SOLITON SOLUTIONS OF (2+1)-DIMENSIONAL NON-LINEAR REACTION-DIFFUSION MODEL VIA RICCATI-BERNOULLI APPROACH

by

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In this study, soliton solutions of the (2+1)-dimensional reaction-diffusion equation are investigated by the extended Kudryashov method based on Riccati-Bernoulli approach. Firstly, we obtained the non-linear ordinary differential form of the (2+1)-dimensional non-linear reaction-diffusion equation by implementing the wave transformation. Then, the extended Kudryashov method has been presented and applied to the non-linear ordinary differential form. By applying the extended Kudryashov method the polynomial form has been gained, solution sets have been obtained and soliton solutions have been formed by taking the appropriate sets. Finally, some graphical representations of the gained results for instance bright, dark, kink and singular solutions are presented and commented. Within the scope of the article, the study on investigating the soliton solutions of the (2+1)-dimensional non-linear reaction-diffusion equation via the extended Kudryashov approach has not been studied and the obtained results have not been reported.

Key words: *wave transform, the extended Kudryashov method, kink soliton, traveling wave solution*

Introduction

The studies of the exact solitary wave solutions of the non-linear differential partial equations (NLPDE) play crucial role in the learning of non-linear physical phenomena in many fields, for instance physics, hydrodynamics, chemistry, wave dynamics, thermodynamics, statistics, quantum mechanics, fluid dynamics, heat and energy transfer and so on. There is a lot of research on the solutions of NLPDE and non-linear optics has a unique importance and research area among these areas. In recent years, many studies have been carried out in this field [1-7]. Moreover, many researchers have concentrated to determine exact solitary wave solutions of NLPDE and many different methods have improved and used in recent years. Such as, the exp-function [8], the Kudryashov R function [9], the trial equation [10], the unified Riccati equation expansion [11], F-expansion [12], (G′/G)-expansion [13], and several others. Apart from these methods, the extended Kudryashov [14] is one of the most effective method to obtain different type solitary wave solutions of NLPDE. In order to model the physical phenomena in the known universe, many equations have been developed especially in the last 50 years and the search for their solutions has been started. As the main reason for this, we can show that software and symbolic computation programs have entered our lives depending on the developments in the computer and electronics

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sector. In this way, both numerical and analytical computational techniques, methods and models have been developed and successfully applied in the solution of many non-linear equations. As an example of these equations, we can give: Schrodinger equation [13], Camassa-Holm equation [15], Kundu–Muhherjee–Naskar model [16], Biswas-Milovic equation [17], Triki-Biswas equation [18], Radhakrishnan–Kundu–Lakshmanan model [19], Kdv forms [20], Zoomeron equation [21], Fornberg-Whitham equation [22], Mikhailov-Novikov-Wang equation [23], Kawahara equation [24], Biswas-Arshed equation [25] and many more. One of these equations is non-linear reaction-diffusion equation which has a large application area. Examples of typical reaction models using the reaction-diffusion equation are, population dynamics, predator-prey system, competition, symbiosis, chemical reactions and so on.

The (2+1)-D non-linear reaction-diffusion equation is given [26]:

$$(P_t + aP^3 + bP^2 + cP - dP_{xx})_x + \lambda P_{yy} = 0 \tag{1}$$

where $P = P(x, y, t)$ is a real valued function, a, b, c, d and λ are the real values. Besides, d is also called diffusion coefficient. Equation (1) has an important role not only in chemical reaction and ecology but also many fields in physics. The NRDE is named in different ways for some special values of the constants λ, a, b, c and d. For example: If $\lambda = 0$, eq. (1) is called reaction-diffusion equation, if $d = 1$, $b = 0$, $c = -a$, eq. (1) becomes 2-D Chaffee-Infante equation [27]. In case that $d = 1$, $a = -b = 1$, $c = 0$, it is called 2-D Huxley equation. If $d = 1$, $a = 1$, $b = -(a+1)$, $c = a$, it is named Fitzhugh–Nagumo equation [28].

There are many particular works on NRDE. Such as, Yun-Quan *et al.* [26] investigated the analytical solutions of the NRDE by using the first integral method. In [29] the dark soliton solutions of the NRDE were examined by using the ansatz method. Biswas [30] examined the 1-Soliton Solution of the NRDE. Triki *et al.* [31] applied the auxiliary equation method to the NRDE.

In this study we investigate the traveling wave solutions of the NRDE given in eq. (1) by using the EKM.

Definition of the wave transformation and derivation of the NODE form

To obtain the NODE form of eq. (1), the wave transformation is considered as:

$$P(x, y, t) = P(\eta), \ \eta = \alpha x + \beta y - \omega t, \tag{2}$$

where η is the new variable and α, β, ω are non-zero real constants.

