) &= 4\int_1^{\infty}\frac{d}{d\tau}\left(\tau^{\frac{3}{2}}O^{(1)}(\tau)\right)\tau^{-\frac{1}{4}}\,\mathrm{e}^{-i\frac{x}{2}\log\tau}\,d\tau \\ &= 4\int_0^{\infty}\frac{d}{dz}\left(e^{3z}O^{(1)}\left(e^{2z}\right)\right)e^{-\frac{1}{2}z}e^{-ixz}d\quad z \\ &= 2\int_0^{\infty}\Phi(z)e^{-ixz}dz,\end{aligned} \tag{16}$$

where $i = \sqrt{-1}$.

So, there exist

$$\Xi(x)=2\int_0^{\infty}\Phi(z)\cos(xz)\,dz=\frac{1}{2}\big(P(x)+Q(x)\big) \tag{17}$$

and

$$\Lambda(x)=2\int_0^{\infty}\Phi(z)\sin(xz)\,dz=\frac{1}{2i}\big(P(x)-Q(x)\big). \tag{18}$$

Since $\Xi(x)$ is an even entire function of order one [4], $P(x)$ and $Q(x)$ are the entire functions of order one. Moreover, $P(x)$ and $Q(x)$ have no zeros in the entire complex plane $x\in\mathbb{C}$. Of cause, $\Lambda(x)$ is of order one and of genus one.

A New Odd Entire Function of Order One

By (6), we see

$$\begin{aligned}\Xi(x,t)&=2\int_0^{\infty}e^{-ty^2}\Phi(y)\cos(xy)\,dy\\&=4\int_1^{\infty}e^{-t\frac{\log^2\tau}{4}}\frac{1}{2\tau}\Phi\left(\frac{\log\tau}{2}\right)\cos\left(\frac{x}{2}\log\tau\right)d\tau\\&=4\int_1^{\infty}e^{-t\frac{\log^2\tau}{4}}\mathrm{A}(\tau)\cos\left(\frac{x}{2}\log\tau\right)d\tau\end{aligned} \tag{19}$$

if we substitute

$$\log\tau=2y \tag{20}$$

into (6), and

$$\begin{aligned}\mathrm{A}(\tau)&=\frac{1}{2\tau}\Phi\left(\frac{\log\tau}{2}\right)\\&=\frac{1}{\tau}\sum_{n=1}^{\infty}\left(2\pi^2n^4e^{\frac{9}{4}\log\tau}-3\pi n^2e^{\frac{5}{4}\log\tau}\right)e^{-\pi n^2e^{\log\tau}}\\&=\sum_{n=1}^{\infty}\left(2\pi^2n^4\tau^{\frac{5}{4}}-3\pi n^2\tau^{\frac{1}{4}}\right)e^{-\pi n^2\tau}\\&=\frac{d}{d\tau}\left(\tau^{\frac{3}{2}}O^{(1)}(\tau)\right)\tau^{-\frac{1}{4}}.\end{aligned} \tag{21}$$

We now define the family of the entire function $\Lambda(x,t):\mathbb{C}\to\mathbb{C}$ for $t\in\mathbb{R}$ by the Fourier sine integral

$$\Lambda(x,t)=2\int_0^{\infty}e^{-ty^2}\Phi(y)\sin(xy)\,dy\,. \tag{22}$$

In view of (22), we find that

$$\Lambda(x,t)=4\int_1^{\infty}e^{-t\frac{\log^2\tau}{4}}\frac{1}{2\tau}\Phi\left(\frac{\log\tau}{2}\right)\sin\left(\frac{x}{2}\log\tau\right)d\tau=4\int_1^{\infty}e^{-t\frac{\log^2\tau}{4}}\mathrm{A}(\tau)\sin\left(\frac{x}{2}\log\tau\right)d\tau. \tag{23}$$

