VARIATIONAL APPROACH FOR THE FRACTIONAL EXOTHERMIC REACTIONS MODEL WITH CONSTANT HEAT SOURCE IN POROUS MEDIUM

by

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In this paper, a new fractional exothermic reactions model with constant heat source in porous media considering the memory effect is proposed. Applying the fractional complex transform, the fractional model is converted into its partner. Then the variational principle of the problem is successfully established. Based on the obtained variational principle, the Ritz method is used to seek the solution of the fractional model. Finally, the correctness and effectiveness of the proposed method are illustrated by the numerical results with the aid of the MATLAB. The obtained results show that the proposed method is easy but effective, and is expected to shed a bright light on practical applications of fractional calculus.

Key words: *He's fractional derivative, fractional model, variational principle, Ritz method*

Introduction

An exothermic reaction is a reaction of chemical or physical type that gives energy in the form of light and heat and dispenses net energy to its locality. Considering a porous material wall thickness with $0 < u < L$, a pseudo-homogeneous model to represent convective driven by an exothermic reaction can be formulated [1-5]:

$$\frac{d^2 y}{du^2} + \beta\varepsilon^2 \left(1 - \frac{y}{\beta}\right) e^{\left(\frac{ky}{k+y}\right)} = 0 \tag{1}$$

where y is the temperature, β – the maximum feasible temperature without free convection, ε^2 – the ratio of the characteristic time for diffusion of heat generator, and k – the stands for the dimensionless activation energy. In the case of the constant heat source, eq. (1) is simplified:

$$\frac{d^2 y}{du^2} + \beta\varepsilon^2 \left(1 - \frac{y}{\beta}\right) = 0 \tag{2}$$

with the boundary conditions:

$$y = 0, \text{ at } u = 1$$

$$\frac{dy}{du} = 0, \text{ at } u = 0$$

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It is well known that integer order derivatives are local in nature, so these derivatives do not accurately describe the problem, especially for processes with historical memory. Recently, the fractal and fractional derivatives have drawn wide attention, and has been used widely to describe many complex phenomenon arising in different fields such as the bioscience [6-8], optics [9, 10], cold plasma [11], vibration [12-14], circuits [15, 16], unsmooth boundary [17-22] and so on [23-29]. Due to the non-local and non-singular properties of the fractional derivatives, the fractional derivatives are more suitable for modelling the complex processes with historical memory than integer derivatives. So we take a modification for eq. (2) to establish a new fractional model with the memory effect via He's fractional derivative, which reads:

$$\frac{\mathrm{d}^{2\alpha}y}{\mathrm{d}u^{2\alpha}}+\beta\varepsilon^2\left(1-\frac{y}{\beta}\right)=0 \tag{3}$$

where

$$0<\alpha\le 1,\ \frac{\mathrm{d}^{2\alpha}}{\mathrm{d}u^{2\alpha}}=\frac{\mathrm{d}^{\alpha}}{\mathrm{d}u^{\alpha}}\frac{\mathrm{d}^{\alpha}}{\mathrm{d}u^{\alpha}},\ \text{and}\ \frac{\mathrm{d}^{\alpha}}{u^{\alpha}}$$

is He's fractional derivative that is defined [30-33]:

$$\frac{\mathrm{d}^{\alpha}y}{\mathrm{d}u^{\alpha}}=\frac{1}{\Gamma(n-\alpha)}\frac{\mathrm{d}^{n}}{\mathrm{d}u^{n}}\int_{u0}^{u}(s-u)^{n-\alpha-1}\left[y_0(s)-y(s)\right]\mathrm{d}s \tag{4}$$

Variational principle

The variational principle shows the energy conservation of the whole solution domain and plays a key role in the numerical and analytical analysis of practical problems. In additional, the variational theory is the basis of the variational iteration method [34, 35]. So a variational-based analytical solution is an optimal one for solving the practical problem. For solving eq. (3), we use the following fractional complex transform [36-38]:

$$U=\frac{u^{\alpha}}{\Gamma(1+\alpha)} \tag{5}$$

Equation (3) can be converted:

$$\frac{\mathrm{d}^2y}{\mathrm{d}U^2}+\beta\varepsilon^2\left(1-\frac{y}{\beta}\right)=0 \tag{6}$$

with the boundary conditions:

$$y=0,\ \text{at}\ U=1 \qquad \frac{\mathrm{d}y}{\mathrm{d}U}=0,\ \text{at}\ U=0 \tag{7}$$