By inserting the wave transformation in eq. (2) into eq. (1), then integrating the gained equation with respect to η and considering the integration constant is zero, we obtain the following equation:

$$c\alpha P + b\alpha P^2 + a\alpha P^3 + (\lambda\beta^2 - \alpha\omega)\frac{\mathrm{d}P}{\mathrm{d}\eta} - \mathrm{d}\alpha^3 \frac{\mathrm{d}^2 P}{\mathrm{d}\eta^2} = 0 \tag{3}$$

where $P = P(\eta)$.

The basics, interpretation of the technique and its utilization

In order to apply the EKM, we offer the solution of eq. (3) in the following form:

$$P(\eta) = A_0 + \sum_{k=1}^{m} \sum_{i+j=k} A_{ij} M^i(\eta) N^j(\eta) + \sum_{k=1}^{m} \sum_{i+j=k} B_{ij} M^{-i}(\eta) N^{-j}(\eta) \tag{4}$$

where A_0, A_{ij}, B_{ij} $(i, j = 0,1,2,...,m)$ are unknown real constants [14]. In eq. (3), considering the terms P'', P^3 and applying the balance rule, the balance number m is calculated as $m = 1$. So, eq. (4) takes the following form (they are taken as $A_{10} = A_1$, $A_{01} = A_2$, $B_{10} = B_1, B_{01} = B_2$):

$$P(\eta) = A_0 + A_1 M(\eta) + A_2 N(\eta) + \frac{B_1}{M(\eta)} + \frac{B_2}{N(\eta)} \tag{5}$$

where $M(\eta)$ and $N(\eta)$ satisfy the following equations:

$$\frac{\mathrm{d}M(\eta)}{\mathrm{d}\eta} = R_2 M^2(\eta) - R_1 M(\eta), \quad \frac{\mathrm{d}N(\eta)}{\mathrm{d}\eta} = S_2 N^2(\eta) + S_1 N(\eta) + S_0 \tag{6}$$

where $M(\eta)$ and $N(\eta)$ produce the following solutions:

$$M(\eta) = \begin{cases} \dfrac{R_1}{R_2 + R_1 \mathrm{e}^{R_1(\eta+\eta_0)}}, & R_1 \neq 0 \\ -\dfrac{1}{R_2(\eta+\eta_0)}, & R_1 = 0 \end{cases}, \quad N(\eta) = \begin{cases} -\dfrac{S_1}{2S_2} - \dfrac{\sqrt{v}}{2S_2}\tanh\left(\dfrac{\sqrt{v}}{2}(\eta+\eta_0)\right), & v > 0 \\ -\dfrac{S_1}{2S_2} - \dfrac{\sqrt{v}}{2S_2}\coth\left(\dfrac{\sqrt{v}}{2}(\eta+\eta_0)\right), & v > 0 \\ -\dfrac{S_1}{2S_2} + \dfrac{\sqrt{-v}}{2S_2}\tan\left(\dfrac{\sqrt{-v}}{2}(\eta+\eta_0)\right), & v < 0 \\ -\dfrac{S_1}{2S_2} - \dfrac{\sqrt{-v}}{2S_2}\cot\left(\dfrac{\sqrt{-v}}{2}(\eta+\eta_0)\right), & v < 0 \\ -\dfrac{S_1}{2S_2} - \dfrac{1}{S_2(\eta+\eta_0)}, & v = 0 \end{cases} \tag{7}$$

where $R_1, R_2, S_0, S_1, S_2, v, \eta_0$ are real constants, $v = S_1{}^2 - 4S_0 S_2$, R_2 and S_2 should not be zero simultaneously. Substituting eq. (5) along with eq. (6) into eq. (3), collecting all the coefficients of $[M(\eta)]^k [N(\eta)]^l, (k, l = -3, -2, ..., 3)$ and setting these coefficients to zero, we obtain the following algebraic equations:

$$M^1(\eta)N^1(\eta): 6\alpha A_2\left(aA_0 + \frac{b}{3}\right)A_1 = 0, \quad M^1(\eta)N^{-1}(\eta): 6\alpha B_2\left(aA_0 + \frac{b}{3}\right)A_1 = 0$$

$$M^{-1}(\eta)N^1(\eta): 6\alpha B_1\left(aA_0 + \frac{b}{3}\right)A_2 = 0, \quad M^{-1}(\eta)N^{-1}(\eta): 6\alpha B_2\left(aA_0 + \frac{b}{3}\right)B_1 = 0$$

$$M^0(\eta)N^3(\eta): -2\alpha^3 dA_2 S_2^2 + a\alpha A_2^3 = 0, \quad M^{-2}(\eta)N^0(\eta): \alpha B_1^2\left(3aA_0 + b\right) = 0$$

$$M^0(\eta)N^{-3}(\eta): -2\alpha^3 dB_2 S_0^2 + a\alpha B_2^3 = 0, \quad M^3(\eta)N^0(\eta): -2\alpha^3 dA_1 R_2^2 + a\alpha A_1^3 = 0$$

