MULTIPLICATIVE CHEBYSHEV DIFFERENTIAL EQUATIONS AND MULTIPLICATIVE CHEBYSHEV POLYNOMIALS

by

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In this work, the definition of the first and second types of multiplicative Chebyshev differential equations is given and the solutions of these equations are investigated with the help of multiplicative power series. Also, the properties of first and second type multiplicative Chebyshev polynomials are given and proved. Finally, these studies are supported with numerical examples.

Key words: *multiplicative analysis, multiplicative Chebyshev polynomials, multiplicative Chebyshev differential equation, multiplicative power series*

Introduction

Differential equations are used by scientists and engineers to model and solve many problems. For this reason, some methods have been developed to find approximate solutions to differential equations. One of these methods is a spectral method in which orthogonal polynomials are used while finding solutions to differential equations. Some of the orthogonal polynomials used in differential equation solutions are Chebyshev, Laguerre, Legendre, and Hermite polynomials.

Chebyshev polynomials were first defined by the famous Russian mathematician Pafnuty Lvovich Chebyshev (1821-1894). Their importance was better understood by Lanczos who took a very important step in approximation theory by introducing a polynomial approach in his work in 1938. And by this approach, practical calculations became much easier. Thus, the number of studies on the use of Chebyshev polynomials and series has increased rapidly. After this approach, many mathematicians, especially Clenshaw (1956, 1957, 1962) and Sezer (1985, 1989, 1996, *etc.*) made applications to the approximate numerical solutions of ordinary differential equations. On the other hand, El-Gendi (1969) introduced new approaches for matrix solutions of linear ordinary differential equations and integral and integro-differential equations with Chebyshev polynomials. This approach has become a subject that has been studied extensively with the programming techniques that emerged with computer technology. The studies of Clenshaw (1957), Mason (1967), Sezer (1996), and many other scientists for the solution of integral and integro-differential equations have increased the importance of Chebyshev polynomials. Second-order linear differential equations are common in practice. The most important second-order linear differential equations are the Airy, Legendre, Hermite, Chebyshev, Bessel, and Laguerre equations. While

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the Chebyshev equation plays a major role in physics and engineering, the Bessel equation appears in the solution of Laplace's equation in cylindrical coordinates. The Laguerre equation is used in quantum mechanics.

There are many reasons for studying multiplicative analysis. First of all, many events in nature change exponentially. For example, the growth of a population or the severity of an earthquake, or the interest rates charged by a bank to its customers change exponentially. Therefore, using multiplicative analysis instead of classical analysis provides a better physical evaluation of such events. The multiplicative analysis also gives better-than-normal results in many fields, including image processing and artificial intelligence, including computer science, finance, economics, biology, and demography [1-10]. In recent years, many researchers have conducted various studies on multiplicative analysis and have obtained effective results [11-22].

The Chebyshev equation has many applications in daily life. Explicit solutions cannot be found in the solution of these equations. It is usually easier to find numerical or series solutions. In this study, multiplicative Chebyshev polynomials and their properties will be given. Also the Chebyshev differential equations and their solutions, which have an important place in classical analysis, will be discussed in the multiplicative analysis of Chebyshev polynomials and their properties. For this, first, let's give the basic definitions and theorems of multiplicative (geometric) analysis below.

Multiplicative calculus and its fundamental properties

Definition 1. Let $g : A \subseteq \mathbb{R} \to \mathbb{R}_{exp}$ be a function. The multiplicative derivative of g is:

$$\frac{\mathrm{d}^* g}{\mathrm{d}t}(t) = g^*(t) = \lim_{h \to 0} \left[\frac{g(t+h)}{g(t)} \right]^{\frac{1}{h}} \tag{1}$$

Here, assuming that g is a positive function, the relationship between the classical derivative and the multiplicative derivative is [8]:

$$g^*(t) = \exp(\ln \circ g)'(t) \Rightarrow g^*(t) = e^{(\ln \circ g)'(t)} = e^{\frac{g'(t)}{g(t)}} \tag{2}$$

