THE 3-D PROBLEM OF TEMPERATURE AND THERMAL FLUX DISTRIBUTION AROUND DEFECTS WITH TEMPERATURE-DEPENDENT MATERIAL PROPERTIES

by

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The analytical solution of 3-D heat conduction problem, including the temperature and thermal flux fields, is one of the important problems that have not been completely solved in solid mechanics. Considering the temperature dependence of material parameters makes the problem more difficult. In this paper, we first reduce the 3-D temperature-dependent heat conduction problem to the solution of 3-D Laplace equation by introducing the intermediate function. Then, the generalized ternary function is proposed, and the general solution of 3-D Laplace equation is given. Finally, the analytical solutions of three specific problems are obtained and the corresponding temperature-thermal flux fields are discussed. The results show that the thermal flux field of 3-D temperature dependent problem is the same as the classical constant thermal conductivity approach result, while the temperature field is different from the classical result. Thermal flux at a planar defect boundary has $r^{-1/2}$ *singularity, and its intensity is proportional to the fourth root of defect width. On the other hand, when blocked by a planar defect, the thermal flux distribution will re-adjusted so that it overflows at the same rate from all parts of the planar defect boundary.*

Key words: temperature-dependent, 3-D problem, thermal conductivity, thermal flux, temperature field

Introduction

Laplace equation is one of the most important PDE, which plays a cornerstone role in heat, electricity, magnetism, soap film, fluid, gravity and even machine learning [1-6]. Because of its special status, scholars have also done a lot of research on the analytical solution of Laplace equation for centuries. For example, the partial general solution of 3-D Laplace equation can be obtained by separating variables [7, 8]. In addition, when the problem has some symmetry and is independent of a co-ordinate, the general solutions of some problems can be obtained in rectangular co-ordinates, cylindrical co-ordinates or spherical co-ordinates [9, 10]. Most importantly, the general solution of 2-D Laplace equation can be obtained by the complex function method, accordingly a large number of classical 2-D elastic and thermoelastic problems have been solved, and the analytical solutions [11, 12] obtained have become the cornerstone of understanding the distribution of 2-D thermal and elastic fields. However, the general solution of 3-D Laplace equation has not been solved, and this motivates our current work.

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On the other hand, the change of temperature affects the micro-structure and chemical composition of the medium, thus affecting its thermal conductivity and other material properties, which also complicated the analysis of heat conduction. Much efforts have been made to study the disturbance of temperature dependence on heat conduction [13, 14], since the temperature dependence of material properties will be very significant when subjected to large temperature gradient. For instance, the temperature dependent thermal conductivity by solving IHCP in infinite region was determined in [15], and the thermal properties of in situ grown graphene reinforced copper matrix laminated composites was researched in [16]. Nevertheless, continuum analysis of 3-D temperature-dependence heat conduction problem is rare, partly due to the lack of a closed form solution for 3-D problem.

Earlier continuum models of heat conduction ignored the temperature dependence of material properties [17, 18], and thus cannot be applied when the materials are subjected to large temperature gradient that are relevant for practical applications. More rigorous calculation has been carried out by some simulation and numerical methods, such as finite element methods [19], 3-D *J*-integral [20], variational method [21], floating random walk Monte-Carlo method and differential transform method, wherein strict analytical derivation is not required. Only very recently, 2-D temperature-dependent problem have been analyzed using generalized complex variable method [22]. It is thus highly desirable if closed-form solutions for 3-D temperature-dependent problem can be derived.

In this paper, 3-D problem of temperature and thermal flux distribution around defects with temperature-dependent material properties are analyzed. The 3-D temperature-dependent heat conduction problem is reduced to the solution of 3-D Laplace equation. The generalized ternary function is proposed to reach the general solution of 3-D Laplace equation. The analytical solutions of three specific problems are obtained, and the corresponding temperature-thermal flux fields are discussed. The results show that the thermal flux field of 3-D temperature dependent problem is the same as the classical constant thermal conductivity approach result, while the temperature field is different from the classical result. Thermal flux at a planar defect boundary has $r^{-1/2}$ singularity, and its intensity is proportional to the fourth root of defect width. On the other hand, when blocked by a planar defect, the thermal flux distribution will re-adjusted so that it overflows at the same rate from all parts of the planar defect boundary. These results provide a powerful tool to analyze the 3-D heat conduction problem, and other problems related to the analytical solution of 3-D Laplace equation.

The 3-D problems

Temperature-dependent heat conduction equation

In a 3-D medium, given that the thermal conductivity of this medium changes with temperature, the heat conduction equation could be expressed:

$$J_{Q_x} = -\kappa(T)\frac{\partial T}{\partial x},\ J_{Q_y} = -\kappa(T)\frac{\partial T}{\partial y},\ J_{Q_z} = -\kappa(T)\frac{\partial T}{\partial z} \tag{1}$$

where T is temperature field, $\kappa(T)$ – the heat conductivity which varies with T arbitrarily, and J_{Qx}, J_{Qy}, and J_{Qz} reprezend the thermal flux components along x-, y-, and z-axes.

In the following analysis, we regard that heat energy is conserved in this system, such that thermal flux is divergence-free:

$$\nabla J_Q = 0 \tag{2}$$

General solution of 3-D temperature-dependent heat conduction equation

Temperature dependence of thermal conductivity greatly complicated the solution of temperature field. Similar to the 2-D problem, we introduce a new function Φ(T) which satisfies:

$$\Phi(T) = \int \kappa(T)\mathrm{d}T \tag{3}$$

According to chain rule, the first order partial derivatives of $\Phi(T)$ to x, y, and z could be explained:

$$\frac{\partial \Phi}{\partial x} = \kappa(T)\frac{\partial T}{\partial x} = -J_{Q_x}, \quad \frac{\partial \Phi}{\partial y} = \kappa(T)\frac{\partial T}{\partial y} = -J_{Q_y}, \quad \frac{\partial \Phi}{\partial z} = \kappa(T)\frac{\partial T}{\partial z} = -J_{Q_z} \tag{4}$$

Substituting eq. (4) into eq. (2), and we obtain a classical Laplace equation:

$$\frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} + \frac{\partial^2 \Phi}{\partial z^2} = 0 \tag{5}$$

Laplace equation is one of the most important PDE, which plays a cornerstone role in electricity, magnetism, heat, soap film, fluid, gravity and machine learning. However, the analytical solution of 3-D Laplace equation has not been completely solved so far. Our goal is to find the general solution of 3-D Laplace equation, and thus determine the distribution of thermal flux and temperature gradient around defects in 3-D matrix. To this end, we introduce the ternary equation here (as detailed in *Appendix A*):

$$u = x + aiy + bjz, \; \overline{u} = x - aiy - bjz, \; u^* = x - aiy + bjz \tag{6}$$

where i, j are the unit imaginary numbers, while u, $\overline{u}$, u^* are three ternary variables which have a unique relationship with the original three rectangular co-ordinate variables. Now $\partial T/\partial x$, $\partial T/\partial y$, and $\partial T/\partial z$ can be re-expressed:

$$\frac{\partial T}{\partial x} = \frac{\partial T}{\partial u} + \frac{\partial T}{\partial \overline{u}} + \frac{\partial T}{\partial u^*}, \frac{\partial T}{\partial y} = ai\left(\frac{\partial T}{\partial u} - \frac{\partial T}{\partial \overline{u}} - \frac{\partial T}{\partial u^*}\right), \; \frac{\partial T}{\partial z} = bj\left(\frac{\partial T}{\partial u} - \frac{\partial T}{\partial \overline{u}} + \frac{\partial T}{\partial u^*}\right) \tag{7}$$

