NOVEL SOLUTIONS FOR THE HEAT EQUATIONS ARISING IN THE ELLIPTIC CURVES OVER THE FIELD OF RATIONAL NUMBERS

by

Xiao-Jun YANG [a,b,c,*], Ping ***CUI*** [d], Feng ***XU*** [e]

[a] State Key Laboratory for Geo-Mechanics and Deep Underground Engineering, and School of Mathematics, China University of Mining and Technology, Xuzhou 221116, China
[b] Department of Mathematics, Faculty of Science, King Abdulaziz University P.O. Box 80257, Jeddah 21589, Saudi Arabia
[c] Department of Mathematics, College of Science, Kyung Hee University, 26 Kyungheedae-ro, Dongdaemun-gu, Seoul 02447, Republic of Korea
[d] School of Mathematics and Statistics, Qujing Normal University, Qujing 655011, Yunnan, China
[e] Business School, Suzhou Vocational University, Suzhou 215104, Jiangsu, China

In this article we consider the solutions of the heat equations with use of the elliptic curves over the field of rational numbers. We propose the entire functions associated with the Hasse-Weil L-function. We show the conjectures that new functions have only real zeros in the entire function plane. The obtained results are proposed as new tool to describe the complex behaviors of the heat problems as well as number theory.

Key words: *heat equation, solution, Hasse-Weil L-function, elliptic curves, entire functions, number theory*

Introduction

Let Θ be an elliptic curve over $\aleph$ with the Weierstrass equation ([1], p.42):

$$x^2 + a_1 xy + a_3 x = y^3 + a_2 y^2 + a_4 y + a_6 , \tag{1}$$

where a_1, a_2, a_3, a_4, a_6 are integers. If $\hbar$ is the conductor of Θ, the Hasse-Weil L-function of an elliptic curve Θ over $\aleph$ is defined as (see Theorem 16.4 in [1])

$$\ell(\Theta, \tau, \hbar) = \sum_{m=1}^{\infty} \beta_\Theta (m, \hbar) n^{-\tau} , \tag{2}$$

where $\Re(\tau) > 3/2$. Let F be a function on the upper-half complex plane

$$\Pi = \left\{ \tau \in \mathbb{C} \middle| \mathrm{Im}(\tau) > 0 \right\} \tag{3}$$

by (see [2], p.115; also see Theorem 16.4 [1], p.451)]

* Corresponding author; e-mail: dyangxiaojun@163.com

$$F\left(x,\hbar\right)=\sum_{m=1}^{\infty}\beta_{\Theta}\left(m,\hbar\right)e^{2\pi imx}\ , \tag{4}$$

for $i=\sqrt{-1}$. In (4), we know that $F\left(x,\hbar\right)$ is non zero.

It is well-known that F is a cusp form for the group [3]

$$\Gamma_0\left(\hbar\right):=\left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix}\in SL_2\left(\mathbb{Z}\right):c\equiv 0\left(\operatorname{mod}\hbar\right)\right\}. \tag{5}$$

As one of results of Breuil-Conrad-Diamond-Taylor [3], Wiles [4] and Taylor-Wiles [5], the complete Hasse-Weil L-function of an elliptic curve Θ over $\aleph$ is defined as is defined as (Silverman [1], p.450)

$$\xi\left(\Theta,\tau,\hbar\right)=\hbar^{\tau/2}\left(2\pi\right)^{-\tau}\Gamma\left(\tau\right)\ell\left(\Theta,\tau,\hbar\right)\qquad\left(s\in\mathbb{C}\right), \tag{6}$$

where the gamma function is denoted as [6]

$$\Gamma\left(\tau\right)=\int_0^{\infty}e^{-\rho}\rho^{\tau-1}d\rho\ .$$

In fact, (6) has the functional equation ([1], Theorem 16.3, p.451)

$$\xi\left(\Theta,\tau,\hbar\right)=ƛ\xi\left(\Theta,2-\tau,\hbar\right), \tag{7}$$

where $ƛ=\pm 1$ is the root number of $\aleph$. We know that (see Theorem 16.4 in [1], p.451)

$$F\left(x,\hbar\right)\in\mathbb{S}_2\left(\Gamma_0\left(\hbar\right)\right), \tag{8}$$

where $\mathbb{S}_2\left(\Gamma_0\left(\hbar\right)\right)$ is the space of cusp forms of weight 2 for $\Gamma_0\left(\hbar\right)$. Moreover, the completed Hasse-Weil L-function has an analytic continuation for the entire complex plane $\tau\in\mathbb{C}$ [1].