The n^{th} order multiplicative derivative of a positive function $g(t)$ at the point t which is defined [3]:

$$g^{*(n)}(t) = \exp(\ln \circ g)^{(n)}(t) \tag{3}$$

Definition 2. Let g and h be differentiable with the multiplicative derivative. If $k > 0$ is an arbitrary constant, then $(k \cdot g)$, $(g \cdot h)$, $(g + h)$, (g/h), (g^h) functions are differentiable with the multiplicative derivative and their multiplicative derivatives can be shown [3]:

i) $$(k.g)^*(t) = g^*(t) \tag{4}$$

ii) $$(gh)^*(t) = g^*(t) \cdot h^*(t) \tag{5}$$

iii) $$(g+h)^*(t) = g^*(t)^{\frac{g(t)}{g(t)+h(t)}} h^*(t)^{\frac{h(t)}{g(t)+h(t)}} \tag{6}$$

iv) $$\left(\frac{g}{h}\right)^{*}(t)=\frac{g^{*}(t)}{h^{*}(t)} \tag{7}$$

v) $$(g^{h})^{*}(t)=g^{*}(t)^{h(t)}g(t)^{h^{'}(t)} \tag{8}$$

Definition 3. Let y be an infinitely multiplicative differentiable function in the neighborhood of a point t_0. In this case, y has a multiplicative power series expansion:

$$y(t)=\prod_{i=0}^{\infty}(a_i)^{(t-t_0)^i},\ (a_i\in\mathbb{R}_{\exp}) \tag{9}$$

Definition 4. Multiplicative homogeneous linear differential equation of order n is given [18]:

$$\left[y^{*(n)}\right]\left[y^{*(n-1)}\right]^{b_{n-1}(t)}\cdots\cdot\left[y^{**}\right]^{b_2(t)}\left[y^{*}\right]^{b_1(t)}y^{b_0(t)}=1 \tag{10}$$

Definition 5. Suppose $y:\mathbb{R}\to\mathbb{R}_{exp}$ is a function on $[a,b]$ and $t_0\in[a,b]$. If there exists a neighborhood of t_0 in the interval $[a,b]$ such that for all points in the neighborhood the function $y(t)$ is equal to the multiplicative power series:

$$y(t)=\prod_{i=0}^{\infty}(a_i)^{(t-t_0)^i},\ (a_i\in\mathbb{R}_{\exp})$$

Then $y(t)$ is called multiplicative-analytic at t_0 [20].

Definition 6. Let $t_0\in[a,b]$. If the functions $b_k(t)$ are analytic (in the classical sense) at t_0 for $k=0,1,2,\cdots,n-1,$, then t_0 is called a multiplicative-ordinary point of eq. (10). If t_0 is not a multiplicative-ordinary point of eq. (10) then it is said to be multiplicative singular point [20].

Theorem 1. Suppose the following initial value problem for the second order multiplicative homogeneous linear differential equation:

$$y^{**}(y^{*})^{A(t)}y^{B(t)}=1 \tag{11}$$

with the initial conditions:

$$y(t_0)=a_0,\quad y^{*}(t_0)=a_1 \tag{12}$$

is given for $a_0,a_1>0.$ If t_0 is a multiplicative-ordinary point of the given eq. (11). Then, there exists a inique analytic solution for the given problem at t_0 [20].

Multiplicative first type Chebyshev differential equations and multiplicative Chebyshev polynomials

Consider the multiplicative Chebyshev differential equation of the first type:

$$(y^{**})^{1-t^2}(y^{*})^{-t}(y)^{r^2}=1 \tag{13}$$

where r is a constant. Now let us examine solutions of eq. (13) as multiplicative power series:

$$y(t)=\prod_{i=0}^{\infty}(a_i)^{t^i}$$

This series has multiplicative derivatives:

$$y^*(t)=\prod_{i=1}^{\infty}(a_i)^{it^{i-1}}$$

$$y^{**}(t)=\prod_{i=2}^{\infty}(a_i)^{i(i-1)t^{i-2}}$$

Replacing these derivatives in eq. (13), we can write:

$$\left[\prod_{i=2}^{\infty}(a_i)^{i(i-1)t^{i-2}}\right]^{1-t^2}\left[\prod_{i=1}^{\infty}(a_i)^{it^{i-1}}\right]^{-t}\left[\prod_{i=0}^{\infty}(a_i)^{t^i}\right]^{r^2}=1$$

$$\left[\prod_{i=2}^{\infty}(a_i)^{i(i-1)t^{i-2}-i(i-1)t^i}\right]\left[\prod_{i=1}^{\infty}(a_i)^{-it^i}\right]\left[\prod_{i=0}^{\infty}(a_i)^{r^2t^i}\right]=1$$

$$(a_1)^{-t}(a_0)^{r^2}(a_1)^{r^2t}\prod_{i=2}^{\infty}(a_1)^{i(i-1)t^{i-2}+(r^2-i^2)t^i}=1$$