Combining eq. (7) with eq. (4) and substituting into thermal conservation eq. (2), we gain:

$$\begin{aligned} &\frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} + \frac{\partial^2 \Phi}{\partial z^2} = \\ &= (1 - a^2 - b^2)\left(\frac{\partial^2 \Phi}{\partial u^2} + \frac{\partial^2 \Phi}{\partial \overline{u}^2} + \frac{\partial^2 \Phi}{\partial u^{*2}}\right) + 2(1 + a^2 + b^2)\frac{\partial^2 \Phi}{\partial u \partial \overline{u}} + \\ &+ 2(1 + a^2 - b^2)\frac{\partial^2 \Phi}{\partial u \partial u^*} + 2(1 - a^2 + b^2)\frac{\partial^2 \Phi}{\partial \overline{u} \partial u^*} \end{aligned} \tag{8}$$

Assuming that a and b satisfy the condition $a^2 + b^2 = 1$, the aforementioned Laplace equation can be simplified:

$$\frac{\partial^2 \Phi}{\partial u \partial \overline{u}} + a^2 \frac{\partial^2 \Phi}{\partial u \partial u^*} + b^2 \frac{\partial^2 \Phi}{\partial \overline{u} \partial u^*} = 0 \tag{9}$$

Obviously, any single variable function (with u or $\overline{u}$ or u^* as its variable) satisfies the 3-D Laplace equation. Thus Φ could be expressed:

$$\Phi = f_1(u) + f_2(\overline{u}) + f_3(u^*) \tag{10}$$

We must point out that any generalized ternary variable function taking $\overline{u}^*$ as its variable is also the solution of 3-D Laplace equation (as detailed in *Appendix B*). Therefore, the general solution can be re-expressed:

$$\Phi = f_1(u) + f_2(\overline{u}) + f_3(u^*) + f_4(\overline{u}^*) \tag{11}$$

Considering a physical field is generally a real function, the solution of the eq. (11) can be finally expanded (as detailed in *Appendix C*):

$$\Phi = f(u) + \overline{f(u)} + f^*(u^*) + \overline{f^*(u^*)} \tag{12}$$

This general solution expressed in ternary variables brings great convenience to the analytical solution of 3-D Laplace equation. According to the relation between Φ and thermal flux in eq. (4), thermal flux components are deduced:

$$\begin{gathered} J_{Q_x} = -\left[f'(u) + \overline{f'(u)} + f^{*\prime}(u) + \overline{f^{*\prime}(u)} \right],\ J_{Q_y} = -ai\left[f'(u) - \overline{f'(u)} - f^{*\prime}(u) + \overline{f^{*\prime}(u)} \right] \\ J_{Qz} = -bj\left[f'(u) - \overline{f'(u)} + f^{*\prime}(u) - \overline{f^{*\prime}(u)} \right] \end{gathered} \tag{13}$$

General treatment of boundary condition

According to the impermeable boundary, the normal thermal flux component at the boundary is zero:

$$J_{Q_\perp} = 0 \tag{14}$$

Substituting thermal flux component eq. (13) into eq. (14), we obtain:

$$\begin{aligned} J_{Q_\perp} &= (J_{Q_x}, J_{Q_y}, J_{Q_z})(n_x, n_y, n_z) = J_{Q_x} n_x + J_{Q_y} n_y + J_{Q_z} n_z = \\ &= -[(n_x + n_y ai + n_z bj) f'(u) + (n_x - n_y ai - n_z bj)\overline{f'(u)} + \\ &+ (n_x - n_y ai + n_z bj) f^{*\prime}(u^*) + (n_x + n_y ai - n_z bj)\overline{f^{*\prime}(u^*)}] = 0 \end{aligned} \tag{15}$$

where n_x, n_y, and n_z are the components of the unit normal vector in the direction of x, y, and z, respectively.

General solution of temperature field

Once $f'(u)$ is derived, eq. (3) can be expanded:

$$\int \kappa(T)\mathrm{d}T = f(u) + \overline{f(u)} + f^*(u^*) + \overline{f^*(u^*)} = 4\,\mathrm{Re}[f(u)] \tag{16}$$

Obviously, the solution of temperature field depends on the concrete form of $\kappa(T)$. In most cases, $\kappa(T)$ should be determined by fitting with the experimental data. For ease of calculation, we expand $\kappa(T)$ to a polynomial of temperature:

$$\kappa(T) = \sum_s^n \kappa_s T^s \tag{17}$$

where κ_s is the coefficient determined by experimental data. In this case, eq. (16) will be transformed into a polynomial equation, and the temperature can be obtained by using the polynomial root formula. For instance, when other coefficients are zero except κ_0 and κ_1, temperature function could be determined:

$$T = -\frac{\kappa_0}{\kappa_1} + \frac{\sqrt{\kappa_0^{\,2} - 2\kappa_1 c_1 + 8\kappa_1\,\mathrm{Re}[f(u)]}}{\kappa_1} \tag{18}$$

where c_1 is the integral constant, which can be determined by the temperature value at the specified position.

Solutions for 3-D problems

To verify the theory, we will consider three specific problems, namely an infinite oblique cylinder cavity, a diamond flake crack and an infinite strip flake crack. Before calculating, we must point out that two-unit imaginary numbers i, j will participate in the calculation. In order to ensure that the general solution of the 3-D Laplace equation is always true, i and j must meet one of the following two relations:

- the i and j are the same imaginary numbers $i = j$ and
- the i and j are completely different imaginary numbers.

In this case $i \pm j$, $i \times j$, and i/j, are all basic quantities and cannot be further simplified.

We emphasize that although the ternary function introduced in this paper is a new mathematical tool, it still exists only in the intermediate process, and the final expression of the heat flow temperature fields can only be real functions. Under the two relations mentioned previously, the process of taking the real part of a ternary function is quite different, which means that the same ternary function has different real parts under different relations. Therefore, in general, the choice of Relation 1 or Relation 2 depends on which can give the correct real part.

In addition, we cannot rule out that a ternary function A under Relation 1 and another ternary function B under Relation 2 have the same real part, but it is obvious that in this case, the real parts of the two are equivalent and both are the solutions of a certain problem, and do not violate the uniqueness of the real function solution of this problem.

Finally, we point out that the operation when applying Relation 1 $i = j$ is much simpler than that when applying Relation 2 $i \neq j$. However, only a few problems can be solved under Relation 1, because only a few boundary equations can be transformed into ternary function equations when $i = j$. Specifically, when $i = j$, then $u = x + i(ay + bz)$, our ternary function is equivalent to replacing y in the classical complex variable function with $ay + bz$. That is, a point on the plane (*e.g.* $x = 1$, $y = 1$) becomes a straight line in 3-D space ($x = 1$, $ay + bz = 1$). Therefore, Relation 1 is more suitable for solving the model whose cross-sectional shape is invariable along a certain direction, while more complex models must be solved under Relation 2.

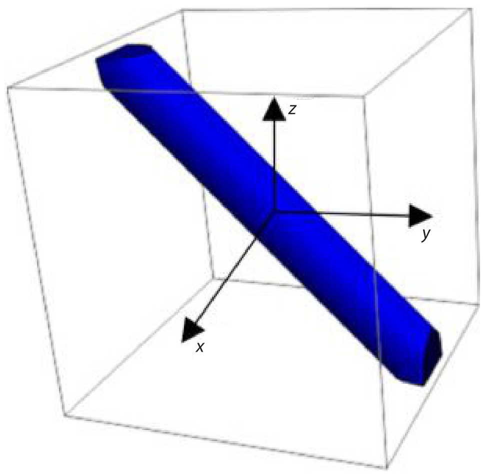

Figure 1. An infinite oblique cylinder cavity embedded in an infinite matrix

Model 1: Infinite oblique cylinder cavity

Firstly, we consider an infinite cylinder cavity, with the surface equation $x^2+(ay+bz)^2=L_1$, embedded in an infinite medium, as shown in fig. 1, where $L_1^{1/2}$ denote the radius of cross-section circle, and the arctangent of b/a is the inclination of the axis of the cylinder in the $y-z$ plane. At the far field, the matrix is subjected to thermal flux J_{Qx}^{∞}, J_{Qy}^{∞}, and J_{Qz}^{∞}.