It has reported that (6) can be represented by (see (2.8.5) in [7], p.42)

$$\xi\left(\Theta,\tau,\hbar\right)=\int_1^{\infty}F\left(\frac{i\varpi}{\sqrt{\hbar}},\hbar\right)\left(\varpi^{\tau-1}+ƛ\varpi^{1-\tau}\right)d\varpi \tag{9}$$

with the relation (see [7], p.42)

$$F\left(\frac{1}{x\hbar},\hbar\right)=ƛ\hbar x^2F\left(x,\hbar\right).$$

It is easy to see that (9) can be decomposed into

$$\begin{aligned}\xi\left(\Theta,\tau,\hbar,ƛ=1\right)&=\int_1^{\infty}F\left(\frac{i\varpi}{\sqrt{\hbar}},\hbar\right)\left(\varpi^{\tau-1}+\varpi^{1-\tau}\right)d\varpi\\&=2\int_1^{\infty}F\left(\frac{i\varpi}{\sqrt{\hbar}},\hbar\right)\cosh\left[\left(\tau-1\right)\log\varpi\right]d\varpi\end{aligned} \tag{10}$$

and

$$\begin{aligned}\xi\left(\Theta,\tau,\hbar,\bar{\lambda}=-1\right)&=\int_{1}^{\infty}F\left(\frac{i\varpi}{\sqrt{\hbar}},\hbar\right)\left(\varpi^{\tau-1}-\varpi^{1-\tau}\right)d\varpi\\&=2\int_{1}^{\infty}F\left(\frac{i\varpi}{\sqrt{\hbar}},\hbar\right)\sinh\left[\left(\tau-1\right)\log\varpi\right]d\varpi.\end{aligned}\tag{11}$$

Putting $\varpi=e^{\eta}$ into (10) and (11) gives

$$\xi\left(\Theta,\tau,\hbar,\bar{\lambda}=1\right)=2\int_{0}^{\infty}F\left(\frac{ie^{\eta}}{\sqrt{\hbar}},\hbar\right)\cosh\left[\left(\tau-1\right)\eta\right]e^{\eta}d\eta\tag{12}$$

and

$$\xi\left(\Theta,\tau,\hbar,\bar{\lambda}=-1\right)=2\int_{0}^{\infty}F\left(\frac{ie^{\eta}}{\sqrt{\hbar}},\hbar\right)\sinh\left[\left(\tau-1\right)\eta\right]e^{\eta}d\eta.\tag{13}$$

Let us denote

$$\Xi\left(\eta\right)=2F\left(\frac{ie^{\eta}}{\sqrt{\hbar}},\hbar\right)e^{\eta}=2e^{\eta}\sum_{m=1}^{\infty}\beta_{\Theta}\left(m,\hbar\right)e^{\frac{-2\pi me^{\eta}}{\sqrt{\hbar}}}=2\sum_{m=1}^{\infty}\beta_{\Theta}\left(m,\hbar\right)e^{\frac{-2\pi me^{\eta}}{\sqrt{\hbar}}+\eta}\tag{14}$$

such that

$$\xi\left(\Theta,\tau,\hbar,\bar{\lambda}=1\right)=\int_{0}^{\infty}\Xi\left(\eta\right)\cosh\left[\left(\tau-1\right)\eta\right]d\eta\tag{15}$$

and

$$\xi\left(\Theta,\tau,\hbar,\bar{\lambda}=-1\right)=\int_{0}^{\infty}\Xi\left(\eta\right)\sinh\left[\left(\tau-1\right)\eta\right]d\eta\,.\tag{16}$$