Let us denote by

$$\begin{aligned} P(x,t) &= 2\int_0^\infty e^{-ty^2}\Phi(y)\mathrm{e}^{ixy}\,dy \\ &= 4\int_1^\infty \frac{d}{d\tau}\left(\tau^{\frac{3}{2}}O^{(1)}(\tau)\right)\tau^{-\frac{1}{4}}\,\mathrm{e}^{i\frac{x}{2}\log\tau}\,d\tau \\ &= 4\int_0^\infty e^{-t\frac{\log^2\tau}{4}}\frac{d}{dz}\left(e^{3z}O^{(1)}\left(e^{2z}\right)\right)e^{-\frac{1}{2}z}e^{ixz}dz = 2\int_0^\infty e^{-t\frac{\log^2\tau}{4}}\Phi(z)e^{ixz}dz \end{aligned} \tag{24}$$

and

$$\begin{aligned} Q(x,t) &= 2\int_0^\infty e^{-ty^2}\Phi(y)\mathrm{e}^{-ixy}\,dy \\ &= 4\int_1^\infty e^{-t\frac{\log^2\tau}{4}}\frac{d}{d\tau}\left(\tau^{\frac{3}{2}}O^{(1)}(\tau)\right)\tau^{-\frac{1}{4}}\,\mathrm{e}^{-i\frac{x}{2}\log\tau}\,d\tau \\ &= 4\int_0^\infty e^{-t\frac{\log^2\tau}{4}}\frac{d}{dz}\left(e^{3z}O^{(1)}\left(e^{2z}\right)\right)e^{-\frac{1}{2}z}e^{-ixz}dz = 2\int_0^\infty e^{-t\frac{\log^2\tau}{4}}\Phi(z)e^{-ixz}dz. \end{aligned} \tag{25}$$

Hence,

$$\begin{aligned} \Xi(x,t) &= 2\int_0^\infty e^{-ty^2}\Phi(y)\cos(xy)\,dy \\ &= 2\int_0^\infty e^{-t\frac{\log^2\tau}{4}}\Phi(z)\cos(xz)\,dz = \frac{1}{2}\left(P(x,t)+Q(x,t)\right) \end{aligned} \tag{26}$$

and

$$\begin{aligned} \Lambda(x,t) &= 2\int_0^\infty e^{-ty^2}\Phi(y)\sin(xy)\,dy \\ &= 2\int_0^\infty e^{-t\frac{\log^2\tau}{4}}\Phi(z)\sin(xz)\,dz = \frac{1}{2i}\left(P(x,t)-Q(x,t)\right). \end{aligned} \tag{27}$$

Since $\Xi(x,t)$ is an even entire function of order one [10], $P(x,t)$ and $Q(x,t)$ are the entire functions of order one. Also, $P(x,t)$ and $Q(x,t)$ have no zeros in the entire complex plane $x\in\mathbb{C}$ and $\Lambda(x,t)$ is an odd entire function of order one because of

$$\Lambda(-x,t) = -\Lambda(x,t) \tag{28}$$

for $t\in\mathbb{R}$ and $x\in\mathbb{C}$.

Conjectures

The series of (27) is expressed as

$$\Lambda(x,t) = 2\int_0^\infty e^{-ty^2}\Phi(y)\sin(xy)\,dy = \sum_{m=0}^\infty \alpha(m,t)\,x^{2m+1}, \tag{29}$$

where

$$\alpha(m,t)=\frac{2(-1)^m}{(2m+1)!}\int_0^\infty e^{-ty^2}\Phi(y)\,y^{2m+1}dy\,. \tag{30}$$

It is well known that

$$\frac{2}{(2m+1)!}\int_0^\infty e^{-ty^2}\Phi(y)\,y^{2m+1}dy>0\,. \tag{31}$$

We now define the Omega function $\Omega(x,t)$ by

$$\Omega(x,t)=\sum_{m=0}^{\infty}\alpha(m,t)\,x^{2m} \tag{32}$$

with the functional equation

$$\Omega(x,t)=\Omega(-x,t)\,. \tag{33}$$

Since Csordas, Norfolk and Varga [10] have said that $\Lambda(x,t)$ is an odd entire function of order one, $\Omega(x,t)$ is an even entire function of order one.

Therefore, there exists the relation

$$\Lambda(x,t)=x\Omega(x,t), \tag{34}$$

which implies that $\Lambda(x,t)$ has a real zero $x=0$.

