AN ODD ENTIRE-FUNCTION SOLUTION FOR ONE-DIMENSIONAL DIFFUSION EQUATION IN THEORY OF MODULAR FORM

by

Xiao-Jun YANG [a,b,c,*], Tasawar ***HAYAT*** [d]

[a] State Key Laboratory for Geo-Mechanics and Deep Underground Engineering, and School of Mathematics, China University of Mining and Technology, Xuzhou 221116, China
[b] Department of Mathematics, Faculty of Science, King Abdulaziz University P.O. Box 80257, Jeddah 21589, Saudi Arabia
[c] Department of Mathematics, College of Science, Kyung Hee University, 26 Kyungheedae-ro, Dongdaemun-gu, Seoul 02447, Republic of Korea
[d] Department of Mathematics, Quaid-i-Azam University, Islamabad 44000, Pakistan

This article addresses a new odd entire function of order one structured by the Fourier sine integral, which is the solution of the one-dimensional diffusion equation in theory of modular form.

Key words: *diffusion equation, odd entire function, Fourier sine integral, modular form*

Introduction

Ramanujan considered a modular form as [1]

$$\Theta(\ell)=q\prod_{m=1}^{\infty}\left(1-q^{m}\right)^{24}=\sum_{k=1}^{\infty}\vartheta(k)q^{k}\ , \tag{1}$$

where $q=e^{2\pi i\ell}$. In fact, (1) is the weight 12 cusp form for $SL_2(\mathbb{Z})$ [2] and $\vartheta(k)$ represents the Ramanujan's arithmetical function [1].

The Ramanujan zeta function $\mathrm{M}(x)$ is defined as [1]

$$\mathrm{M}(\mu)=\sum_{n=1}^{\infty}\frac{\vartheta(n)}{n^{\mu}} \tag{2}$$

for $\mathrm{Re}(\mu)>13/2$. Here, (2) has an Euler product of the form [3]

$$\mathrm{M}(\mu)=\prod_{p}\frac{1}{1-\vartheta(\ell)\,p^{-\mu}+p^{11-2\mu}}\ . \tag{3}$$

Clearly, (1) has the functional equation [4]

$$\Theta(\ell)=\frac{1}{\ell^{12}}\Theta\left(-\frac{1}{\ell}\right) \tag{4}$$

* Corresponding author; e-mail: dyangxiaojun@163.com

for $\ell > 0$.

The Ramanujan xi function reads [4]

$$\hat{\xi}(\mu) = \int_0^\infty \tau^{\mu-1}\Theta(i\tau)\,d\tau . \tag{5}$$

It follows from (5) that [4]

$$\hat{\xi}(\mu) = \int_0^1 \Theta(i\tau)\left(\tau^{\mu-1} + \tau^{11-\mu}\right)d\tau . \tag{6}$$

with the functional equation [4]

$$\hat{\xi}(\mu) = \hat{\xi}(12-\mu). \tag{7}$$

It is easy to see that [5]

$$\hat{\xi}(\mu) = 2\int_0^1 \Theta(i\tau)\tau^5 \cosh\left[(\mu-6)\log\tau\right]d\tau . \tag{8}$$

On putting $\mu = 6 + ix$ in (8), we show that [5]

$$\hat{\Xi}(x) = 2\int_0^1 \Theta(i\tau)\tau^5 \cos(x\log\tau)\,d\tau \tag{9}$$

is an even entire function of order one [5].

Substituting $\tau = e^{-z}$ into (9) gives

$$\hat{\Xi}(x) = 2\int_0^1 \Theta(i\tau)\tau^5 \cos(x\log\tau)\,d\tau = \int_0^\infty \Lambda(z)\cos(xz)\,dz , \tag{10}$$

where

$$\Lambda(z) = 2e^{-2\pi e^{-z}} e^{-6z} \prod_{m=1}^\infty \left(1 - e^{-2\pi m e^{-z}}\right)^{24} . \tag{11}$$

Let us denote by

$$\mathbb{P}(x) = \int_0^\infty \Lambda(z) e^{ixz}\,dz \tag{12}$$

and

$$\mathbb{Q}(x) = \int_0^\infty \Lambda(z) e^{-ixz}\,dz . \tag{13}$$

It is clear that $\mathbb{P}(x)$ and $\mathbb{Q}(x)$ are the entire functions of order one. From (10), (12) and (13) we obtain

$$\hat{\Xi}(x) = \int_0^\infty \Lambda(z)\cos(xz)\,dz = \frac{1}{2}\left(\mathbb{P}(x) + \mathbb{Q}(x)\right). \tag{14}$$

By (12) and (13), we show

$$\Omega(x)=\frac{1}{2i}\left(\mathbb{P}(x)-\mathbb{Q}(x)\right)=\int_0^\infty \Lambda(z)\sin(xz)\,dz\,. \tag{15}$$

Thus, $\Omega(x)$ is an odd entire function of order one.