Taking $\tau=1+i\theta$ into (15) and (16) and using the well-known relations

$$\cosh\left(ix\right)=\cos\left(x\right)\tag{17}$$

and

$$\sinh\left(ix\right)=i\sin\left(x\right),\tag{18}$$

we show

$$\mathbb{G}\left(\Theta,\theta,\hbar,\bar{\lambda}=1\right)=\xi\left(\Theta,\tau=1+i\theta,\hbar,\bar{\lambda}=1\right)=\int_{0}^{\infty}\Xi\left(\eta\right)\cos\left(\theta\eta\right)d\eta\tag{19}$$

and

$$\xi\left(\Theta,\tau=1+i\theta,\hbar,\bar{\lambda}=1\right)=i\int_{0}^{\infty}\Xi\left(\eta\right)\sin\left(\theta\eta\right)d\eta\,.\tag{20}$$

We now consider (18) as

$$\mathbb{F}\left(\Theta,\tau,\hbar,\bar{\lambda}=-1\right)=i\xi\left(\Theta,\tau=1+i\theta,\hbar,\bar{\lambda}=1\right)=\int_{0}^{\infty}\Xi\left(\eta\right)\sin\left(\theta\eta\right)d\eta\,.\tag{21}$$

In our paper we consider a family of the function $\mathbb{G}_{t}\left(\Theta,\theta,\hbar,\bar{\lambda}=1\right):\mathbb{C}\to\mathbb{C}$ for $t\in\mathbb{R}$

by the Fourier cosine integral

$$\mathbb{G}_t(\Theta,\theta,\hbar,\lambda=1)=\int_0^\infty e^{-t\eta^2}\Xi(\eta)\cos(\theta\eta)\,d\eta \tag{22}$$

and a family of the function $\mathbb{F}_t(\Theta,\theta,\hbar,\lambda=-1):\mathbb{C}\to\mathbb{C}$ for $t\in\mathbb{R}$ by the Fourier sine integral

$$\mathbb{F}_t(\Theta,\theta,\hbar,\lambda=-1)=\int_0^\infty e^{-t\eta^2}\Xi(\eta)\sin(\theta\eta)\,d\eta\,. \tag{23}$$

Let $J=(\upsilon,\nu)$. By using (22) and (23), we reduce to the heat equations (see [8], p.15)

$$\frac{\partial\mathbb{G}_t(\Theta,\theta,\hbar,\lambda=1)}{\partial t}=\frac{\partial^2\mathbb{G}_t(\Theta,\theta,\hbar,\lambda=1)}{\partial\theta^2} \tag{24}$$

and

$$\frac{\partial\mathbb{G}_t(\Theta,\theta,\hbar,\lambda=1)}{\partial t}=\frac{\partial^2\mathbb{G}_t(\Theta,\theta,\hbar,\lambda=1)}{\partial\theta^2}, \tag{25}$$

for $t>0$ and $\theta\in J$, respectively. Further we consider more general cases by

$$\mathbb{X}_t(\Theta,\theta,\hbar,\lambda=1,\kappa)=\int_0^\infty e^{-\kappa t\eta^2}\Xi(\eta)\cos(\theta\eta)\,d\eta \tag{26}$$

and

$$\mathbb{Y}_t(\Theta,\theta,\hbar,\lambda=-1,\kappa)=\int_0^\infty e^{-\kappa t\eta^2}\Xi(\eta)\sin(\theta\eta)\,d\eta \tag{27}$$

to obtain the heat equations (see [8], p.15)

$$\frac{\partial\mathbb{G}_t(\Theta,\theta,\hbar,\lambda=1)}{\partial t}=\kappa\frac{\partial^2\mathbb{G}_t(\Theta,\theta,\hbar,\lambda=1)}{\partial\theta^2} \tag{28}$$

and

$$\frac{\partial\mathbb{G}_t(\Theta,\theta,\hbar,\lambda=1)}{\partial t}=\kappa\frac{\partial^2\mathbb{G}_t(\Theta,\theta,\hbar,\lambda=1)}{\partial\theta^2}, \tag{29}$$

where the positive constant $\kappa>0$ stands for the thermal diffusivity of the material.