In order to establish the variational principle of eq. (6), we first re-write eq. (6) in the form:

$$\frac{\mathrm{d}^2y}{\mathrm{d}U^2}-\varepsilon^2y+\beta\varepsilon^2=0 \tag{8}$$

The variational principle of eq. (8) can be easily established as [39-45]:

$$J(y)=\int_0^1\left\{-\frac{1}{2}y'^2-\frac{1}{2}\varepsilon^2y^2+\beta\varepsilon^2y\right\}\mathrm{d}U \tag{9}$$

The obtained variational principle in eq. (9) is the theoretical basis of Ritz method. In the following content, we will use the Ritz method to solve eq. (8).

The Ritz method

We assume the solution of eq. (8) taking the form:

$$y(U) = aU^3 + bU^2 + cU + d \tag{10}$$

Applying the boundary conditions of eq. (7), we have:

$$\begin{aligned} a + b + c + d &= 0 \\ c &= 0 \\ 6a + 2b + \beta\varepsilon^2 &= 0 \end{aligned} \tag{11}$$

which leads to:

$$\begin{aligned} b &= \frac{-\beta\varepsilon^2 - 6a}{2} \\ c &= 0 \\ d &= 2a + \frac{\beta\varepsilon^2}{2} \end{aligned} \tag{12}$$

So we get the expression of $y(U)$ with the variable of a:

$$y(U) = aU^3 + \left(\frac{-\beta\varepsilon^2 - 6a}{2}\right)U^2 + 2a + \frac{\beta\varepsilon^2}{2} \tag{13}$$

Substituting it into eq. (9):

$$J(y) = \int_0^1 \left\{-\frac{1}{2}y\left(U'^2\right) - \frac{1}{2}\varepsilon^2 y(U)^2 + \beta\varepsilon^2 y(U)\right\}\mathrm{d}U$$

The Ritz method is also called the variational direct method. It can transform the stationary condition of a functional into the stationary condition of a function, so as to obtain the approximate solution. Applying the Ritz method [46-48], we require:

$$\frac{\mathrm{d}J}{\mathrm{d}a} = 0 \tag{14}$$

which results:

$$-\frac{24a}{5} - \frac{68a\varepsilon^2}{35} - \frac{61\beta\varepsilon^4}{120} = 0 \tag{15}$$

Then we can get the value of a:

$$\alpha = -\frac{427\beta\varepsilon^4}{4032 + 1632\varepsilon^2} \tag{16}$$

The solution of eq. (8) is obtained:

$$y(U) = -\frac{427\beta\varepsilon^4}{4032 + 1632\varepsilon^2}U^3 + \left(\frac{-\beta\varepsilon^2}{2} + \frac{1281\beta\varepsilon^4}{4032 + 1632\varepsilon^2}\right)U^2 - \frac{854\beta\varepsilon^4}{4032 + 1632\varepsilon^2} + \frac{\beta\varepsilon^2}{2} \tag{17}$$

Correspondingly, we can get the solution of eq. (3) via the transform given by eq. (5):

$$y(u)=-\frac{427\beta\varepsilon^4}{4032+1632\varepsilon^2}\left[\frac{u^\alpha}{\Gamma(1+\alpha)}\right]+\left(\frac{-\beta\varepsilon^2}{2}+\frac{1281\beta\varepsilon^4}{4032+1632\varepsilon^2}\right)\left[\frac{u^\alpha}{\Gamma(1+\alpha)}\right]-\\-\frac{854\beta\varepsilon^4}{4032+1632\varepsilon^2}+\frac{\beta\varepsilon^2}{2} \tag{18}$$

Obviously, when $\alpha = 1$, the aforementioned expression becomes the solution of eq. (2):

$$y(u)=-\frac{427\beta\varepsilon^4}{4032+1632\varepsilon^2}u^3+\left(\frac{-\beta\varepsilon^2}{2}+\frac{1281\beta\varepsilon^4}{4032+1632\varepsilon^2}\right)u^2-\frac{854\beta\varepsilon^4}{4032+1632\varepsilon^2}+\frac{\beta\varepsilon^2}{2} \tag{19}$$

Results and discussion

When $\alpha = 1$, $\varepsilon = 0.5$, and $\beta = 12$, the solution of our method compared with FDM [1], HATM [2] are shown in tab. 1. It can be seen that the different methods present a well agreement, but our method is simple.