If we rearrange the terms in the last equation we get:

$$\left[(a)^2(a_0)^{r^2}\right]\left[(a_3)^{6x}(a_1)^{-t}(a_1)^{r^2t}\right]\prod_{i=2}^{\infty}\left[(a_{i+2})^{(i+1)(i+2)}(a_i)^{(r^2-i^2)}\right]^{t^i}=1$$

Afterward, we have the following equations:

$$(a_2)^2(a_0)^{r^2}=1$$

$$(a_3)^6(a_1)^{r^2-1}=1$$

$$(a_{i+2})^{(i+1)(i+2)}(a_i)^{(r^2-i^2)}=1$$

With the help of these recurrence equations, we write the bases (a_{2i}) which have an even index:

$$a_2=(a_0)^{\frac{-r^2}{2!}}$$

$$a_4=(a_0)^{\frac{-r^2(2^2-r^2)}{4!}}$$

$$a_6=(a_0)^{\frac{-r^2(2^2-r^2)(4^2-r^2)}{6!}}$$

$$\vdots$$

$$a_{2i}=(a_0)^{\frac{-r^2(2^2-r^2)(4^2-r^2)\cdots[(2i-2)^2-r^2]}{(2i)!}}$$

The bases (a_{2i+1}) which have an odd index can be written:

$$a_3 = (a_1)^{\frac{1-r^2}{3!}}$$

$$a_5 = (a_1)^{\frac{(1^2-r^2)(3^2-r^2)}{5!}}$$

$$a_7 = (a_1)^{\frac{(1^2-r^2)(3^2-r^2)(5^2-r^2)}{7!}}$$

$$\vdots$$

$$a_{2i+1} = (a_1)^{\frac{(1^2-r^2)(3^2-r^2)(5^2-r^2)\cdots[(2i-1)^2-r^2]}{(2i+1)!}}$$

Thus we get the solution to eq. (13):

$$y(t) = a_0 \prod_{i=1}^{\infty} (a_0)^{\frac{-r^2(2^2-r^2)(4^2-r^2)\cdots[(2i-2)^2-r^2]}{(2i)!}t^{2i}} (a_1)^t \prod_{i=1}^{\infty} (a_1)^{\frac{(1^2-r^2)(3^2-r^2)(5^2-r^2)\cdots[(2i-1)^2-r^2]}{(2i+1)!}t^{2i+1}}$$

where a_0 and a_1 are arbitrary numbers. For $a_0 = e$ and $a_1 = 1$ we have the solution of eq. (3) as $F_*(t)$ shown below:

$$F_*(t) = e e^{\frac{-r^2}{2}t^2} e^{\frac{-r^2(2^2-r^2)}{4!}t^4} e^{\frac{-r^2(2^2-r^2)(4^2-r^2)}{6!}t^6} \cdots$$

Similarly for $a_0 = 1$ and $a_1 = e$, we have the solution to eq. (13) as $G_*(t)$:

$$G_*(t) = e^t e^{\frac{1-r^2}{3!}t^3} e^{\frac{(1^2-r^2)(3^2-r^2)}{5!}t^5} e^{\frac{(1^2-r^2)(3^2-r^2)(5^2-r^2)}{7!}t^7} \cdots$$

When r is an integer, then one of the two functions $F_*(t)$ or $G_*(t)$ has a finite number of terms. If r is an even integer then $F_*(t)$ has a finite number of terms. On the contrary, if r is an odd integer then $G_*(t)$ has a finite number of terms. One of the functions $F_*(t)$ or $G_*(t)$ which we get by this way is a r^{th} degree multiplicative polynomial. By using these polynomials, r^{th} degree first type multiplicative Chebyshev polynomial is defined:

$$(\mathcal{T}_*)_r(t) = [F_*(t)]^{(-1)^{r/2}} \text{ when } r \text{ is even and}$$

$$(\mathcal{T}_*)_r(t) = [G_*(t)]^{(-1)^{\frac{r-1}{2}}r} \text{ when } r \text{ is odd}$$