$$M^0(\eta)N^1(\eta): A_2(-2\alpha^3 dS_0 S_2 - \alpha^3 dS_1^2 + 3a\alpha A_0^2 + 6a\alpha A_1 B_1 +$$
$$+3a\alpha A_2 B_2 + \beta^2 \lambda S_1 + 2\alpha bA_0 - \alpha\omega S_1 + \alpha c) = 0$$

$$M^0(\eta)N^{-1}(\eta): B_2(-2\alpha^3\mathrm{d}S_0S_2 - \alpha^3\mathrm{d}S_1^2 + 3a\alpha A_0^2 + 6a\alpha A_1B_1 + \\ +3a\alpha A_2B_2 - \beta^2\lambda S_1 + 2\alpha bA_0 + \alpha\omega S_1 + \alpha c) = 0$$

$$M^0(\eta)N^0(\eta): \{[(6aA_0 + 2b)B_2 - S_0\omega]A_2 + (6aA_0A_1 + 2bA_1 + \omega R_2)B_1 + \\ +B_2S_2\omega + A_0(aA_0^2 + bA_0 + c)\}\alpha + \beta^2\lambda\left(A_2S_0 - B_1R_2 - B_2S_2\right) - \\ -d\left(A_2S_0S_1 - B_1R_1R_2 + B_2S_1S_2\right)\alpha^3 = 0$$

$$M^1(\eta)N^0(\eta): 3A_1\left\{-\frac{R_1^2d\alpha^3}{3} + \left[\frac{R_1\omega}{3} + (A_0^2 + A_1B_1 + 2A_2B_2)a + \frac{2bA_0}{3} + \frac{c}{3}\right]\alpha - \frac{R_1\beta^2\lambda}{3}\right\} = 0$$

$$M^{-1}(\eta)N^0(\eta): 3B_1\left\{-\frac{R_1^2d\alpha^3}{3} + \left[-\frac{R_1\omega}{3} + (A_0^2 + A_1B_1 + 2A_2B_2)a + \frac{2bA_0}{3} + \frac{c}{3}\right]\alpha + \frac{R_1\beta^2\lambda}{3}\right\} = 0$$

$$M^{-2}(\eta)N^{-1}(\eta): 3a\alpha B_1^2B_2 = 0,\ \ M^1(\eta)N^{-2}(\eta): 3a\alpha A_1B_2^2 = 0,\ \ M^{-1}(\eta)N^2(\eta): 3a\alpha A_2^2B_1 = 0$$
$$M^2(\eta)N^{-1}(\eta): 3a\alpha A_1^2B_2 = 0,\ \ M^{-2}(\eta)N^1(\eta): 3a\alpha A_2B_1^2 = 0,\ \ M^2(\eta)N^1(\eta): 3a\alpha A_1^2A_2 = 0$$

$$M^1(\eta)N^2(\eta): 3a\alpha A_1A_2^2 = 0,\ \ M^{-1}(\eta)N^{-2}(\eta): 3a\alpha B_1B_2^2 = 0,\ \ M^{-3}(\eta)N^0(\eta): a\alpha B_1^3 = 0$$

$$M^0(\eta)N^{-2}(\eta): B_2(-3\alpha^3dS_0S_1 + 3a\alpha A_0B_2 - \beta^2\lambda S_0 + \alpha bB_2 + \alpha\omega S_0) = 0$$

$$M^0(\eta)N^2(\eta): A_2(-3\alpha^3dS_1S_2 + 3a\alpha A_0A_2 + \beta^2\lambda S_2 + \alpha bA_2 - \alpha\omega S_2) = 0$$

$$M^2(\eta)N^0(\eta): A_1(3\alpha^3\mathrm{d}R_1R_2 + 3a\alpha A_0A_1 + \beta^2\lambda R_2 + \alpha bA_1 - \alpha\omega R_2) = 0 \tag{8}$$

Solving the algebraic system in eq. (8), we yield the following sets and the solution functions of eq. (1) for each case.

Case 1: $R_1 \neq 0, v = S_1{}^2 - 4S_0S_2 > 0$:

$$Set_1 = \left[\begin{array}{l} \beta = \beta, \omega = \dfrac{\lambda\beta^2}{\alpha}, S_1 = -\dfrac{\Delta}{3da\alpha}, S_2 = \dfrac{9ab^2c - 2b^4}{18\Omega\alpha^2S_0}, B_1 = 0, B_2 = -\dfrac{\Omega S_0\alpha}{\Delta ab} \\ A_0 = \dfrac{-18abcd + 4b^3d}{3ab^2d - 9\tau}, A_1 = 0, A_2 = -\dfrac{9abc - 2b^3}{18\Delta\alpha aS_0}, R_1 = R_1, R_2 = R_2 \end{array}\right] \tag{9}$$