The target of this paper is to consider an odd entire function

$$\Omega(x,t)=\int_0^\infty e^{-tz^2}\Lambda(z)\sin(xz)\,dz\,, \tag{16}$$

to reduce to the classical diffusion equation in the one-dimensional space [6]

$$\frac{\partial\Lambda(x,t)}{\partial t}-\frac{\partial^2\Lambda(x,t)}{\partial x^2}=0 \tag{17}$$

for $x\in\mathbb{R}$ and $t\in\mathbb{R}$.

The structure of the paper is presented as follows. In Section 2 we guess the behavior of the real zeros for (16). In Section 3 we derive the solution (16) for the classical diffusion equation (17). Finally we get the conclusion in Section 4.

The Real Zeros for (16)

We now define a family of the function $\hat{\Xi}(x,t):\mathbb{C}\to\mathbb{C}$ for $t\in\mathbb{R}$ by the Fourier sine integral

$$\Omega(x,t)=\int_0^\infty e^{-tz^2}\Lambda(z)\sin(xz)\,dz\,, \tag{18}$$

which can be decomposed as

$$\Omega(x,t)=\frac{1}{2i}\left(\mathbb{P}(x,t)+\mathbb{Q}(x,t)\right), \tag{19}$$

where

$$\mathbb{P}(x,t)=\int_0^\infty e^{-tz^2}\Lambda(z)e^{ixz}\,dz \tag{20}$$

and

$$\mathbb{Q}(x,t)=\int_0^\infty e^{-tz^2}\Lambda(z)e^{-ixz}\,dz\,. \tag{21}$$

It is seen from (18) that

$$\Omega(-x,t)=-\Omega(x,t), \tag{22}$$

which implies that $\Omega(x,t)$ is an odd function.

Here, $\Omega(x,t)$ is also an odd entire function of order one. It is easy to see that $\Omega(x,t)$ is of genus one. Although $\mathbb{P}(x,t)$ and $\mathbb{Q}(x,t)$ are the entire functions of order one, $\mathbb{P}(x,t)$ and $\mathbb{Q}(x,t)$ have no zeros in the entire complex plane $x\in\mathbb{C}$. So, the convergence exponent of the zeros of $\Omega(x,t)$ is one (see [7], p.19).

We have the result as follows:

Conjecture I. $\Omega(x,t)$ has purely real zeros in the entire complex plane $x \in \mathbb{C}$.

A Special Solution of the One-dimensional Diffusion Equation

Making use of (18), we arrive at

$$\frac{\partial \Omega(x,t)}{\partial t} = \frac{\partial}{\partial t}\left\{\int_0^\infty e^{-tz^2}\Lambda(z)\sin(xz)\,dz\right\} = -\int_0^\infty z^2 e^{-tz^2}\Lambda(z)\sin(xz)\,dz\,, \tag{23}$$

$$\frac{\partial \Omega(x,t)}{\partial x} = \frac{\partial}{\partial x}\left\{\int_0^\infty e^{-tz^2}\Lambda(z)\sin(xz)\,dz\right\} = \int_0^\infty z e^{-tz^2}\Lambda(z)\cos(xz)\,dz \tag{24}$$

and

$$\frac{\partial^2 \Omega(x,t)}{\partial x^2} = \frac{\partial^2}{\partial x^2}\left\{\int_0^\infty e^{-tz^2}\Lambda(z)\sin(xz)\,dz\right\} = -\int_0^\infty z^2 e^{-tz^2}\Lambda(z)\sin(xz)\,dz\,. \tag{25}$$

If $x \in \mathbb{R}$ and $t \in \mathbb{R}$, it follows from (23) and (25) that

$$\frac{\partial \Omega(x,t)}{\partial t} = \frac{\partial^2 \Omega(x,t)}{\partial x^2}\,, \tag{26}$$

which is a classical diffusion equation in the one-dimensional space [6] for $\alpha < x < \beta$ and $t > 0$.

By (18) we present the initial value condition

$$\Omega(x,t=0) = \int_0^\infty \Lambda(z)\sin(xz)\,dz\,, \tag{27}$$

and the Dirichlet-type boundary value conditions

$$\Omega(x=\alpha,t) = \int_0^\infty e^{-tz^2}\Lambda(z)\sin(\alpha z)\,dz\,, \tag{28}$$

and

$$\Omega(x=\beta,t) = \int_0^\infty e^{-tz^2}\Lambda(z)\sin(\beta z)\,dz\,. \tag{29}$$

Conclusion

In the present paper we have obtained the odd entire function of order one via theory of modular form. This has been structured by the Fourier sine integral. We have conjectured that it has purely real zeros in the entire complex plane. We have derived the one-dimensional diffusion equation with the Dirichlet-type boundary value conditions.

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Nomenclature

$\mathbb{R}$ -set of real numbers, [$-$]	$\mathbb{C}$ -set of complex numbers, [$-$]
t -space coordinate, [s]	x -time coordinate, [m]

$\mathrm{Re}(\mu)$ -real part, [-]

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