The structure of the paper is given as follows. In Section 2 we introduce the entire functions associated with the Hasse-Weil L-function. In Section 3 we use them to deduce the heat equations. Finally we show the decision in Section 4.

Entire Functions of Elliptic Curves over the Field of Rational Numbers

The properties and conjecture for $\mathbb{G}_t(\Theta,\theta,\hbar,\lambda=1)$

A family of the function $\mathbb{G}_t(\Theta,\theta,\hbar,\lambda=1):\mathbb{C}\to\mathbb{C}$ for $t\in\mathbb{R}$ is defined by the Fourier cosine integral

$$\mathbb{G}_t\left(\Theta,\theta,\hbar,\bar{\lambda}=1\right)=\int_0^\infty e^{-t\eta^2}\Xi\left(\eta\right)\cos\left(\theta\eta\right)d\eta\,. \tag{30}$$

Substituting $t=0$ into (30), we obtain

$$\mathbb{G}_{t=0}\left(\Theta,\theta,\hbar,\bar{\lambda}=1\right)=\mathbb{G}\left(\Theta,\theta,\hbar,\bar{\lambda}=1\right)=\int_0^\infty \Xi\left(\eta\right)\cos\left(\theta\eta\right)d\eta\,. \tag{31}$$

There is

$$\mathbb{G}_t\left(\Theta,\theta,\hbar,\bar{\lambda}=1\right)=\mathbb{G}_t\left(\Theta,-\theta,\hbar,\bar{\lambda}=1\right) \tag{32}$$

for $\theta\in\mathbb{C}$.

Since $\mathbb{G}\left(\Theta,\theta,\hbar,\bar{\lambda}=1\right)$ is an entire function of order one [9], (30) is an entire function of order one and is of genus one. We see that the convergence exponent of the zeros of (30) is one [10]. It is easy to see the following result:

Conjecture I. $\mathbb{G}_t\left(\Theta,\theta,\hbar,\bar{\lambda}=1\right)$ has purely real zeros in the entire complex plane $\theta\in\mathbb{C}$.

The properties and conjecture for $\mathbb{F}_t\left(\Theta,\theta,\hbar,\bar{\lambda}=-1\right)$

A family of the function $\mathbb{F}_t\left(\Theta,\theta,\hbar,\bar{\lambda}=-1\right):\mathbb{C}\to\mathbb{C}$ for $t\in\mathbb{R}$ is defined by the Fourier sine integral

$$\mathbb{F}_t\left(\Theta,\theta,\hbar,\bar{\lambda}=-1\right)=\int_0^\infty e^{-t\eta^2}\Xi\left(\eta\right)\sin\left(\theta\eta\right)d\eta\,. \tag{33}$$

Putting $t=0$ into (33), we obtain

$$\mathbb{F}_{t=0}\left(\Theta,\theta,\hbar,\bar{\lambda}=-1\right)=\mathbb{F}\left(\Theta,\theta,\hbar,\bar{\lambda}=-1\right)=\int_0^\infty \Xi\left(\eta\right)\sin\left(\theta\eta\right)d\eta \tag{34}$$

for $\theta\in\mathbb{C}$.

It is seen that

$$\mathbb{F}_t\left(\Theta,-\theta,\hbar,\bar{\lambda}=-1\right)=-\mathbb{F}_t\left(\Theta,\theta,\hbar,\bar{\lambda}=-1\right) \tag{35}$$

for $\theta\in\mathbb{C}$.

Since $\mathbb{F}\left(\Theta,\theta,\hbar,\bar{\lambda}=-1\right)$ is an entire function of order one [9], (33) is an entire function of order one and is of genus one. We see that the convergence exponent of the zeros of (33) is one [10]. So, we suggest the following result:

Conjecture II. $\mathbb{G}_t\left(\Theta,\theta,\hbar,\bar{\lambda}=1\right)$ has purely real zeros in the entire complex plane $\theta\in\mathbb{C}$.