Obviously, when $i=j$, the surface equation can be simply expressed as $u\bar{u}=L_1$. According to the remote thermal flux, $f'(u)$ can be obtained:

$$f'(u)=A+f_0'(u) \tag{19}$$

where A is tehe determined by remote thermal loads according to eq. (13), and $f_0'(u)$ satisfies the condition:

$$\lim_{u\to\infty} f_0'(u)=0 \tag{20}$$

Substituting boundary condition and $f'(u)$ to insulation boundary condition eq. (15) yields:

$$[n_x+(n_ya+n_zb)i]f'(u)+[n_x-(n_ya+n_zb)i]\overline{f'(u)}=2\left[f_0'(u)u+\bar{A}\frac{L_1}{u}+\overline{f_0'(u)}\frac{L_1}{\bar{u}}+Au\right]=0 \tag{21}$$

thus we obtain:

$$f_0'(u)=-\bar{A}\frac{L_1}{u^2} \tag{22}$$

Then function $f'(u)$ could be finally expressed:

$$f'(u)=A-\bar{A}\frac{L_1}{u^2} \tag{23}$$

Now we just need to solve A. Noting that A has only two independent parts, the real part and the imaginary part (since we letting $i=j$ in the solving process), which seems to contradict the independent loading in three directions of the 3-D problem. To explain this clearly, we introduce a new orthogonal co-ordinate, as shown in fig. 2, where x'-axis is coincident with the original x-axis, y'-axis is along the axis of the cylinder and z'-axis is perpendicular to the x' and y' axes.

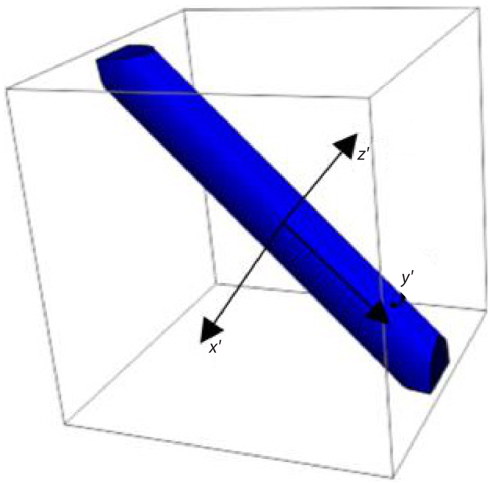

Figure 2. An infinite cylinder cavity in $x'y'z'$ rectangular co-ordinates

The transforms between two co-ordinates are:

$$x' = x, y' = by - az, \ z' = ay + bz \tag{24}$$

and the remote thermal fluxes along x', y', z' axes $J_{Qx}^{\infty\prime}$, $J_{Qy}^{\infty\prime}$, and $J_{Qz}^{\infty\prime}$ are:

$$J_{Q_x}^{\infty}{}' = J_{Q_x}^{\infty}, \ \ J_{Q_y}^{\infty}{}' = bJ_{Q_y}^{\infty} - aJ_{Q_z}^{\infty}, \ \ J_{Q_z}^{\infty}{}' = aJ_{Q_y}^{\infty} + bJ_{Q_z}^{\infty} \tag{25}$$

Obviously, the thermal flux along y'-axis (axis direction of cylinder) in the whole medium will not be disturbed by the cylinder and is always constant. Therefore, we do not need to solve the thermal flux field brought by $J_{Qy}^{\infty\prime}$, but only need to determine the solution of the $J_{Qx}^{\infty\prime}$, $J_{Qz}^{\infty\prime}$. In other words, if the final solution is inconsistent with $J_{Qy}^{\infty\prime}$, it is only necessary to superimpose the uniform thermal flux field in y'-direction $J_{Qy}^{\infty\prime\prime}$, to make the final solution consistent with the loading.

Now, substituting $f'(u)$ of eq. (23) into the thermal flux component we obtain:

$$A = -\frac{1}{4}\left(J_{Q_x}^{\infty}{}' - J_{Q_z}^{\infty}{}' i\right) \tag{26}$$

Thus, the temperature and thermal flux can be obtained:

$$\begin{aligned}
T &= -\frac{\kappa_0}{\kappa_1} + \frac{\sqrt{\kappa_0^2 - 2\kappa_1 c + 2\kappa_1 \operatorname{Re}\left[-\left(J_{Q_x}^{\infty}{}' - \dfrac{bJ_{Q_y}^{\infty}{}'' + aJ_{Q_z}^{\infty}{}'}{a} i\right)u - \left(J_{Q_x}^{\infty}{}' + \dfrac{bJ_{Q_y}^{\infty}{}'' + aJ_{Q_z}^{\infty}{}'}{a} i\right)\dfrac{L_1}{u}\right]}}{\kappa_1} \\
J_{Q_x} &= \frac{1}{2}\left[2J_{Q_x}^{\infty}{}' - \left(J_{Q_x}^{\infty}{}' + \frac{bJ_{Q_y}^{\infty}{}'' + aJ_{Q_z}^{\infty}{}'}{a} i\right)\frac{L_1}{u^2} - \left(J_{Q_x}^{\infty}{}' - \frac{bJ_{Q_y}^{\infty}{}'' + aJ_{Q_z}^{\infty}{}'}{a} i\right)\frac{L_1}{\bar{u}^2}\right] \\
J_{Q_y} &= -\frac{1}{2}ai\left[\frac{2\left(bJ_{Q_y}^{\infty}{}'' + aJ_{Q_z}^{\infty}{}'\right)}{a} i + \left(J_{Q_x}^{\infty}{}' + \frac{bJ_{Q_y}^{\infty}{}'' + aJ_{Q_z}^{\infty}{}'}{a} i\right)\frac{L_1}{u^2} - \left(J_{Q_x}^{\infty}{}' - \frac{bJ_{Q_y}^{\infty}{}'' + aJ_{Q_z}^{\infty}{}'}{a} i\right)\frac{L_1}{\bar{u}^2}\right] \\
J_{Q_z} &= -\frac{1}{2}bi\left[\frac{2\left(bJ_{Q_y}^{\infty}{}'' + aJ_{Q_z}^{\infty}{}'\right)}{a} i + \left(J_{Q_x}^{\infty}{}' + \frac{bJ_{Q_y}^{\infty}{}'' + aJ_{Q_z}^{\infty}{}'}{a} i\right)\frac{L_1}{u^2} - \left(J_{Q_x}^{\infty}{}' - \frac{bJ_{Q_y}^{\infty}{}'' + aJ_{Q_z}^{\infty}{}'}{a} i\right)\frac{L_1}{\bar{u}^2}\right]
\end{aligned} \tag{27}$$

and in rectangular co-ordinates, they are:

$$\begin{aligned}
T &= -\frac{\kappa_0}{\kappa_1} + \frac{\sqrt{\kappa_0^2 - 2\kappa_1 c - 2\kappa_1\left[J_{Q_x}^{\infty}{}' x' + \dfrac{bJ_{Q_y}^{\infty}{}'' + aJ_{Q_z}^{\infty}{}'}{a} z'\right]\left[1 + \dfrac{L_1}{x'^2 + z'^2}\right]}}{\kappa_1} \\
J_{Q_x} &= J_{Q_x}^{\infty}{}' - \frac{aJ_{Q_x}^{\infty}{}' L_1(x'^2 - z'^2) + 2(bJ_{Q_y}^{\infty}{}'' + aJ_{Q_z}^{\infty}{}')L_1 x'z'}{a(x'^2 + z'^2)^2} \\
J_{Q_y} &= bJ_{Q_y}^{\infty}{}'' + aJ_{Q_z}^{\infty}{}' + \frac{(bJ_{Q_y}^{\infty}{}'' + aJ_{Q_z}^{\infty}{}')L_1(x'^2 - z'^2) - 2aL_1 J_{Q_x}^{\infty}{}' x'z'}{(x'^2 + z'^2)^2} \\
J_{Q_z} &= b\left[\frac{bJ_{Q_y}^{\infty}{}'' + aJ_{Q_z}^{\infty}{}'}{a} + \frac{(bJ_{Q_y}^{\infty}{}'' + aJ_{Q_z}^{\infty}{}')L_1(x'^2 - z'^2) - 2J_{Q_x}^{\infty}{}' aL_1 x'z'}{a(x'^2 + z'^2)^2}\right]
\end{aligned} \tag{28}$$

When the thermal conductivity is constant, the temperature field:

$$T=\frac{J_{Q_x}^{\infty}}{\kappa}\mathrm{Re}\left[i\sqrt{-(x'+iz')^2-L_3}\right] \tag{29}$$

Model 2: A finite diamond flake crack

Then we consider a finite diamond flake crack $x = 0(ay + bz)^2 \le L_2$, $(ay - bz)^2 \le L_2$ embedded in an infinite matrix, as shown in fig. 3, where $L_2^{1/2}/a$ and $L_2^{1/2}/b$ denote the length of half diamond diagonals along y-axis and z-axis. The origin of the co-ordinate is located at the center of the crack. At the far field, the material is subjected to imposed thermal flow J_{Qx}^{∞}.