where $\tau = \sqrt{-b^2a^2d^2(4ac - b^2)}, \Delta = \sqrt{-18a^2cd + 5ab^2d - 3\tau}, \Omega = ab^2d - 3\tau$:

$$P_1(x,y,t) = A_0 + \frac{A_1R_1}{R_2 + R_1\mathrm{e}^{R_1(\alpha x + \beta y - \omega t + \eta_0)}} - A_2\left\{\frac{S_1 + \sqrt{v}\tanh[Y(x,y,t)]}{2S_2}\right\} + \\ + \frac{B_1[R_2 + R_1\mathrm{e}^{R_1(\alpha x + \beta y - \omega t + \eta_0)}]}{R_1} - \frac{2S_2B_2}{S_1 + \sqrt{v}\tanh[Y(x,y,t)]} \tag{10}$$

or

$$P_2(x,y,t) = A_0 + \frac{A_1 R_1}{R_2 + R_1 \mathrm{e}^{R_1(\alpha x+\beta y-\omega t+\eta_0)}} - A_2 \left\{ \frac{S_1 + \sqrt{v}\coth[Y(x,y,t)]}{2S_2} \right\} + \\ + \frac{B_1[R_2 + R_1 \mathrm{e}^{R_1(\alpha x+\beta y-\omega t+\eta_0)}]}{R_1} - \frac{2S_2 B_2}{S_1 + \sqrt{v}\coth[Y(x,y,t)]} \tag{11}$$

here $A_0, A_1, A_2, B_1, B_2, R_1, R_2, S_1, S_2, \beta, \omega$ are defined in Set_1 given by eq. (9) and

$$Y(x,y,t) = \frac{\sqrt{v}\,(\alpha x + \beta y - \omega t + \eta_0)}{2}$$

Case 2: $R_1 \neq 0, v = S_1{}^2 - 4S_0 S_2 < 0$:

$$Set_2 = \left[\begin{array}{l} \beta = \beta, \omega = \dfrac{(b\alpha^2\sqrt{2ad} + 2a\lambda\beta^2)\sigma + 2\lambda\beta^2 ab\sqrt{\dfrac{2}{da}} - 12\alpha^2 ca + 2\alpha^2 b^2}{2a\alpha\delta}, R_1 = \dfrac{\delta}{4\alpha} \\ R_2 = R_2, S_1 = S_1, S_2 = S_2, A_0 = 0, A_1 = -\dfrac{R_2\left(\sqrt{2ad}\,\sigma + 2b\right)\alpha}{a\delta}, A_2 = 0, B_1 = 0, B_2 = 0 \end{array} \right] \tag{12}$$

where $\delta = b\sqrt{\dfrac{2}{da}} + \sqrt{\dfrac{-8ac + 2b^2}{ad}}$, $\sigma = \sqrt{\dfrac{-8ac + 2b^2}{ad}}$

$$P_3(x,y,t) = A_0 + \frac{A_1 R_1}{R_2 + R_1 \mathrm{e}^{R_1(\alpha x+\beta y-\omega t+\eta_0)}} + \frac{A_2\left\{-S_1 + \sqrt{-v}\tan[F(x,y,t)]\right\}}{2S_2} + \\ + \frac{B_1[R_2 + R_1 \mathrm{e}^{R_1(\alpha x+\beta y-\omega t+\eta_0)}]}{R_1} + \frac{2S_2 B_2}{-S_1 + \sqrt{-v}\tan[F(x,y,t)]} \tag{13}$$

or

$$P_4(x,y,t) = A_0 + \frac{A_1 R_1}{R_2 + R_1 \mathrm{e}^{R_1(\alpha x+\beta y-\omega t+\eta_0)}} - \frac{A_2\left\{S_1 + \sqrt{-v}\cot[F(x,y,t)]\right\}}{2S_2} + \\ + \frac{B_1[R_2 + R_1 \mathrm{e}^{R_1(\alpha x+\beta y-\omega t+\eta_0)}]}{R_1} - \frac{2S_2 B_2}{S_1 + \sqrt{-v}\cot[F(x,y,t)]} \tag{14}$$

where A_0, A_1, A_2, B_1, B_2, R_1 , R_2, S_1, S_2, β, ω are defined in Set_2 given by eq. (12) and

$$F(x,y,t) = \frac{\sqrt{-v}\,(\alpha x + \beta y - \omega t + \eta_0)}{2}$$

Case 3: $R_1 \neq 0, v = S_1{}^2 - 4S_0S_2 = 0$:

$$Set_3 = \left[\begin{array}{l} \beta = \dfrac{\sqrt{2}\sqrt{a\lambda\alpha\left(\sqrt{2}\sqrt{da}\,\alpha b + 2a\omega\right)}}{2a\lambda}, \omega = \omega, R_1 = \dfrac{\sqrt{-ad(4ac-b^2)}}{\sqrt{2}da\alpha}, R_2 = \dfrac{\sqrt{2}\sqrt{da}\,A_1}{2d\alpha} \\ S_1 = S_1, S_2 = S_2, A_0 = \dfrac{-bad - \sqrt{-ad(4ac-b^2)}\sqrt{da}}{2a^2d}, A_1 = A_1, A_2 = 0, B_1 = 0, B_2 = 0 \end{array} \right] \tag{15}$$

$$\begin{aligned} P_5(x,y,t) = A_0 + \frac{A_1R_1}{R_2 + R_1 \mathrm{e}^{R_1(\alpha x + \beta y - \omega t + \eta_0)}} - A_2\left[\frac{S_1(\alpha x + \beta y - \omega t + \eta_0) + 2}{2S_2(\alpha x + \beta y - \omega t + \eta_0)}\right] + \\ + B_1\left[\frac{R_2 + R_1 \mathrm{e}^{R_1(\alpha x + \beta y - \omega t + \eta_0)}}{A_1R_1}\right] - B_2\frac{2S_2(\alpha x + \beta y - \omega t + \eta_0)}{S_1(\alpha x + \beta y - \omega t + \eta_0) + 2} \end{aligned} \tag{16}$$

where $A_0, A_1, A_2, B_1, B_2, R_1, R_{2,} S_1, S_2, \beta, \omega$ are defined in Set_3 given by eq. (15).

Case 4: $R_1 = 0, v = S_1{}^2 - 4S_0S_2 > 0$:

$$Set_4 = \left[\begin{array}{l} \lambda = \lambda, \beta = \sqrt{\dfrac{\sqrt{2}\sqrt{ad}\,\alpha^2 b + 2\omega a\alpha}{2a\lambda}}, R_1 = R_1, R_2 = R_2, S_1 = S_1, S_2 = \dfrac{\theta}{8\alpha^2 daS_0} \\ A_0 = -\dfrac{\left(-2\alpha dS_1 a + \sqrt{2}\sqrt{ad}\,b\right)\sqrt{2}}{4a\sqrt{ad}}, A_1 = 0, A_2 = \dfrac{\theta\sqrt{2}\sqrt{ad}}{8\alpha a^2 S_0 d}, B_1 = 0, B_2 = 0 \end{array} \right] \tag{17}$$

where $\theta = (2\alpha^2 dS_1^2 + 4c)a - b^2$.

$$\begin{aligned} P_6(x,y,t) = A_0 - \frac{A_1}{R_2(\alpha x + \beta y - \omega t + \eta_0)} - A_2\left\{\frac{S_1 + \sqrt{v}\tanh[Y(x,y,t)]}{2S_2}\right\} - \\ - B_1R_2(\alpha x + \beta y - \omega t + \eta_0) - \frac{2S_2B_2}{S_1 + \sqrt{v}\tanh[Y(x,y,t)]} \end{aligned} \tag{18}$$

or

$$\begin{aligned} P_7(x,y,t) = A_0 - \frac{A_1}{R_2(\alpha x + \beta y - \omega t + \eta_0)} - A_2\left\{\frac{S_1 + \sqrt{v}\coth[Y(x,y,t)]}{2S_2}\right\} - \\ - B_1R_2(\alpha x + \beta y - \omega t + \eta_0) - \frac{2S_2B_2}{S_1 + \sqrt{v}\coth[Y(x,y,t)]} \end{aligned} \tag{19}$$

where A_0, A_1, A_2, B_1, B_2, R_1, $R_{2,}$ S_1, S_2, β, λ are defined in Set_4 given by eq. (17) and

$$Y(x,y,t) = \frac{\sqrt{v}\,(\alpha x + \beta y - \omega t + \eta_0)}{2}$$

Case 5: $R_1 = 0, v = S_1^2 - 4S_0S_2 < 0$:

$$Set_5 = [\lambda = \lambda, \omega = \omega, R_2 = R_2, S_1 = S_1, S_2 = S_2, \\ A_0 = \frac{-b + \sqrt{-4ac + b^2}}{2a}, A_1 = 0, A_2 = 0, B_1 = 0, B_2 = 0\Bigg] \tag{20}$$

$$P_8(x,y,t) = A_0 - \frac{A_1}{R_2(\alpha x + \beta y - \omega t + \eta_0)} + \frac{A_2\left\{-S_1 + \sqrt{-v}\tan[F(x,y,t)]\right\}}{2S_2} - \\ -B_1R_2(\alpha x + \beta y - \omega t + \eta_0) + \frac{2S_2B_2}{-S_1 + \sqrt{-v}\tan[F(x,y,t)]} \tag{21}$$