The properties and conjectures for $\mathbb{X}_t\left(\Theta,\theta,\hbar,\bar{\lambda}=1,\kappa\right)$ *and* $\mathbb{Y}_t\left(\Theta,\theta,\hbar,\bar{\lambda}=-1,\kappa\right)$

A family of the function $\mathbb{X}_t\left(\Theta,\theta,\hbar,\bar{\lambda}=1,\kappa\right):\mathbb{C}\to\mathbb{C}$ for $t\in\mathbb{R}$ is defined by the Fourier cosine integral

$$\mathbb{X}_t\left(\Theta,\theta,\hbar,\bar{\lambda}=1,\kappa\right)=\int_0^\infty e^{-\kappa t\eta^2}\Xi\left(\eta\right)\cos\left(\theta\eta\right)d\eta\,. \tag{36}$$

where the positive constant $\kappa > 0$ stands for the thermal diffusivity of the material.

It is clear that (36) is an entire function of order one and is of genus one. The convergence exponent of its zeros is one [10].

Conjecture III. $\mathbb{X}_t\left(\Theta,\theta,\hbar,\lambda=1,\kappa\right)$ has purely real zeros in the entire complex plane $\theta \in \mathbb{C}$.

A family of the function $\mathbb{Y}_t\left(\Theta,\theta,\hbar,\lambda=-1,\kappa\right):\mathbb{C}\to\mathbb{C}$ for $t\in\mathbb{R}$ is defined by the Fourier sine integral

$$\mathbb{Y}_t\left(\Theta,\theta,\hbar,\lambda=-1,\kappa\right)=\int_0^\infty e^{-\kappa t\eta^2}\Xi\left(\eta\right)\sin\left(\theta\eta\right)d\eta, \tag{37}$$

where the positive constant $\kappa > 0$ stands for the thermal diffusivity of the material.

It is seen that (37) is an entire function of order one and is of genus one. The convergence exponent of its zeros is one [10].

Conjecture IV. $\mathbb{Y}_t\left(\Theta,\theta,\hbar,\lambda=-1,\kappa\right)$ has purely real zeros in the entire complex plane $\theta \in \mathbb{C}$.

Remark. There are the following relations

$$\mathbb{X}_t\left(\Theta,\theta,\hbar,\lambda=1,\kappa=1\right)=\mathbb{G}_t\left(\Theta,\theta,\hbar,\lambda=1\right) \tag{38}$$

and

$$\mathbb{Y}_t\left(\Theta,\theta,\hbar,\lambda=-1,\kappa=1\right)=\mathbb{F}_t\left(\Theta,\theta,\hbar,\lambda=-1\right). \tag{39}$$

Moreover,

$$\mathbb{X}_{t=0}\left(\Theta,\theta,\hbar,\lambda=1,\kappa=1\right)=\mathbb{G}_{t=0}\left(\Theta,\theta,\hbar,\lambda=1\right)=\mathbb{G}\left(\Theta,\theta,\hbar,\lambda=1\right) \tag{40}$$

and

$$\mathbb{Y}_{t=0}\left(\Theta,\theta,\hbar,\lambda=-1,\kappa=1\right)=\mathbb{F}_{t=0}\left(\Theta,\theta,\hbar,\lambda=-1\right)=\mathbb{F}\left(\Theta,\theta,\hbar,\lambda=-1\right). \tag{41}$$

The Solutions for the Heat Equations

By using (36) and considering $\theta \in J$, we show that

$$\frac{\partial \mathbb{X}_t}{\partial t}=\frac{\partial}{\partial t}\left\{\int_0^\infty e^{-\kappa t\eta^2}\Xi\left(\eta\right)\cos\left(\theta\eta\right)d\eta\right\}=-\kappa\int_0^\infty \eta^2 e^{-\kappa t\eta^2}\Xi\left(\eta\right)\cos\left(\theta\eta\right)d\eta, \tag{42}$$

