A NEW FRACTAL MODEL OF THE CONVECTIVE-RADIATIVE FINS WITH TEMPERATURE-DEPENDENT THERMAL CONDUCTIVITY

by

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Original scientific paper
https://doi.org/10.2298/TSCI220917207W

In this paper, the convective-radiative fins of rectangular profile with temperature-dependent thermal conductivity are considered. By studying the conventional heat transfer equation, its modified fractal form, which can describe the problem in the porous medium, is presented based on He's fractal derivative for the first time. The fractal two-scale transform method together with the Taylor series are applied to deal with fractal model, and an analytical approximate solution is obtained. The impact of the different fractal orders on the thermal behavior of the fins is also elaborated in detail. In addition, a comparison between our solution and the existing one is given to prove the correctness of the proposed method, which shows that the proposed method is easy but effective, and are expected to shed a bright light on practical applications of fractal calculus.

Key words: *He's fractal derivative, porous medium, Taylor series, fractal two-scale transform method*

Introduction

The heat transfer devices with high heat transfer rates, low cost and small size are much required in many engineering applications. As a heat transfer rate device, the Fins have been used widely in many engineering systems as heat exchangers to dissipate the heat [1-5]. At present, many research results have been achieved in the theoretical research of fins, such as the perturbation techniques [6-8], the decomposition method [9, 10], the differential transformation method [11, 12], integral equation method [13, 14] and so on. The general shape of rectangular fin is shown in the fig. 1, the 1-D governing ODE for the 1-D convective-radiative straight fin of rectangular profile with cross-sectional area, A, and length, L [15]:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(kA\frac{\mathrm{d}T}{\mathrm{d}x}\right)-\varepsilon\sigma P\left(T^4-T_s^4\right)-hP\left(T-T_a\right)=0,\ \ 0<x<L \tag{1}$$

where h is the heat convection transfer coefficient, k – the thermal conductivity, σ – the Stefan-Boltzmann constant, ε – the surface emissivity, and x – the distance measured from the fin tip.

The thermal conductivity (TC) $k(T)$, which is the function of the local temperature change at any position, can be expressed [15]:

$$k\left(T\right)=k_a\left[1+\lambda\left(T-T_a\right)\right] \tag{2}$$

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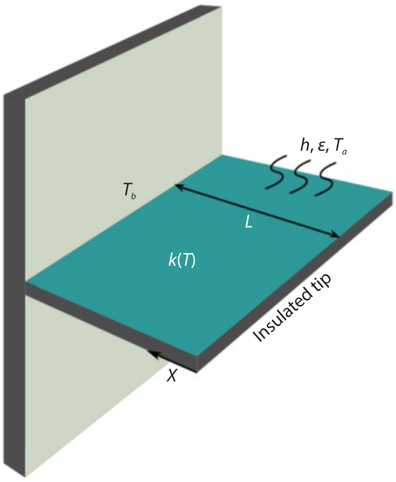

Figure 1. The rectangular convective-radiative fin with an insulated tip

The boundary conditions of the the base and the fin tip are given [15]:

$$\frac{\mathrm{d}T}{\mathrm{d}x} = 0 \ \text{ for } \ x = 0 \tag{3}$$

and

$$T = T_b \ \text{ for } \ x = L \tag{4}$$

By using the dimensionless transform, eq. (1) can be re-written [15]:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[(1+\alpha\theta)\frac{\mathrm{d}\theta}{\mathrm{d}x}\right] - m2\theta - \varepsilon_r\theta4 = 0, \ 0 < x < 1 \tag{5}$$

with the boundary condition:

$$\theta'(0) = 0, \ \theta(1) = 1 \tag{6}$$

where the dimensionless transform takes the form [15]:

$$\theta = \frac{T - T_a}{T_b - T_a}, \ k(\theta) = \frac{k(T)}{k_a}, \ x = \frac{X}{L}$$

$$\alpha = \lambda(T_b - T_a), \ M = L\sqrt{\frac{hP}{k_a A}}, \ \varepsilon_r = \frac{\varepsilon\sigma P L^2 T_b^3}{k_a A}$$

where k_a is the thermal conductivity at temperature of a surrounding fluid, T_a – the temperature of a surrounding fluid, T_b – the temperature at fin's base, P – the perimeter of cross-section, and λ – the measure of thermal conductivity variation with temperature.