In this case, we choose $i \neq j$. According to the symmetry, we suppose that a and b have same signs and discuss the region in the first quadrant. Referring to the solution of complex variable method for 2-D problem, when the medium is subjected to imposed thermal flux J_{Qx}^{∞}, $\Phi(T)$ could be chosen:

$$\Phi(T)=\frac{J_{Q_x}^{\infty}}{4}\left[(i+j)\sqrt{-u^2-L_2}+\overline{(i+j)\sqrt{-u^2-L_2}}+(i^*+j^*)\sqrt{-u^{*2}-L_2}+\overline{(i^*+j^*)\sqrt{-u^{*2}-L_2}}\right] \tag{30}$$

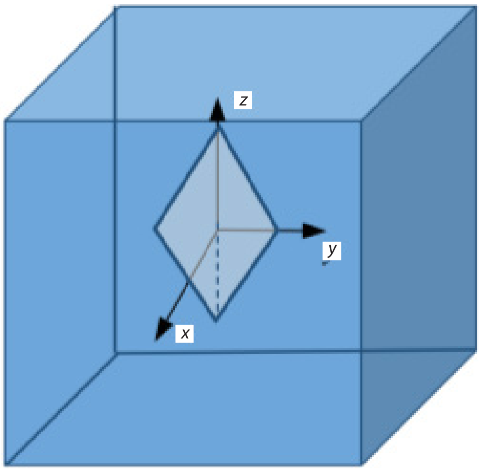

Figure 3. A diamond flake crack embedded in an infinite matrix

Obviously, each term of $\Phi(T)$ is the solution of 3-D Laplace equation, and the sum of the four terms ensure it a real function. Considering the value of the square root, the previous equation can be expressed at infinity:

$$\begin{aligned}\lim_{u\to\infty}\Phi(T)&=\frac{J_{Q_x}^{\infty}}{4}\left[(i+j)\times iu+\overline{(i+j)}\times\overline{iu}+(i^*+j^*)\times i^*u^*+\overline{(i^*+j^*)}\times\overline{i^*u^*}\right]\\&=\frac{J_{Q_x}^{\infty}}{4}\left(-u+iju-\overline{u}+\overline{iju}-u^*+i^*j^*u^*-\overline{u^*}+\overline{i^*j^*u^*}\right)=-J_{Q_x}^{\infty}x\end{aligned} \tag{31}$$

which is consistent with the remote thermal flux J_{Qx}^{∞}.

According to thermal flux component eq. (13), the distribution of thermal flux along the x, y, and z directions are obtained:

$$\begin{gathered}J_{Q_x}=-\frac{\partial\Phi}{\partial x}=\\=\frac{J_{Q_x}^{\infty}}{4}\left[(i+j)\frac{u}{\sqrt{-u^2-L_2}}+\overline{(i+j)\frac{u}{\sqrt{-u^2-L_2}}}+(i^*+j^*)\frac{u^*}{\sqrt{-u^{*2}-L_2}}+\overline{(i^*+j^*)\frac{u^*}{\sqrt{-u^{*2}-L_2}}}\right]\\J_{Q_y}=-\frac{\partial\Phi}{\partial y}=\\=\frac{J_{Q_x}^{\infty}a}{4}\left[(i+j)\frac{ui}{\sqrt{-u^2-L_2}}+\overline{(i+j)\frac{ui}{\sqrt{-u^2-L_2}}}+(i^*+j^*)\frac{u^*i^*}{\sqrt{-u^{*2}-L_2}}+\overline{(i^*+j^*)\frac{u^*i^*}{\sqrt{-u^{*2}-L_2}}}\right]\\J_{Q_z}=-\frac{\partial\Phi}{\partial z}=\\=\frac{J_{Q_x}^{\infty}b}{4}\left[(i+j)\frac{uj}{\sqrt{-u^2-L_2}}+\overline{(i+j)\frac{uj}{\sqrt{-u^2-L_2}}}+(i^*+j^*)\frac{u^*j^*}{\sqrt{-u^{*2}-L_2}}+\overline{(i^*+j^*)\frac{u^*j^*}{\sqrt{-u^{*2}-L_2}}}\right]\end{gathered}\tag{32}$$

The corresponding solutions in rectangular co-ordinates are (as detailed in *Appendix D*):

$$\begin{gathered}T=-\frac{\kappa_0}{\kappa_1}+\\+\frac{\sqrt{\kappa_0{}^2-2\kappa_1c+\sqrt{2}\kappa_1J_{Q_x}^{\infty}\sqrt{L_2+x^2-(ay+bz)^2+\sqrt{[L_2+x^2-(ay+bz)^2]^2+4x^2(ay+bz)^2}}}}{\kappa_1}\\J_{Q_x}=-\frac{\partial\Phi}{\partial x}=-\frac{J_{Q_x}^{\infty}x}{\sqrt{2}}\frac{1+\dfrac{L_2+x^2+(ay+bz)^2}{\sqrt{[L_2+x^2-(ay+bz)^2]^2+4x^2(ay+bz)^2}}}{\sqrt{L_2+x^2-(ay+bz)^2+\sqrt{[L_2+x^2-(ay+bz)^2]^2+4x^2(ay+bz)^2}}}\\J_{Q_y}=-\frac{\partial\Phi}{\partial y}=\frac{J_{Q_x}^{\infty}a}{\sqrt{2}}\frac{ay+bz+\dfrac{[L_2-x^2-(ay+bz)^2](ay+bz)}{\sqrt{[L_2+x^2-(ay+bz)^2]^2+4x^2(ay+bz)^2}}}{\sqrt{L_2+x^2-(ay+bz)^2+\sqrt{[L_2+x^2-(ay+bz)^2]^2+4x^2(ay+bz)^2}}}\\J_{Q_z}=-\frac{\partial\Phi}{\partial z}=\frac{J_{Q_x}^{\infty}b}{\sqrt{2}}\frac{ay+bz+\dfrac{[L_2-x^2-(ay+bz)^2](ay+bz)}{\sqrt{[L_2+x^2-(ay+bz)^2]^2+4x^2(ay+bz)^2}}}{\sqrt{L_2+x^2-(ay+bz)^2+\sqrt{[L_2+x^2-(ay+bz)^2]^2+4x^2(ay+bz)^2}}}\end{gathered}\tag{33}$$

When the thermal conductivity is constant, the temperature is:

$$T=\frac{J_{Q_x}^{\infty}}{\sqrt{2}\kappa}\sqrt{L_2+x^2-(ay+bz)^2+\sqrt{[L_2+x^2-(ay+bz)^2]^2+4x^2(ay+bz)^2}}\tag{34}$$

At the crack surface $x = 0(ay + bz)^2 \leq L_2$, $(ay - bz)^2 \leq L_2$, u, $\overline{u}$, u^*, and $\overline{u}^*$ denote pure imaginary term and $(-u^2 - L^2)^{1/2}$ is also imaginary term, we can deduce that $(i+j)u/(-u^2 - L^2)^{1/2}$ is pure imaginary. Accordingly, the conjugate, * operation, the conjugate of * operation are all pure imaginary. Thus, we can conclude that the thermal flux J_{Qx}^{∞} must be zero at the crack surface.