$$\frac{\partial \mathbb{X}_t}{\partial \theta}=\frac{\partial}{\partial \theta}\left\{\int_0^\infty e^{-\kappa t\eta^2}\Xi\left(\eta\right)\cos\left(\theta\eta\right)d\eta\right\}=-\int_0^\infty \eta e^{-\kappa t\eta^2}\Xi\left(\eta\right)\sin\left(\theta\eta\right)d\eta, \tag{43}$$

and

$$\frac{\partial^2 \mathbb{X}_t}{\partial \theta^2}=\frac{\partial^2}{\partial \theta^2}\left\{\int_0^\infty e^{-\kappa t\eta^2}\Xi\left(\eta\right)\cos\left(\theta\eta\right)d\eta\right\}=-\int_0^\infty \eta^2 e^{-\kappa t\eta^2}\Xi\left(\eta\right)\cos\left(\theta\eta\right)d\eta. \tag{44}$$

From (42) and (44) we get the heat equation (see [8], p.15)

$$\frac{\partial \mathbb{X}_t}{\partial t}=\kappa\frac{\partial^2 \mathbb{X}_t}{\partial \theta^2}\quad\left(\theta\in J,\, t>0\right), \tag{45}$$

with the initial condition

$$\mathbb{Y}_{t=0}\left(\Theta,\theta,\hbar,\lambda=-1,\kappa\right)=\int_{0}^{\infty}\Xi\left(\eta\right)\sin\left(\theta\eta\right)d\eta \tag{46}$$

and the Dirichlet-type boundary conditions

$$\mathbb{X}_{t}\left(\Theta,\theta=\upsilon,\hbar,\lambda=1,\kappa\right)=\int_{0}^{\infty}e^{-\kappa t\eta^{2}}\Xi\left(\eta\right)\cos\left(\upsilon\eta\right)d\eta \tag{47}$$

and

$$\mathbb{X}_{t}\left(\Theta,\theta=\nu,\hbar,\lambda=1,\kappa\right)=\int_{0}^{\infty}e^{-\kappa t\eta^{2}}\Xi\left(\eta\right)\cos\left(\nu\eta\right)d\eta\,. \tag{48}$$

By using (30), we obtain that

$$\frac{\partial\mathbb{G}_{t}}{\partial t}=\frac{\partial}{\partial t}\left\{\int_{0}^{\infty}e^{-t\eta^{2}}\Xi\left(\eta\right)\cos\left(\theta\eta\right)d\eta\right\}=-\int_{0}^{\infty}\eta^{2}e^{-t\eta^{2}}\Xi\left(\eta\right)\cos\left(\theta\eta\right)d\eta\,, \tag{49}$$

$$\frac{\partial\mathbb{G}_{t}}{\partial\theta}=\frac{\partial}{\partial\theta}\left\{\int_{0}^{\infty}e^{-t\eta^{2}}\Xi\left(\eta\right)\cos\left(\theta\eta\right)d\eta\right\}=-\int_{0}^{\infty}\eta e^{-t\eta^{2}}\Xi\left(\eta\right)\sin\left(\theta\eta\right)d\eta\,, \tag{50}$$

and

$$\frac{\partial^{2}\mathbb{G}_{t}}{\partial\theta^{2}}=\frac{\partial^{2}}{\partial\theta^{2}}\left\{\int_{0}^{\infty}e^{-t\eta^{2}}\Xi\left(\eta\right)\cos\left(\theta\eta\right)d\eta\right\}=-\int_{0}^{\infty}\eta^{2}e^{-\kappa t\eta^{2}}\Xi\left(\eta\right)\cos\left(\theta\eta\right)d\eta\,. \tag{51}$$