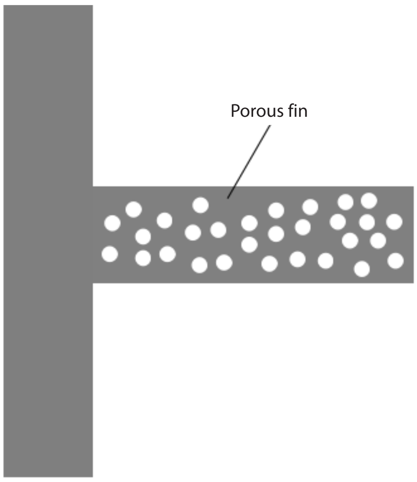

Figure 2. Schematic of a porous fin

The fractal modification

Through the previous introduction, we found that eq. (5) can well describe the steady heat conduction model of convective-radiative fins with temperature dependent thermal conductivity. However, when the fins are the porous medium, fig. 2, it becomes invalid, so we need to give a modification of it. Recently, the fractal and fractional calculus are adopted to model many complex phenomenon arising in the extreme conditions such as the un-smooth boundary [16-21], microgravity space [22, 23], fractal media [24], porous media [25], and so on [26-28]. Inspired by these research results, here we apply the fractal calculus to eq. (5) to derive its fractal modification form which can model the problem in the porous medium:

$$\frac{\mathrm{d}}{\mathrm{d}x^\zeta}\left[(1+\alpha\theta)\frac{\mathrm{d}\theta}{dx^\zeta}\right] - m^2\theta - \varepsilon_r\theta^4 = 0, \ 0 < x < 1 \tag{7}$$

where $0 < \zeta \le 1$, $\mathrm{d}/\mathrm{d}x^\zeta$ is He's fractal derivative that is defined [29-31]:

$$\frac{\partial\theta}{\partial x^{\zeta}}(x)=\Gamma(1+\zeta)\lim_{\substack{x-x_0=\Delta x\to\psi\\ \Delta x\neq 0}}\frac{\theta(x)-\theta(x_0)}{(x-x_0)^{\zeta}}$$

where ψ is the smallest porous size and ζ – the the fractal dimension of the porous structure. In the following content, we mainly try to solve eq. (7) by using the fractal two-scale transform and Taylor series method [32-35].

Solution of the fractal modification

The two-scale transform is first proposed by He and used widely to solve the fractal PDEs [36, 37]. Using the fractal two-scale transform with the form [38-40]:

$$S=x^{\zeta} \tag{8}$$

The eq. (7) can be converted into the form:

$$\frac{\mathrm{d}}{\mathrm{d}S}\left[(1+\alpha\theta)\frac{\mathrm{d}\theta}{\mathrm{d}S}\right]-M^2\theta-\varepsilon_r\theta^4=0 \tag{9}$$

with the boundary conditions:

$$\theta'(0)=0,\ \theta(1)=1 \tag{10}$$

In order to use Taylor series method, we set:

$$\theta(0)=n \tag{11}$$

Then we can get the value of $\theta''(0)$ by setting $S=0$ for eq. (9):

$$\theta''(0)=\frac{\varepsilon_r n^4+M^2 n}{\alpha n+1} \tag{12}$$

Then, we differentiate eq. (9) with respect to S, which gives:

$$\left[\alpha\theta(S)+1\right]\theta'''(S)-M^2\theta'(S)+3\alpha\theta'(S)\theta''(S)-4\varepsilon_r\theta(S)^3\theta'(S)=0 \tag{13}$$