When approaching the crack surface from the outside, according to eq. (30), there is 0/0 in the expression:

$$\frac{x}{\left(L_2-(ay+bz)^2+\sqrt{\left((ay+bz)^2-L_2\right)^2}\right)^{1/2}}$$

For solving it, we use the L'Hospital's rules and obtain:

$$J_{Q_x}=J_{Q_x}^{\infty}\sqrt{\frac{(ay+bz)^2}{(ay+bz)^2-L_2}} \tag{35}$$

Assuming $ay + bz = L_2^{1/2} + r$, we have:

$$\lim_{r\to 0}J_{Q_x}=J_{Q_x}^{\infty}\sqrt{\frac{L_2}{2\sqrt{L_2}r}}=\frac{J_{Q_x}^{\infty}}{\sqrt{2}}\sqrt[4]{L_2}\sqrt{\frac{1}{r}} \tag{36}$$

which indicates that thermal flux J_{Qx}^{∞} has $r^{-1/2}$ singularity.

We define the thermal flux intensity factor at the crack surface as $K_{JQx} = J_{Qx}^{\infty}\times r^{1/2}$, and then we obtain:

$$K_{J_{Q_x}}=\frac{J_{Q_x}^{\infty}}{\sqrt{2}}\sqrt[4]{L_2}\sqrt{\frac{1}{r}}\times\sqrt{r}=\frac{J_{Q_x}^{\infty}}{\sqrt{2}}\sqrt[4]{L_2} \tag{37}$$

It can be seen that the thermal flux intensity factor is proportional to remote thermal flux J_{Qx}^{∞} and $L_2^{1/4}$, where $L_2^{1/2}/a$ and $L_2^{1/2}/b$ denote the length of half diamond diagonals along y-axis and z-axis. This is an interesting result because there is the same intensity of thermal flux anywhere on the diamond boundary, although geometrically different areas of the diamond have different widths. This indicates that when blocked by the diamond surface defect, the thermal flux distribution is re-adjusted so that it overflows at the same rate from all parts of the diamond boundary.

Model 3: A finite strip flake crack

At last, we consider an infinite strip flake crack $x = 0$, $(ay + bz)^2 < L_3$ embedded in an infinite medium, as shown in fig. 4, where $2L_3^{1/2}$ denote the width of crack, $L_3^{1/2}/a$ and $L_3^{1/2}/b$ denote the intercept in y-axis and z-axis. The origin of the co-ordinate is located at the center of the crack. At the far field, the material is subjected to imposed thermal flux J_{Qx}^{∞}.

We again discuss the special case with $i = j$ in this example. Similar to the diamond flake crack, the general solution of the temperature can be selected according to thermal flux boundary condition and the remote condition:

$$\Phi=\frac{J_{Q_x}^{\infty}}{4}\left[i\sqrt{-u^2-L_3}+\overline{i\sqrt{-u^2-L_3}}+i^*\sqrt{-u^{*2}-L_3}+\overline{i^*\sqrt{-u^{*2}-L_3}}\right] \tag{38}$$

Accordingly, the expression of thermal flux J_{Qx}^{∞}, J_{Qy}^{∞}, and J_{Qz}^{∞} could be deduced:

$$\begin{aligned}
J_{Q_x}&=\frac{J_{Q_x}^{\infty}}{4}\left[\frac{iu}{\sqrt{-u^2-L_3}}+\overline{\frac{iu}{\sqrt{-u^2-L_3}}}+\frac{i^*u^*}{\sqrt{-u^{*2}-L_3}}+\overline{\frac{i^*u^*}{\sqrt{-u^{*2}-L_3}}}\right]\\
J_{Q_y}&=-\frac{J_{Q_x}^{\infty}a}{4}\left[\frac{u}{\sqrt{-u^2-L_3}}-\overline{\frac{u}{\sqrt{-u^2-L_3}}}-\frac{u^*}{\sqrt{-u^{*2}-L_3}}+\overline{\frac{u^*}{\sqrt{-u^{*2}-L_3}}}\right]\\
J_{Q_z}&=-\frac{J_{Q_x}^{\infty}b}{4}\left[\frac{u}{\sqrt{-u^2-L_3}}-\overline{\frac{u}{\sqrt{-u^2-L_3}}}-\frac{u^*}{\sqrt{-u^{*2}-L_3}}+\overline{\frac{u^*}{\sqrt{-u^{*2}-L_3}}}\right]
\end{aligned} \tag{39}$$

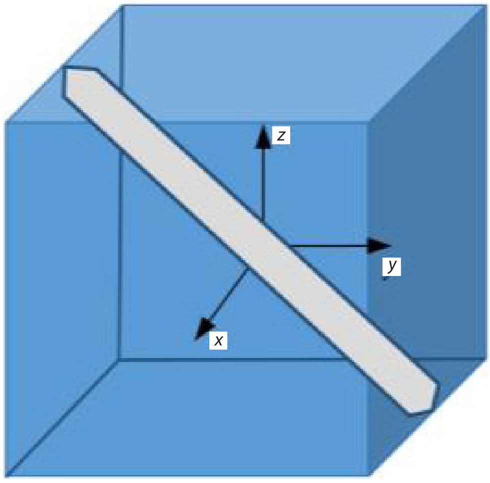

Figure 4. A strip flake crack embedded in an infinity medium

For the crack surface $x = 0$, $(ay + bz)^2 < L_3$, $(-u^2 - L_3)^{1/2}$ contains only i term. Thus, we conclude that $J_{Q_x}^{\infty}$ is zero at the crack surface.

Taking the same method with *Appendix E*, the solutions in rectangular co-ordinates are:

$$T = -\frac{\kappa_0}{\kappa_1} + \frac{\sqrt{\kappa_0^{\ 2} - 2\kappa_1 c + \sqrt{2}\kappa_1 J_{Q_x}^{\infty}\sqrt{L_3 + x^2 - (ay+bz)^2 + \sqrt{[L_3 + x^2 - (ay+bz)^2]^2 + 4x^2(ay+bz)^2}}}}{\kappa_1}$$

$$J_{Q_x} = -\frac{\partial \Phi}{\partial x} = -\frac{J_{Q_x}^{\infty} x}{\sqrt{2}} \frac{1 + \dfrac{L_3 + x^2 + (ay+bz)^2}{\sqrt{[L_3 + x^2 - (ay+bz)^2]^2 + 4x^2(ay+bz)^2}}}{\sqrt{L_3 + x^2 - (ay+bz)^2 + \sqrt{[L_3 + x^2 - (ay+bz)^2]^2 + 4x^2(ay+bz)^2}}} \tag{40}$$

$$J_{Q_y} = -\frac{\partial \Phi}{\partial y} = \frac{J_{Q_x}^{\infty} a}{\sqrt{2}} \frac{ay + bz + \dfrac{[L_3 - x^2 - (ay+bz)^2](ay+bz)}{\sqrt{[L_3 + x^2 - (ay+bz)^2]^2 + 4x^2(ay+bz)^2}}}{\sqrt{L_3 + x^2 - (ay+bz)^2 + \sqrt{[L_3 + x^2 - (ay+bz)^2]^2 + 4x^2(ay+bz)^2}}}$$

$$J_{Q_z} = -\frac{\partial \Phi}{\partial z} = \frac{J_{Q_x}^{\infty} b}{\sqrt{2}} \frac{ay + bz + \dfrac{[L_3 - x^2 - (ay+bz)^2](ay+bz)}{\sqrt{[L_3 + x^2 - (ay+bz)^2]^2 + 4x^2(ay+bz)^2}}}{\sqrt{L_3 + x^2 - (ay+bz)^2 + \sqrt{[L_3 + x^2 - (ay+bz)^2]^2 + 4x^2(ay+bz)^2}}}$$

When the thermal conductivity is constant, the temperature and thermal flux fields:

$$T = \frac{J_{Q_x}^{\infty}}{\sqrt{2}} \sqrt{L_3 + x^2 - (ay+bz)^2 + \sqrt{\left[L_3 + x^2 - (ay+bz)^2\right]^2 + 4x^2(ay+bz)^2}} \tag{41}$$

When approaching the crack surface from the outside, the thermal flux can be simplified:

$$J_{Q_x} = J_{Q_x}^{\infty}\left[\frac{(ay+bz)^2}{(ay+bz)^2} - L_3\right]^{1/2} \tag{42}$$

assuming $ay + bz = L_3^{1/2} + r$, we obtain:

$$\lim_{r\to 0} J_{Q_x} = J_{Q_x}^{\infty}\sqrt{\frac{L_3}{2\sqrt{L_3}r}} = \frac{J_{Q_x}^{\infty}}{\sqrt{2}}\sqrt[4]{L_3}\sqrt{\frac{1}{r}} \tag{43}$$

which also indicates that thermal flux J_{Qx}^{∞} has $r^{-1/2}$ singularity.