The first type of multiplicative Chebyshev polynomials are:

$$(\mathcal{T}_*)_0(t) = e$$

$$(\mathcal{T}_*)_1(t) = e^t$$

$$(\mathcal{T}_*)_2(t) = e^{-1+2t^2}$$

$$(\mathcal{T}_*)_3(t) = e^{-3t+4t^3}$$

$$(\mathcal{T}_*)_4(t) = e^{1-8t^2+8t^4}$$

$$\vdots$$

Corollary. Let $(\mathcal{T})_r(t)$ be the Chebyshev polynomial in classical analysis. Then:

$$(\mathcal{T}_*)_r(t) = e^{(\mathcal{T})_r(t)} \tag{14}$$

The properties of the first type multiplicative Chebyshev polynomials

1. $(\mathcal{T}_*)_r(t) = [(\mathcal{T}_*)_{r-1}(t)]^{2t}[(\mathcal{T}_*)_{r-2}(t)]^{-1}, \quad r \ge 2$
2. For $r \ge 1,$ the exponent of e^{t^r} in $(\mathcal{T}_*)_r(t)$ is equal to 2^{r-1}
3. $(\mathcal{T}_*)_r(t) = e^{\cos(r \cdot \arccos t)}$ for $-1 \le t \le 1, \quad r \ge 0$
4. For $r \ge 2,$ the following equality holds:

$$(\mathcal{T}_*)_r(t) = \frac{[D^*(\mathcal{T}_*)_{r+1}(t)]^{\frac{1}{2}\left(\frac{1}{r+1}\right)}}{[D^*(\mathcal{T}_*)_{r-1}(t)]^{\frac{1}{2}\left(\frac{1}{r-1}\right)}} \tag{15}$$

5. $[(\mathcal{T}_*)_n(t)]^{2\ln(\mathcal{T}_*)_m(t)} = (\mathcal{T}_*)_{m+n}(t)(\mathcal{T}_*)_{|m-n|}(t),$ for $m, n \ge 0.$
6. For $r \ge 1,$ there are *r* distinct multiplicative roots of the equation $(\mathcal{T}_*)_r(t) = 1$ in the interval:

   $-1 \le t \le 1$ and these roots are $t_k = \cos\left[\frac{(2k-1)\pi}{2r}\right]$ for $k = 1, 2, \cdots, r$

Proof.

1. If *r* is an even integer then $r-1$ is odd and $r-2$ is even. So:

$$(\mathcal{T}_*)_{r-1}(t) = [G_*(t)]^{(-1)^{\frac{r-2}{2}}(r-1)}$$

$$(\mathcal{T}_*)_{r-2}(t) = [F_*(t)]^{(-1)^{\frac{r-2}{2}}}$$

Therefore we have:

$$[(\mathcal{T}_*)_{r-1}(t)]^{2t}[(\mathcal{T}_*)_{r-2}(t)]^{-1} = [G_*(t)]^{(-1)^{\frac{r-2}{2}}(r-1)2t}[F_*(t)]^{(-1)^{\frac{r-2}{2}}(-1)}$$

$$[(\mathcal{T}_*)_{r-1}(t)]^{2t}[(\mathcal{T}_*)_{r-2}(t)]^{-1} = \underbrace{\left[e^{1-\frac{r^2}{2}t^2-\frac{r^2(2^2-r^2)}{4!}t^4-\frac{r^2(2^2-r^2)(4^2-r^2)}{6!}t^6+\cdots}\right]}_{F(t)}{}^{(-1)^{r/2}}$$

$$[F_*(t)]^{(-1)^{\frac{r}{2}}} = (\mathcal{T}_*)_r(t)$$

On the other hand, if r is an odd integer then $r-1$ is even and $r-2$ is odd. Therefore, we have:

$$[(\mathcal{T}_*)_{r-1}(t)]^{2t}[(\mathcal{T}_*)_{r-2}(t)]^{-1} = [F_*(t)]^{(-1)^{\frac{r-1}{2}}2t}[G_*(t)]^{-(-1)^{\frac{r-3}{2}}(r-2)} =$$

$$= \left[e^{rt+\frac{r(1-r^2)}{3!}t^3+\frac{r(1^2-r^2)(3^2-r^2)}{5!}t^5+\frac{r(1^2-r^2)(3^2-r^2)(5^2-r^2)}{7!}t^7+\cdots} \right]^{(-1)^{\frac{r-1}{2}}}$$

$$= \left\{ \left[\underbrace{e^{rt+\frac{r(1-r^2)}{3!}t^3+\frac{r(1^2-r^2)(3^2-r^2)}{5!}t^5+\frac{r(1^2-r^2)(3^2-r^2)(5^2-r^2)}{7!}t^7+\cdots}}_{G(t)} \right]^r \right\}^{(-1)^{\frac{r-1}{2}}}$$

$$[G(t)]^{r(-1)^{\frac{r-1}{2}}} = (\mathcal{T}_*)_r(t)$$

This completes the proof.