By letting $S=0$ and using the boundary conditions in eqs. (10)-(12) for the previous equation, we can get:

$$\theta'''(0)=0$$

In the same manner, we have the following expression by differentiating eq. (13) with respect to S:

$$\left[\alpha\theta(S)+1\right]\theta^{(4)}(S)+3\alpha\theta'(S)\theta''(S)^2-M^2\theta''(S)-12\varepsilon_r\theta(S)^2\theta'(S)^2+$$
$$+4\alpha\theta'(S)\theta'''(S)-4\varepsilon_r\theta(S)^3\theta''(S)=0$$

which leads to

$$\theta^{(4)}(0)=\frac{4\varepsilon_r\theta(0)^3\theta''(0)-3\alpha\theta''(0)^2+M^2\theta''(0)}{\alpha n+1}$$

where $\theta(0)$ and $\theta''(0)$ are given in eqs. (11) and (12), respectively.

Similarly, we can get $\theta^{(5)}(0)$, $\theta^{(6)}(0)$, $\theta^{(7)}(0)$... and so on.

Then the approximate solution of $\theta(S)$ in the form of Taylor series can be written:

$$\theta(S)=\theta(0)+\theta'(0)S+\frac{1}{2}\theta''(0)S^2+\frac{1}{3!}\theta'''(0)S^3+\frac{1}{4!}\theta^{(4)}(0)S^4+\frac{1}{5!}\theta^{(5)}(0)S^5+ +\frac{1}{6!}\theta^{(6)}(0)S^6+\frac{1}{7!}\theta^{(7)}(0)S^7+... \tag{14}$$

If the values of α, M, and ε_r are given once, the value of n can be determined by using the boundary condition $\theta(1) = 1$. Therefore, we can get the corresponding approximate solution of $\theta(x)$ with the form:

$$\theta(x)=\theta(0)+\theta'(0)x^{\zeta}+\frac{1}{2}\theta''(0)x^{2\zeta}+\frac{1}{3!}\theta'''(0)x^{3\zeta}+\frac{1}{4!}\theta^{(4)}(0)x^{4\zeta}+\frac{1}{5!}\theta^{(5)}(0)x^{5\zeta}+ +\frac{1}{6!}\theta^{(6)}(0)x^{6\zeta}+\frac{1}{7!}\theta^{(7)}(0)x^{7\zeta}+... \tag{15}$$

The solution process can continue if a higher order approximate solution is needed. When $\zeta = 1$, the aforementioned expression becomes the solution of eq. (1). Next, we are going to use two examples to verify the correctness of our proposed method.

Example 1. Consider $M = 0.5$, $\varepsilon_r = 0.8$, and $\alpha = 0.2$, the value of n can be obtained by using the boundary condition $\theta(1) = 1$ with 6[th] Taylor series:

$$n = 0.76961489$$

Then the solution of $\theta(x) = 1$ can be obtained in the form of 6[th] Taylor series:

$$\theta(x)=0.76961489+0.20498156x^{2\zeta}+0.021653242x^{4\zeta}+0.0037503027x^{6\zeta} \tag{16}$$

We present the behavior of the previous equation in fig. 3 for ζ =0.02, 0.2, 0.4, 0.7, 0.9, and 1.0. For the case $\zeta = 1$, the results obtained of our method agree well with that of the existing exact method (the green line) in [15], the compared results are shown in tab. 1, which proves the correctness of our method.