From (49) and (51) we get the heat equation (see [8], p.15)

$$\frac{\partial\mathbb{G}_{t}}{\partial t}=\frac{\partial^{2}\mathbb{G}_{t}}{\partial\theta^{2}}\quad\left(\theta\in J,\,t>0\right), \tag{52}$$

with the initial condition

$$\mathbb{G}_{t=0}\left(\Theta,\theta,\hbar,\lambda=1\right)=\int_{0}^{\infty}\Xi\left(\eta\right)\cos\left(\theta\eta\right)d\eta \tag{53}$$

and the Neumann-type boundary conditions

$$\frac{\partial\mathbb{G}_{t}}{\partial\theta}\Big|_{\theta=\upsilon}=-\int_{0}^{\infty}\eta e^{-t\eta^{2}}\Xi\left(\eta\right)\sin\left(\upsilon\eta\right)d\eta \tag{54}$$

and

$$\frac{\partial\mathbb{G}_{t}}{\partial\theta}\Big|_{\theta=\nu}=-\int_{0}^{\infty}\eta e^{-t\eta^{2}}\Xi\left(\eta\right)\sin\left(\nu\eta\right)d\eta\,. \tag{55}$$

Using (37) and $\theta\in J$, we arrive at

$$\frac{\partial\mathbb{Y}_{t}}{\partial t}=\frac{\partial}{\partial t}\left\{\int_{0}^{\infty}e^{-\kappa t\eta^{2}}\Xi\left(\eta\right)\sin\left(\theta\eta\right)d\eta\right\}=-\kappa\int_{0}^{\infty}\eta^{2}e^{-\kappa t\eta^{2}}\Xi\left(\eta\right)\sin\left(\theta\eta\right)d\eta\,, \tag{56}$$

$$\frac{\partial\mathbb{Y}_{t}}{\partial\theta}=\frac{\partial}{\partial\theta}\left\{\int_{0}^{\infty}e^{-\kappa t\eta^{2}}\Xi\left(\eta\right)\sin\left(\theta\eta\right)d\eta\right\}=\int_{0}^{\infty}\eta e^{-\kappa t\eta^{2}}\Xi\left(\eta\right)\cos\left(\theta\eta\right)d\eta\,, \tag{57}$$

and

$$\frac{\partial^2 \mathbb{Y}_t}{\partial \theta^2} = \frac{\partial^2}{\partial \theta^2}\left\{\int_0^\infty e^{-\kappa t \eta^2}\Xi(\eta)\sin(\theta\eta)\,d\eta\right\} = -\int_0^\infty \eta^2 e^{-\kappa t\eta^2}\Xi(\eta)\sin(\theta\eta)\,d\eta\,. \tag{58}$$

It follows from (56) and (58) that the heat equation reads (see [8], p.15)

$$\frac{\partial \mathbb{Y}_t}{\partial t} = \kappa \frac{\partial^2 \mathbb{Y}_t}{\partial \theta^2} \quad (\theta \in J,\, t>0) \tag{59}$$

with the initial condition

$$\mathbb{Y}_{t=0}(\Theta,\theta,\hbar,\lambda=-1,\kappa) = \int_0^\infty \Xi(\eta)\sin(\theta\eta)\,d\eta \tag{60}$$

and the Dirichlet-type boundary conditions

$$\mathbb{Y}_t(\Theta,\theta=\upsilon,\hbar,\lambda=-1,\kappa) = \int_0^\infty e^{-\kappa t\eta^2}\Xi(\eta)\sin(\upsilon\eta)\,d\eta \tag{61}$$

and

$$\mathbb{Y}_t(\Theta,\theta=\nu,\hbar,\lambda=-1,\kappa) = \int_0^\infty e^{-\kappa t\eta^2}\Xi(\eta)\sin(\nu\eta)\,d\eta\,. \tag{62}$$

Making use of (33), we also present

$$\frac{\partial \mathbb{F}_t}{\partial t} = \frac{\partial}{\partial t}\left\{\int_0^\infty e^{-t\eta^2}\Xi(\eta)\sin(\theta\eta)\,d\eta\right\} = -\int_0^\infty \eta^2 e^{-\kappa t\eta^2}\Xi(\eta)\sin(\theta\eta)\,d\eta\,, \tag{63}$$

$$\frac{\partial \mathbb{F}_t}{\partial \theta} = \frac{\partial}{\partial \theta}\left\{\int_0^\infty e^{-t\eta^2}\Xi(\eta)\sin(\theta\eta)\,d\eta\right\} = \int_0^\infty \eta e^{-t\eta^2}\Xi(\eta)\cos(\theta\eta)\,d\eta\,, \tag{64}$$

and

$$\frac{\partial^2 \mathbb{F}_t}{\partial \theta^2} = \frac{\partial^2}{\partial \theta^2}\left\{\int_0^\infty e^{-t\eta^2}\Xi(\eta)\sin(\theta\eta)\,d\eta\right\} = -\int_0^\infty \eta^2 e^{-t\eta^2}\Xi(\eta)\sin(\theta\eta)\,d\eta\,. \tag{65}$$