or

$$P_9(x,y,t) = A_0 - \frac{A_1}{R_2(\alpha x + \beta y - \omega t + \eta_0)} - \frac{A_2\left\{S_1 + \sqrt{-v}\cot[F(x,y,t)]\right\}}{2S_2} - \\ -B_1R_2(\alpha x + \beta y - \omega t + \eta_0) - \frac{2S_2B_2}{S_1 + \sqrt{-v}\cot[F(x,y,t)]} \tag{22}$$

where A_0, A_1, A_2, B_1, B_2, R_1, R_2, S_1, S_2, λ, ω are defined in Set_5 given by eq. (20) and

$$F(x,y,t) = \frac{\sqrt{-v}\,(\alpha x + \beta y - \omega t + \eta_0)}{2}$$

Case 6: $R_1 = 0, v = S_1^2 - 4S_0S_2 = 0$:

$$Set_6 = [\lambda = \lambda, \omega = \omega, R_2 = R_2, S_1 = S_1, S_2 = S_2 \\ A_0 = \frac{-b + \sqrt{-4ac + b^2}}{2a}, A_1 = 0, A_2 = 0, B_1 = 0, B_2 = 0\Bigg] \tag{23}$$

$$P_{10}(x,y,t) = A_0 - \frac{A_1}{R_2(\alpha x + \beta y - \omega t + \eta_0)} - A_2\left[\frac{S_1(\alpha x + \beta y - \omega t + \eta_0) + 2}{2S_2(\alpha x + \beta y - \omega t + \eta_0)}\right] - \\ -B_1R_2(\alpha x + \beta y - \omega t + \eta_0) - B_2\frac{2S_2(\alpha x + \beta y - \omega t + \eta_0)}{S_1(\alpha x + \beta y - \omega t + \eta_0) + 2} \tag{24}$$

where $A_0, A_1, A_2, B_1, B_2, R_1, R_2, S_1, S_2, \lambda, \omega$ are defined in Set_6 given by eq. (23).

Result and discussion

In this section, graphical demonstrations related to the soliton solutions that obtained in the article are given.

Figures 1 and 2 are the graphs of $P_1(x,y,t)$ given by eq. (10), and their 3-D, 2-D figures are given in (a) and (b), respectively. In fig. 1, the graph of $P_1(x,y,t)$ according to the Set_1 in eq. (9) and for some special parameter values as $a = 0.5, b = -2, c = -1,$

$d=\lambda=\alpha=\beta=\eta_0=y=1$. Figure 1(a) shows the bright soliton shape of $P_1(x,y,t)$ and fig. 1(b) represents the rightward traveling wave feature for the values $t=1,2,3$. Figure 2 represented the graph of $P_1(x,y,t)$ by choosing the Set_1 in eq. (9) and for the values $a=0.5, b=2, c=-1, d=\lambda=\alpha=\beta=\eta_0=y=1$. Figure 2(a) characterizes the dark soliton shape of $P_1(x,y,t)$, and fig. 2(b) illustrates the rightward traveling wave property at the values $t=1,2,3$.

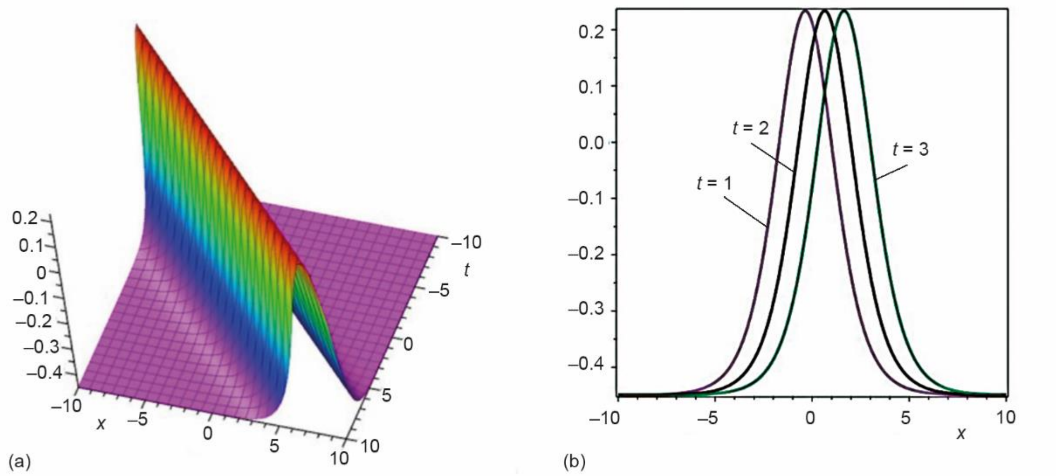

Figure 1. Graphical representations of $P_1(x, y, t)$ given by eq. (10) with Set_1 in eq. (9) and a = 0.5, b = –2, c = –1, $d=\lambda=\alpha=\beta=\eta_0=y$ = 1; (a) 3-D profile and (b) 2-D projection