Table 1. Numerical comparison when $\zeta = 1$

	$M = 0.5$	$\varepsilon r = 0.8$	$\alpha = 0.2$	$M = 0.5$	$\varepsilon r = 0.8$	$\alpha = 0.2$
x	Our method	[15]	[15]	Our method	[15]	[15]
0	0.7696	0.7692	0.7688	0.6306	0.6294	0.6448
0.1	0.7717	0.7713	0.7711	0.6339	0.6327	0.6483
0.2	0.7778	0.7774	0.7781	0.6438	0.6426	0.6590
0.3	0.7882	0.7878	0.7896	0.6605	0.6593	0.6767
0.4	0.8030	0.8025	0.8058	0.6842	0.6829	0.7016
0.5	0.8223	0.8218	0.8266	0.7151	0.7137	0.7336
0.6	0.8464	0.8459	0.8521	0.7538	0.7523	0.7726
0.7	0.8757	0.8752	0.8821	0.8007	0.7992	0.8188
0.8	0.9107	0.9102	0.9168	0.8567	0.8552	0.8721
0.9	0.9518	0.9515	0.9561	0.9227	0.9216	0.9325
1.0	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000

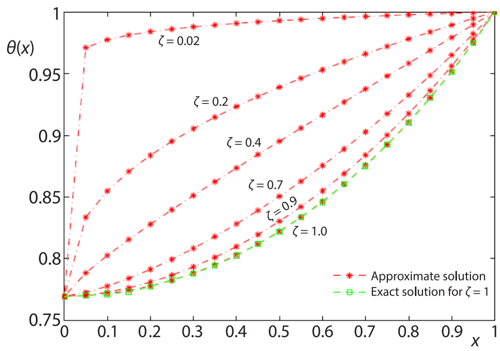

Figure 3. The solution property for *Example 1* (0 < ζ ≤ 1)

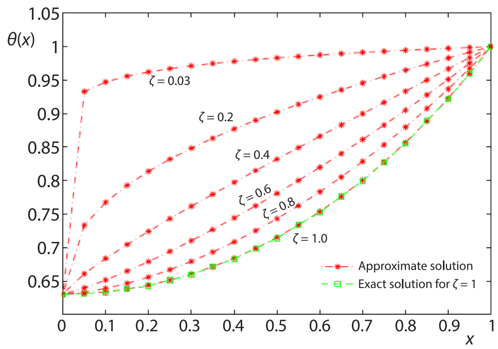

Figure 4. The solution property for *Example 2* (0 < ζ ≤ 1)

Example 2. When $M = 1$, $\varepsilon_r = 1.5$, and $\alpha = 0.5$, the value of n can be determined by using the 6th Taylor series:

$$n = 0.6307221768$$

Correspondingly, we get the solution of $\theta(x)$ with 6th Taylor series:

$$0(x) = 0.630572215768 + 0.32986218014x^{2\zeta} + 0.0316579414721x^{4\zeta} + \\ + 0.00790766261934x^{6\zeta} \tag{17}$$

Figure 4 plots the solution of eq. (17) for $\zeta = 0.03$, 0.2, 0.4, 0.6, 0.8, and 1.0. As expected, the results of our method and the existed exact method [15] have a good agreement in case $\zeta = 1$ (the results are presented in tab. 1). Moreover, figs. 3 and 4 have a common feature, that is, the smaller the value of ζ is, the higher the temperature value is, which means the worse the heat transfer performance of fin. This is related to the porous structure of the fin. Here there is $\zeta = 1 - \beta$, where β represents the porosity of the fin. For $\beta \rightarrow 0$, it means the fins of non-porous media, which exhibition a better heat transfer performance.

Conclusion

This paper proposed a fractal model of the convective-radiative porous fins with temperature-dependent thermal conductivity in the porous medium based on He's fractal derivative. A simple method based on the fractal two-scale transform and the Taylor series is suggested to solve the problem. The impact of the fractal dimension ζ on the thermal behavior is elaborated in detailed. Furthermore, our obtained solution for $\zeta = 1$ is also compared with the existing one, and a well agreement is reached. The obtained results in this paper are expected to shed a bright light on practical applications of the fractal calculus.

Acknowledgment

This work is supported by the Key Programs of Universities in Henan Province of China (22A140006), the Fundamental Research Funds for the Universities of Henan Province (NSFRF210324), Program of Henan Polytechnic University (B2018-40).

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Paper submitted: September 17, 2022
Paper revised: October 6, 2022
Paper accepted: October 12, 2022