Table 1. Comparison of different method for $\alpha = 1$, $\varepsilon^2 = 0.5$, and $\beta = 12$

u	Our method	FDM [1]	HATM [2]
0	2.47153	2.4804	2.48066
0.1	2.44920	2.4566	2.45685
0.2	2.38113	2.3851	2.38531
0.3	2.26574	2.2655	2.26567
0.4	2.10146	2.0972	2.09734
0.5	1.88668	1.8793	1.87948
0.6	1.61983	1.6109	1.61099
0.7	1.29932	1.2904	1.29054
0.8	0.92357	0.9164	0.91652
0.9	0.49099	0.4870	0.48705
1.0	0	0	0

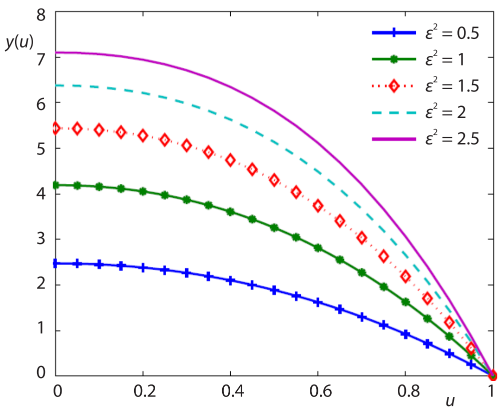

Figure 1. Plots of $y(u)$ with different value of ε^2 when $\alpha = 1$, $\beta = 12$

When $\alpha = 1$, $\beta = 12$, the behavior of $y(u)$ with different value of ε^2 is plotted in fig. 1, where it can be noticed that, if the value of ε is increased, then the enhancement in temperature profile is caused.

For choosing $\varepsilon^2 = 0.5$, $\beta = 12$, the influence of different fractional order α on the temperature profile of $y(u)$ is shown in fig. 2, where it can be found that an increase in the value of α can cause the enhancement in temperature profile.

By using $\varepsilon^2 = 1$, $\beta = 12$, the behavior of the solution with different fractional order is plotted in fig. 3. In this case, we can observe that the increase of α will lead to the enhancement of temperature distribution, which is the same with the conclusion that drawn when selecting $\varepsilon^2 = 0.5$, $\beta = 12$.

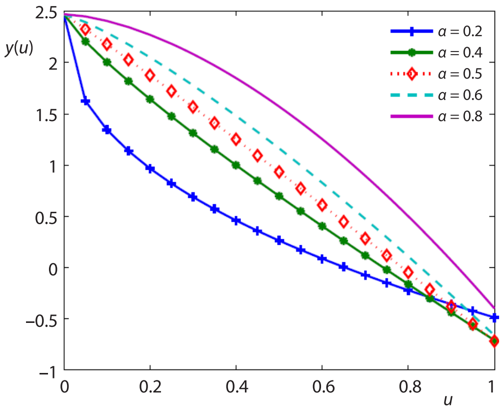

Figure 2. Plots of $y(u)$ with different fractional order α when $\varepsilon^2 = 0.5$, $\beta = 12$

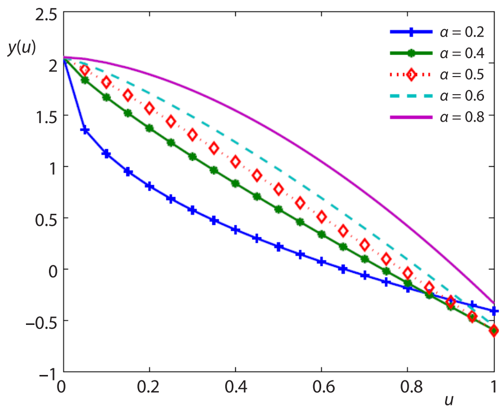

Figure 3. Plots of $y(u)$ with different fractional order α when $\varepsilon^2 = 1$, $\beta = 10$

Conclusion

Taking into account of the memory effect, a new fractional exothermic reactions model with constant heat source and porous media is proposed in this paper. With the help of the fractional complex transform, the fractional differential equation is converted into a partial differential equation. Then the variational principle is developed and the Ritz method is employed to solve the problem. Compared with the existed methods, a well agreement is reached, which shows that the proposed method is easy but effective.

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