With (63) and (65), we obtain the heat equation (see [8], p.15)

$$\frac{\partial \mathbb{F}_t}{\partial t} = \frac{\partial^2 \mathbb{F}_t}{\partial \theta^2} \tag{66}$$

with the initial condition

$$\mathbb{F}_{t=0}(\Theta,\theta,\hbar,\lambda=-1) = \int_0^\infty \Xi(\eta)\sin(\theta\eta)\,d\eta \tag{67}$$

and the Neumann-type boundary conditions

$$\frac{\partial \mathbb{F}_t}{\partial \theta}\Big|_{\theta=\upsilon} = \int_0^\infty \eta e^{-t\eta^2}\Xi(\eta)\cos(\upsilon\eta)\,d\eta \tag{68}$$

and

$$\frac{\partial \mathbb{F}_t}{\partial \theta}\Big|_{\theta=\nu} = \int_0^\infty \eta e^{-t\eta^2} \Xi(\eta) \cos(\nu\eta) d\eta . \tag{69}$$

Conclusion

In the present work we have proposed the entire functions associated with the Hasse-Weil L-function. We have guessed that the entire functions of the elliptic curves over the field of rational numbers have purely real zeros in the entire complex plane. We have discovered that the entire functions are considered as the solutions for the heat equations.

Acknowledgements:

This work is supported by the Yue-Qi Scholar of the China University of Mining and Technology (No. 102504180004), the Joint Special Fund for Fundamental Research of Local Undergraduate Universities(Partial) in Yunnan Province, China (No. 202001BA070001-024), the Science Research Fund of the Education Department of Yunnan Province, China (No. 2020J0630), Scientific and Technological Innovation Team of Nonlinear Analysis and Algebra with Their Applications in Universities of Yunnan Province, China (No. 2020CXTD25).

Nomenclature

$\mathbb{R}$ -set of real numbers, [$-$]
$\mathbb{C}$ -set of complex numbers, [$-$]
θ -space coordinate, [m]
t -time coordinate, [s]
$\Re(\tau)$ -real part, [-]
$\aleph$ -set of rational numbers, [$-$]

References

[1] Silverman, J. H., *The Arithmetic of Elliptic Curves*, Springer, New York, 2009
[2] Diamond, F., and Shurman, J. M., *A First Course in Modular Forms*, Springer, New York, 2005
[3] Breuil, C., *et al.*, On the Modularity of Elliptic Curves over Q: Wild 3-adic Exercises, *Journal of the American Mathematical Society*, 14 (2001), 4, pp.843-939
[4] Wiles, A., Modular Elliptic Curves and Fermat's Last Theorem, *Annals of mathematics*, 141 (1995), 3, pp.443-551
[5] Taylor, R. and Wiles, A., Ring-theoretic Properties of Certain Hecke Algebras, *Annals of Mathematics*, 141 (1995), 3, pp.553-572
[6] Yang, X. J., *Theory and Applications of Special Functions for Scientists and Engineers*, Springer Singapore, 2021
[7] Cremona, J. E., *Algorithms for Modular Elliptic Curves*, Cambridge University Press, Cambridge, 1997
[8] Cannon, J. R., *The One-dimensional Heat Equation*, Vol. 23, Cambridge University Press, Cambridge, 1984
[9] Ogg, A., *Modular Forms and Dirichlet Series*, Vol. 39, WA Benjamin, New York, 1969
[10] Boas, R. P., *Entire Functions*, Academic Press, New York, USA, 1954

Paper submitted: August 7, 2022
Paper revised: November 12, 2022
Paper accepted: November 22, 2022