We define the thermal flux intensity factor of strip crack as $K_{JQx} = J_{Qx}^{\infty} \times r^{1/2}$, and then obtain:

$$K_{J_{Q_x}} = \frac{J_{Q_x}^{\infty}}{\sqrt{2}}\sqrt[4]{L_3}\sqrt{\frac{1}{r}}\times\sqrt{r} = \frac{J_{Q_x}^{\infty}}{\sqrt{2}}\sqrt[4]{L_3} \tag{44}$$

Similar to the diamond crack, the thermal flux intensity factor is proportional to remote thermal flux and the fourth root of crack width, and the thermal flux overflows at the same rate from all parts of the strip crack boundary.

Result and discussion

Numerical calculations were carried out to demonstrate the analysis and search for meaningful results. Assuming that the medium is 45 carbon steel, with the corresponding material constants shown in tab. 1. The parameter with $a = b = 1/(2)^{1/2}$, $L_1 = L_2 = L_3 = 1$ mm, $J_{Qx}^{\infty} = 2000$ W/m^2, and $J_{Qy}^{\infty} = J_{Qz}^{\infty} = 0$ are considered, with J_{Qx}^{∞} varying.

Table 1. Material parameters of 45 carbon steel

Temperature [K]	473.15	573.15	673.15	773.15	873.15
$\kappa(T)$ [Wm^{-1}K^{-1}]	46.9	45.2	42.3	39.4	35.5

We first plot the thermal flux vector fields around the three defects in figs. 5-7. In fig. 5 it can be seen that the thermal flux flow around the cylinder cavity, and the thermal flux density at the cavity surface changes significantly. In figs. 6 and 7 it can be seen that the thermal flux at the crack surface parallel to it, which confirms the insulation boundary condition. In addition, there are strong normal thermal flux at the crack boundary, which also coincide with the theoretical results eqs. (33) and (40).

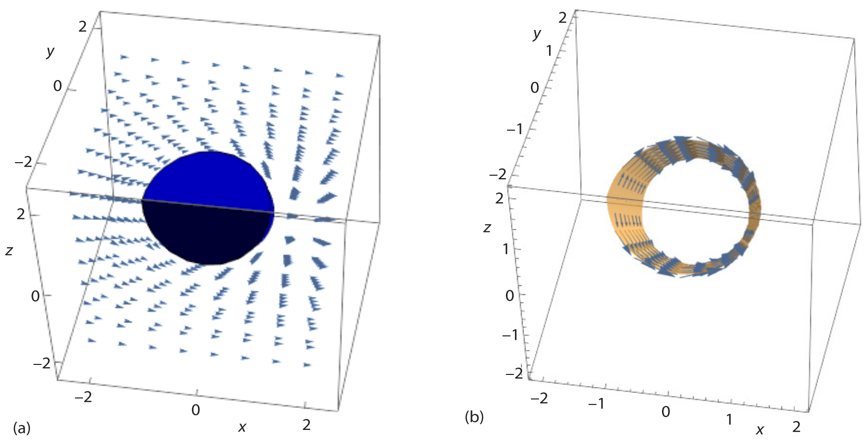

Figure 5. Distribution of thermal flux; (a) around infinite cylinder cavity and (b) at the cavity surface

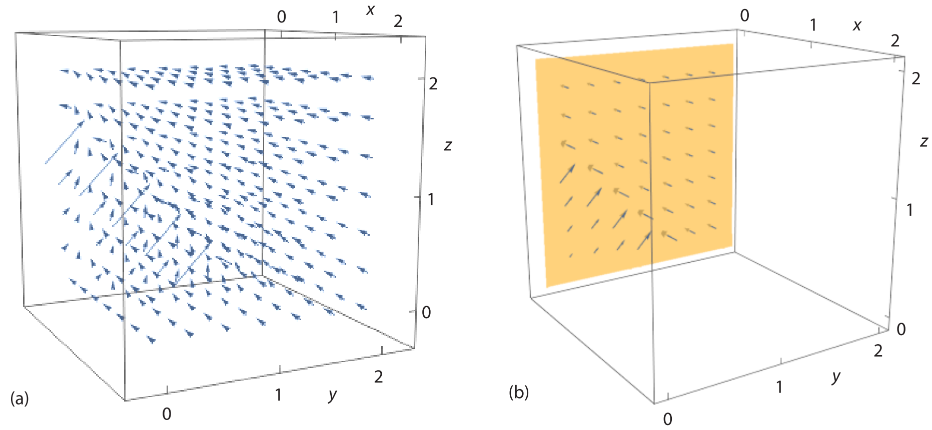

Figure 6. Distribution of thermal flux; (a) around the diamond flake crack and (b) at the crack surface plane

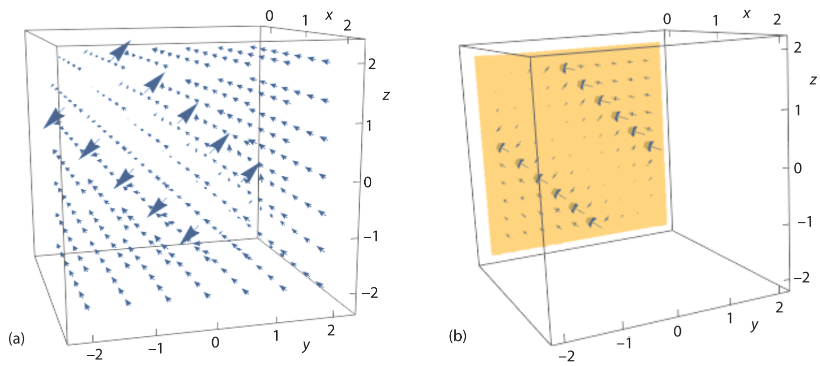

Figure 7. Distribution of thermal flux; (a) around the strip flake crack and (b) at the crack surface plane

We then plot the contours of temperature field around the three defects under the assumption of thermal conductivity being (a) constant, (b) linearly with temperature in figs. 8-11. It can be seen from figs. 8 and 9 that the distant temperature field is hardly affected by the cylinder cavity, and the corresponding isothermal surface is almost planar, while the isothermal surface around the cavity is significantly distorted due to the existence of the cavity,

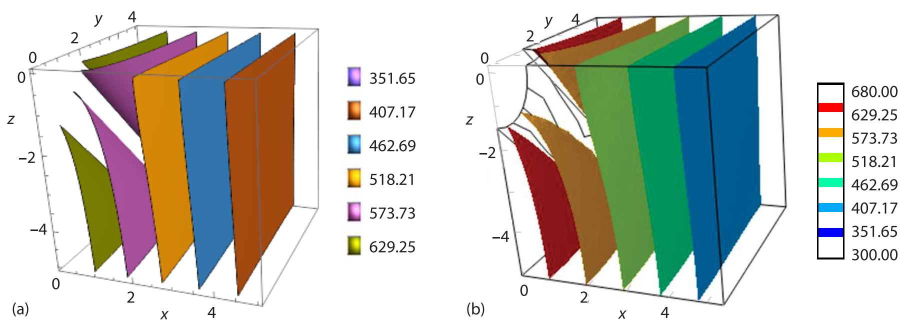

Figure 8. Temperature distribution with constant conductivity around the infinite cylinder cavity under the condition; (a) analytical solution and (b) finite element solution