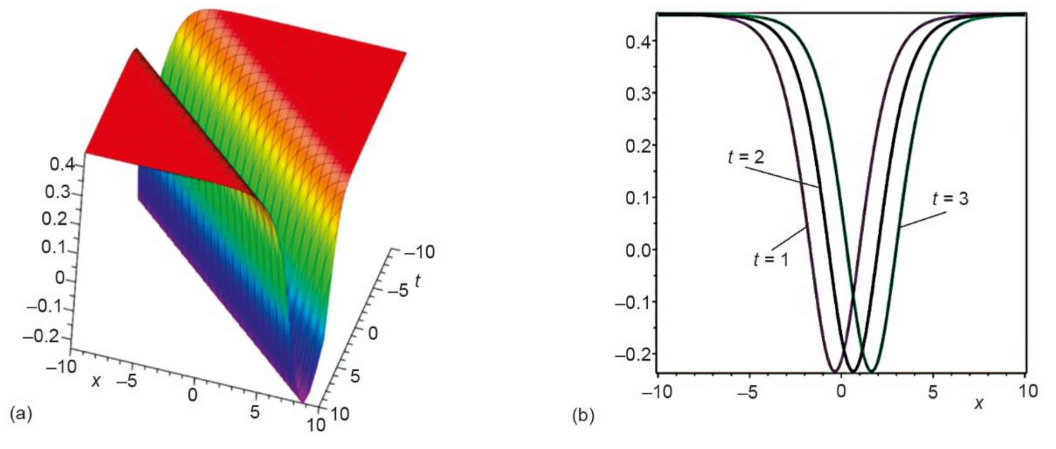

Figure 2. Graphical representations of $P_1(x, y, t)$ given by eq. (10) with Set_1 in eq. (9) and a = 0.5, b = 2, c = –1, $d=\lambda=\alpha=\beta=\eta_0=y$ = 1; (a) 3-D profile and (b) 2-D projection

In fig.3, the 3-D and 2-D figures of $P_3(x,y,t)$ in eq. (13) are given. Here, we select the Set_2 in eq. (12) and $a=-0.5, b=2, c=1, d=-1, R_2=\lambda=\alpha=\beta=\eta_0=y=1$. Figure 3(a) indicates the kink soliton which is one of the basic soliton types and fig. 3(b) depicts the state of the wave at the values $x=1,2,3$. The movement of the wave to the left is observed.

Figure 4 presents the some illustrations of $P_6(x,y,t)$ given by eq. (18) for the values $a=0.5$, $b=2$, $c=-1$, $\alpha=2$, $d=\omega=\lambda=\eta_0=y=1$ and Set_4 in eq. (17). Figure 4(a) represents the kink soliton and fig. 4(b) shows the state of the wave at the values $t=1,2,3$. The wave has the character of a rightward traveling wave.

In fig. 5, 3-D and 2-D figures of $P_7(x,y,t)$ in eq. (19) are exhibited related to the Set_4 in eq. (17) and $a=0.5, b=2$, $c=d=\omega=\lambda=\alpha=\eta_0=y=1$. As a result, we have a singular soliton in fig. 5(a). We show the state of the wave at the values $t=1,2,3$ in fig. 5(b) and obtain that the wave has the character of a rightward traveling wave.

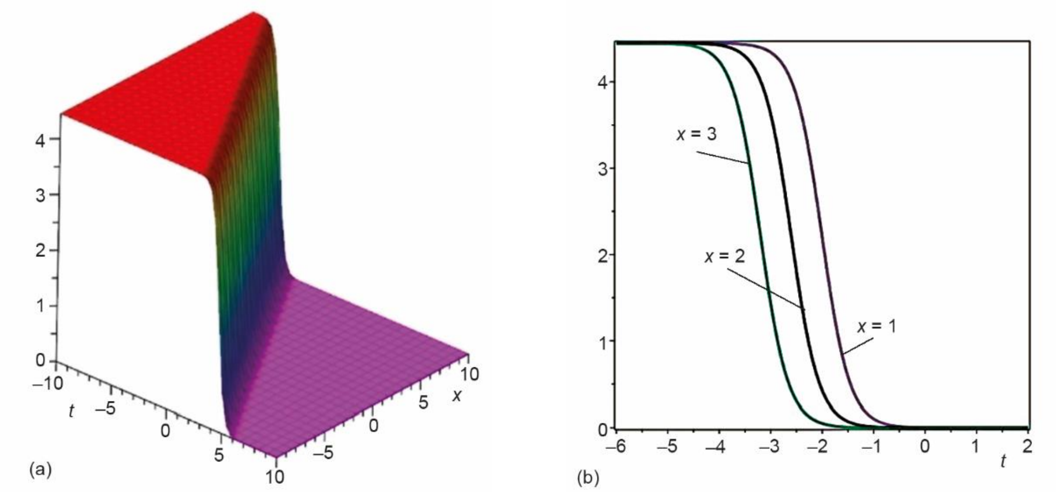

Figure 3. Graphical representations of $P_3(x, y, t)$ given by eq. (13) with Set_2 in eq. (12) and $a = -0.5$, $b = 2$, $c = 1$, $d = -1$, $R_2 = \lambda = \alpha = \beta = \eta_0 = y = 1$; (a) 3-D profile and (b) 2-D projection