Clearly, $\Lambda(x,t)$ and $\Omega(x,t)$ have the same nontrivial zeros in the entire complex plane because $\Lambda(x,t)$ and $\Omega(x,t)$ are of order one and of genus one. It implies that $\Lambda(x,t)$ has infinity of zeros because the convergence exponent of the zeros of $\Lambda(x,t)$ is one (see [11], p.19). Hence, we have the following conjecture as follows:

Conjecture 1. The function $\Lambda(x,t)$ has only real zeros in the entire complex plane.

Conjecture 1 is equivalent to the following result:

Conjecture 2. The function $\Omega(x,t)$ has only real zeros in the entire complex plane.

The Solution for the Heat Equation Associated with the Odd Entire Function of Order One

Applying (22), we find that

$$\frac{\partial\Lambda(x,t)}{\partial t}=-2\int_0^\infty y^2e^{-ty^2}\Phi(y)\sin(xy)\,dy\,, \tag{35}$$

$$\frac{\partial\Lambda(x,t)}{\partial x}=2\int_0^\infty e^{-ty^2}\Phi(y)\,y\cos(xy)\,dy \tag{36}$$

and

$$\frac{\partial^2\Lambda(x,t)}{\partial x^2}=-2\int_0^\infty e^{-ty^2}\Phi(y)\,y^2\sin(xy)\,dy\,. \tag{37}$$

Making use of (35) and (37), we obtain (see [9], p.15)

$$\frac{\partial \Lambda(x,t)}{\partial t} = \frac{\partial^2 \Lambda(x,t)}{\partial x^2} \tag{38}$$

for $t > 0$ and $x \in \mathbb{R}$.

We now consider three cases of the initial and boundary conditions of Dirichlet-type, Neumann-type and Cauchy-type as follows.

There is the initial value condition

$$\Lambda(x, t=0) = 2\int_0^\infty \Phi(y)\sin(xy)\,dy\,. \tag{39}$$

Case 1: The Dirichlet-type boundary conditions for $a < x < b$.

By (22), we have

$$\Lambda(x,t) = 2\int_0^\infty e^{-ty^2}\Phi(y)\sin(xy)\,dy \tag{40}$$

such that

$$\Lambda(x=a,t) = 2\int_0^\infty e^{-ty^2}\Phi(y)\sin(ay)\,dy \tag{41}$$

and

$$\Lambda(x=b,t) = 2\int_0^\infty e^{-ty^2}\Phi(y)\sin(by)\,dy\,. \tag{42}$$

Case 2: The Neumann-type boundary conditions for $a < x < b$.

With (36) we obtain

$$\frac{\partial \Lambda(x,t)}{\partial t} = 2\int_0^\infty e^{-ty^2}\Phi(y)\,y\cos(xy)\,dy \tag{43}$$

such that

$$\frac{\partial \Lambda(x=a,t)}{\partial t} = 2\int_0^\infty e^{-ty^2}\Phi(y)\,y\cos(ay)\,dy \tag{44}$$

and

$$\frac{\partial \Lambda(x=b,t)}{\partial t} = 2\int_0^\infty e^{-ty^2}\Phi(y)\,y\cos(by)\,dy\,. \tag{45}$$

Case 3: The Cauchy-type boundary conditions for $a < x < b$.

It follows from (22) and (22) that

$$\Lambda(x=a,t) = 2\int_0^\infty e^{-ty^2}\Phi(y)\sin(ay)\,dy \tag{46}$$

and

$$\frac{\partial\Lambda(x=a,t)}{\partial t}=2\int_0^\infty e^{-ty^2}\Phi(y)\,y\cos(ay)\,dy\,. \tag{47}$$

Conclusion

In this work we have suggested an odd entire function of order one, which is the solution of the heat equation. We have investigated the Dirichlet-type, Neumann-type and Cauchy-type boundary conditions for the heat equation. We have conjectured that this function has only real zeros. This can be considered as a new direction in the heat equation and number theory.

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Nomenclature

$\mathbb{R}$ -set of real numbers, [$-$]	$\mathbb{C}$ -set of complex numbers, [$-$]
t -time coordinate, [s]	x -space coordinate, [m]

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