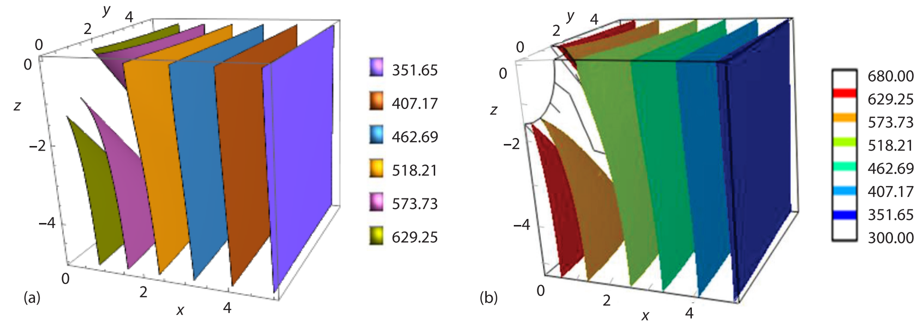

Figure 9. Temperature distribution with linearly conductivity with temperature around the infinite cylinder cavity under the condition; (a) analytical solution and (b) finite element solution

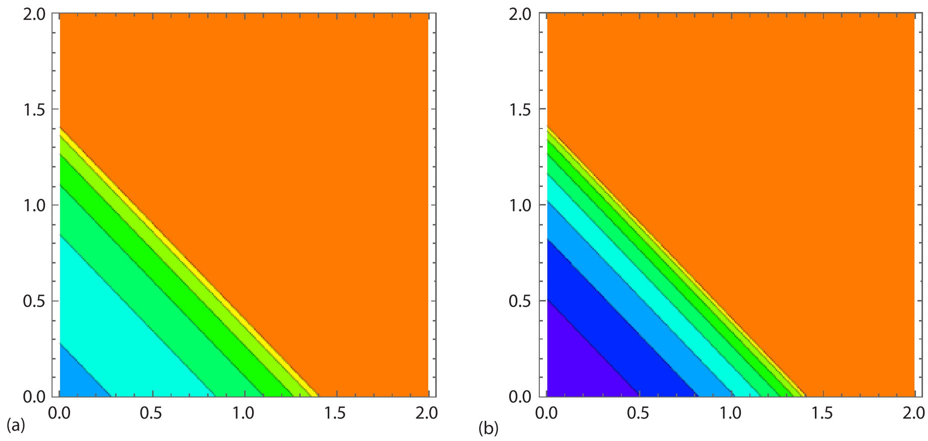

Figure 10. The temperature distribution at the diamond flake crack surface under the assumption of thermal conductivity being; (a) constant and (b) linearly with temperature

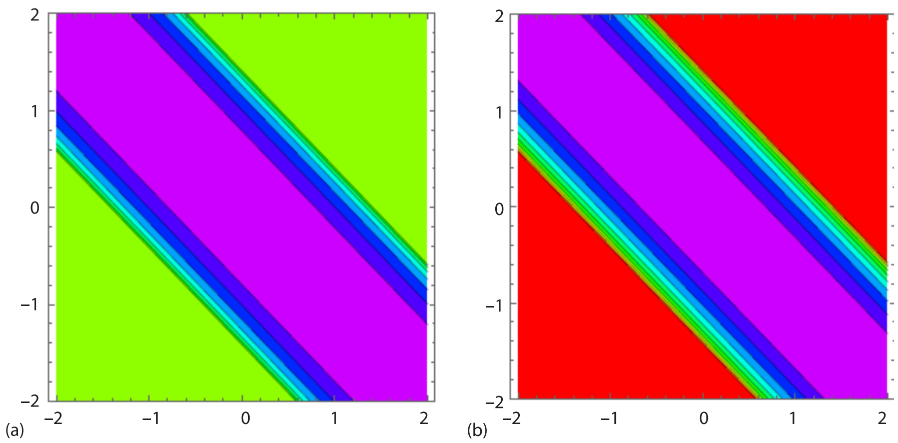

Figure 11. The temperature distribution at the strip flake crack surface under the assumption of thermal conductivity being (a) constant and (b) linearly with temperature

which is consistent with the theoretical results eq. (28). In addition, we take the finite element software ABAQUS to check the temperature field in figs. 8(b) and 9(b), as can be seen, the simulation results and the analytical results are in perfect agreement. Figures 10 and 11 indicate the temperature gradient at the crack edge increases sharply, which is consistent with the huge thermal flux at the crack edge.

Conclusions

The 3-D problem of temperature and thermal flux distribution around defects with temperature-dependent material properties are analyzed in this paper. The 3-D temperature-dependent heat conduction problem is reduced to the solution of 3-D Laplace equation. The generalized ternary function is proposed to reach the general solution of 3-D Laplace equation. The analytical solutions of three specific problems are obtained, and the corresponding temperature-thermal flux fields are discussed. The results are as follows.

- The general solution of 3-D Laplace equation can be obtained by constructing ternary functions.
- The thermal flux field of 3-D temperature dependent problem is the same as classical constant thermal conductivity approach result, while the temperature field is different from the classical result.
- When blocked by a planar defect, the thermal flux distribution will re-adjusted so that it overflows at the same rate from all parts of the planar defect boundary.
- Thermal flux at the planar defect boundary have $r^{-1/2}$ singularity, and its intensity is proportional to the fourth root of defect width.
- In our examples, the temperature gradient in the temperature-dependent case is larger than that in the classical case.
- The temperature gradient at the crack edge increases sharply, which is consistent with the huge thermal flux at the crack edge.

Acknowledgment

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Appendix A

Based on complex variable function $x + iy$ we establish the ternary function $u = x + iy + jz$ by adding the term jz. For the generalization, we add two coefficients a, b to form the generalized ternary variable, which is denoted:

$$u = x + aiy + bjz \tag{A1}$$

where i and j are two independent imaginary units, a and b are real numbers. Similarly, the conjugation of ternary function u can be expressed:

$$\overline{u} = x - aiy - bjz \tag{A2}$$

It is obvious that there are three co-ordinate variables x, y, z. In order to satisfy the unique relation between the real variables and the ternary variables, we introduce the third ternary variable:

$$u^* = x - aiy + bjz \tag{A3}$$

We add * above the ternary number to indicate the same real term and j term as u, the contrast i term. Obviously, u, $\overline{u}$, and u^* are linearly independent. Corresponding to u^*, its conjugation can be expressed:

$$\overline{u}^* = x + aiy - bjz = u + \overline{u} - u^* \tag{A4}$$

It is can be seen that $\overline{u}^*$ depends on u, $\overline{u}$, and u^*. In addition, we establish the differential operation rules of generalized ternary function according to the differential rule of complex variable function. Assume T is an arbitrary function, the partial derivations with respect to x, y, z are given:

$$\frac{\partial T}{\partial x} = \frac{\partial T}{\partial u} + \frac{\partial T}{\partial \overline{u}} + \frac{\partial T}{\partial u^*}, \ \frac{\partial T}{\partial y} = ai\left(\frac{\partial T}{\partial u} - \frac{\partial T}{\partial \overline{u}} - \frac{\partial T}{\partial u^*}\right), \ \frac{\partial T}{\partial z} = bj\left(\frac{\partial T}{\partial u} - \frac{\partial T}{\partial \overline{u}} + \frac{\partial T}{\partial u^*}\right) \tag{A5}$$