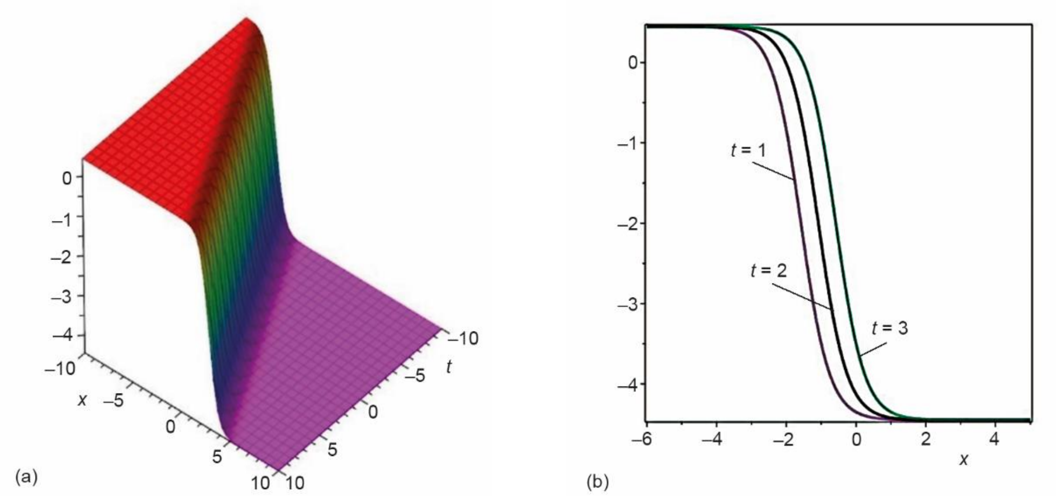

Figure 4. Graphical representations of $P_6(x, y, t)$ given by eq. (18) with Set_4 in eq. (17) and $a = 0.5$, $b = 2$, $c = -1$, $\alpha = 2$, $d = \omega = \lambda = \eta_0 = y = 1$; (a) 3-D profile and (b) 2-D projection

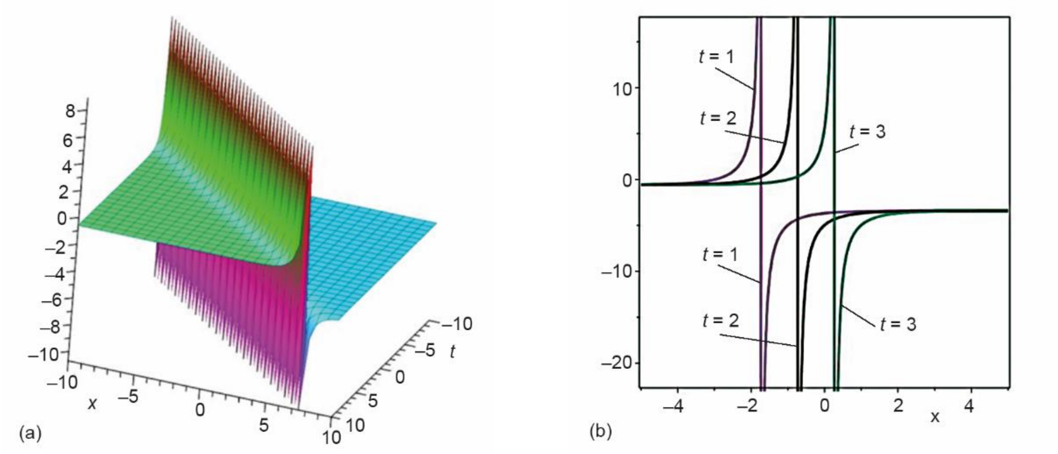

Figure 5. Graphical representations of $P_7(x, y, t)$ given by eq. (19) with Set_4 in eq. (17) and $a = 0.5$, $b = 2$, $c = d = \omega = \lambda = \alpha = \eta_0 = y = 1$; (a) 3-D profile and (b) 2-D projection

Conclusion

In this paper, we propose to examine the soliton solutions of the (2+1)-D reaction--diffusion equation (NRDE) which has a virtual role in physics and engineering. To obtain the solutions, we applied the extended Kudryashov method based on Riccati-Bernoulli approach to the NODE form of the NRDE. As the result, we reached bright, dark, kink and singular soliton solution of the investigated problem and their graphic depictions and characterizations were examined according to the special parameter values.

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