Corresponding $\partial T/\partial u$, $\partial T/\partial \overline{u}$, and $\partial T/u^*$ can be deduced:

$$\frac{\partial T}{\partial u} = \frac{1}{2}\left(\frac{\partial T}{\partial x} + \frac{1}{ai}\frac{\partial T}{\partial y}\right), \ \frac{\partial T}{\partial \overline{u}} = \frac{1}{2}\left(\frac{\partial T}{\partial x} - \frac{1}{bj}\frac{\partial T}{\partial z}\right), \ \frac{\partial T}{\partial u^*} = \frac{1}{2}\left(\frac{1}{bj}\frac{\partial T}{\partial z} - \frac{1}{ai}\frac{\partial T}{\partial y}\right) \tag{A6}$$

Substituting u, $\bar{u}$, u^*, $\overline{u}^*$ into the aforementioned equation, we have:

$$\begin{gathered}
\frac{\partial u}{\partial u} = \frac{1}{2}\left(1 + \frac{1}{ai}ai\right) = 1, \ \frac{\partial u}{\partial \overline{u}} = \frac{1}{2}\left(1 - \frac{1}{bj}bj\right) = 0, \ \frac{\partial u}{\partial u^*} = \frac{1}{2}\left(\frac{1}{bj}bj - \frac{1}{ai}ai\right) = 0 \\
\frac{\partial \overline{u}}{\partial u} = \frac{1}{2}\left(1 - \frac{1}{ai}ai\right) = 0, \ \frac{\partial \overline{u}}{\partial \overline{u}} = \frac{1}{2}\left(1 + \frac{1}{bj}bj\right) = 1, \ \frac{\partial}{\partial u^*}\overline{u} = \frac{1}{2}\left(-\frac{1}{bj}bj + \frac{1}{ai}ai\right) = 0 \\
\frac{\partial u^*}{\partial u} = \frac{1}{2}\left(1 - \frac{1}{ai}ai\right) = 0, \ \frac{\partial u^*}{\partial \overline{u}} = \frac{1}{2}\left(1 - \frac{1}{bj}bj\right) = 0, \ \frac{\partial u^*}{\partial u^*} = \frac{1}{2}\left(\frac{1}{bj}bj + \frac{1}{ai}ai\right) = 1 \\
\frac{\partial \overline{u}^*}{\partial u} = \frac{1}{2}\left(1 + \frac{1}{ai}ai\right) = 1, \ \frac{\partial \overline{u}^*}{\partial \overline{u}} = \frac{1}{2}\left(1 + \frac{1}{bj}bj\right) = 1, \ \frac{\partial \overline{u}^*}{\partial u^*} = \frac{1}{2}\left(-\frac{1}{bj}bj - \frac{1}{ai}ai\right) = -1
\end{gathered} \tag{A7}$$

Appendix B

Substituting $f(\overline{u}^*)$ into eq. (9), we have:

$$\begin{gathered}
\frac{\partial^2 f(\overline{u}^*)}{\partial u \partial \overline{u}} + a^2\frac{\partial^2 f(\overline{u}^*)}{\partial u \partial u^*} + b^2\frac{\partial^2 f(\overline{u}^*)}{\partial \overline{u} \partial u^*} = \frac{\partial f'(\overline{u}^*)}{\partial \overline{u}} + a^2\frac{\partial f'(\overline{u}^*)}{\partial u^*} + b^2\frac{\partial f'(\overline{u}^*)}{\partial u^*} = \\
= f''(\overline{u}^*) - a^2 f''(\overline{u}^*) - b^2 f''(\overline{u}^*) = 0
\end{gathered} \tag{A8}$$

Appendix C

Being similar to the algebraic operation of the complex number, we establish the operation rules of ternary numbers. Let:

$$u_1 = a_1 + ib_1 + jc_1 + ijd_1, \ u_2 = a_2 + ib_2 + jc_2 + ijd_2 \tag{A9}$$

noting that the real term, i term, j term, and ij term are linearly independent, thus we obtain:

$$\begin{gathered}
u_1 \pm u_2 = a_1 \pm a_2 + i(b_1 \pm b_2) + j(c_1 \pm c_2) + ij(d_1 \pm d_2) \\
u_1u_2 = u_2u_1 = a_1a_2 - b_1b_2 - c_1c_2 + d_1d_2 + i(a_1b_2 + a_2b_1 - c_1d_2 - c_2d_1) + \\
+ j(a_1c_2 + a_2c_1 - b_1d_2 - b_2d_1) + ij(a_1d_2 + a_2d_1 + b_1c_2 + b_2c_1) \\
\frac{u_1}{u_2} = \frac{(a_1 + ib_1 + jc_1 + ijd_1)(a_2 - ib_2 - jc_2 + ijd_2)(a_2 - ib_2 + jc_2 - ijd_2)(a_2 + ib_2 - jc_2 - ijd_2}{(a_2 + ib_2 + jc_2 + ijd_2)(a_2 - ib_2 - jc_2 + ijd_2)(a_2 - ib_2 + jc_2 - ijd_2)(a_2 + ib_2 - jc_2 - ijd_2)}
\end{gathered} \tag{A10}$$

Here we can see that the ternary number still follows the Commutative Law of Multiplication. In addition, we add a horizontal line above the ternary number to indicate the conjugate. The conjugate of the product of two ternary numbers can be determined:

$$\begin{gathered}\overline{u_1 u_2} = a_1 a_2 - b_1 b_2 - c_1 c_2 + d_1 d_2 - i(a_1 b_2 + a_2 b_1 - c_1 d_2 - c_2 d_1) - \\ - j(a_1 c_2 + a_2 c_1 - b_1 d_2 - b_2 d_1) + ij(a_1 d_2 + a_2 d_1 + b_1 c_2 + b_2 c_1) = \overline{u_1 u_2}\end{gathered} \tag{A11}$$

The sum along with its conjugate can be written:

$$u_1 u_2 + \overline{u_1 u_2} = 2\left[a_1 a_2 - b_1 b_2 - c_1 c_2 + d_1 d_2 + ij(a_1 d_2 + a_2 d_1 + b_1 c_2 + b_2 c_1)\right] \tag{A12}$$

Obviously, the operation rules of ternary number are consistent with complex number, except that j term and ij term are added. In addition, we define the * operation as: the same real term and j term, the contrast i term and ij term with primitive function:

$$u_1^* u_2^* + \overline{u_1^* u_2^*} = 2\left[a_1 a_2 - b_1 b_2 - c_1 c_2 + d_1 d_2 - ij(a_1 d_2 + a_2 d_1 + b_1 c_2 + b_2 c_1)\right] \tag{A13}$$

Here we can see that the sum of arbitrary function and its conjugate is comprised of real term and ij term. By comparing eqs. (A13) and (A10), we realize that the sum of four operations like one function as well as its conjugate, * operation and conjugate of * operation is real number. So far, we have obtained general method of inverting generalized ternary variable function into a real function, that is, the expression:

$$f(u) + \overline{f(u)} + f^*(u^*) + \overline{f^*(u^*)}$$

must be a real function.

Appendix D

According to eq. (A1), $(-u^2 - L_2)^{1/2}$ can be expressed:

$$\sqrt{-u^2 - L_2} = \sqrt{-(x^2 - a^2 y^2 - b^2 z^2 + 2axyi + 2bxzj + 2abyzij) - L_2} \tag{A14}$$

assuming:

$$\sqrt{-u^2 - L_2} = m + ni + pj + qij \tag{A15}$$

According to the boundary condition and remote thermal flux, we have the solution:

$$\begin{gathered}m = -\frac{1}{2}\left(\sqrt{\frac{w_1 + w_2}{2}} + \sqrt{\frac{v_1 + v_2}{2}}\right),\ n = -\frac{1}{2}\left(\sqrt{\frac{-w_1 + w_2}{2}} + \sqrt{\frac{-v_1 + v_2}{2}}\right) \\ p = -\frac{1}{2}\left(\sqrt{\frac{-w_1 + w_2}{2}} - \sqrt{\frac{-v_1 + v_2}{2}}\right),\ q = -\frac{1}{2}\left(-\sqrt{\frac{w_1 + w_2}{2}} + \sqrt{\frac{v_1 + v_2}{2}}\right)\end{gathered} \tag{A16}$$

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