2. This property can be seen by the recursive relation in the first property. If we take $(2t)-th$ power of $(\mathcal{T}_*)_{r-1}(t)$ then the property is proofed.
3. For proof of this property the trigonometric properties:

$$\cos(r\alpha) = \cos[\alpha + (r-1)\alpha] =$$

$$= \cos(\alpha)\cdot\cos[(r-1)\alpha] - \sin(\alpha)\cdot\sin[(r-1)\alpha] =$$

$$= \cos(\alpha)\cdot\cos[(r-1)\alpha] + \cos(\alpha)\cdot\cos[(r-1)\alpha] - \cos[(r-2)\alpha]$$

$$\cos(r\alpha) = 2\cos\alpha\cdot\cos[(r-1)\alpha] - \cos[(r-2)\alpha]$$

Finally, taking $\alpha = \arccos t$ for $-1 \le t \le 1$ we get:

$$2t\cdot\cos[(r-1)\arccos t] - \cos[(r-2)\arccos t] = \cos(r\cdot\arccos t) \tag{16}$$

Now if we take the exponential function of both sides of equation (16) we have:

$$e^{2t\cdot\cos[(r-1)\arccos t]-\cos[(r-2)\arccos t]} = e^{\cos(r\cdot\arccos t)} \tag{17}$$

We will use mathematical induction to show the validity of eq. (15). The first two first types of multiplicative Chebyshev polynomials are:

$$(\mathcal{T}_*)_0(t) = e^{\cos(0\cdot\arccos t)} = e^1, \quad (\mathcal{T}_*)_1(t) = e^{\cos(1\cdot\arccos t)} = e^t$$

and they satisfy the property. Let us assume:

$$(\mathcal{T}_*)_0(t) = e^{\cos(k\cdot\arccos t)}$$

for $k = 2,3,\cdots,r-1$. Thinking together eq. (17) and the first property, we have:

$$(\mathcal{T}_*)_r(t) = \left[(\mathcal{T}_*)_{r-1}(t)\right]^{2t}\left[(\mathcal{T}_*)_{r-2}(t)\right]^{-1} = e^{2t\cdot\cos[(r-1)\arccos t]-\cos[(r-2)\arccos t]}$$

$$(\mathcal{T}_*)_r(t) = e^{\cos(r\cdot\arccos t)}$$

4. We have the following equalities:

$$[D^*(\mathcal{T}_*)_{r+1}(t)]^{\frac{1}{r+1}} = e^{\frac{\sin(r+1)\alpha}{\sin\alpha}}$$

and

$$[D^*(\mathcal{T}_*)_{r-1}(t)]^{\frac{1}{r-1}} = e^{\frac{\sin(r-1)\alpha}{\sin\alpha}}$$

from the third property. Here D^* is the multiplicative differential operator. So we can write:

$$\frac{[D^*(\mathcal{T}_*)_{r+1}(t)]^{\frac{1}{2}\left(\frac{1}{r+1}\right)}}{[D^*(\mathcal{T}_*)_{r-1}(t)]^{\frac{1}{2}\left(\frac{1}{r-1}\right)}} = e^{\frac{1}{2}\left[\frac{\sin(r+1)\alpha}{\sin\alpha}-\frac{\sin(r-1)\alpha}{\sin\alpha}\right]} = e^{\cos r\alpha} = (\mathcal{T}_*)_r(t)$$

which is needed to show.

5.

$$[(\mathcal{T}_*)_n(t)]^{2\ln(\mathcal{T}_*)_m(t)} = [e^{\cos(n\alpha)}]^{2\ln e^{\cos(m\alpha)}} = e^{2\cos(m\alpha)\cos(n\alpha)} = e^{\cos[(m+n)\alpha]+\cos[(m-n)\alpha]} =$$

$$= e^{\cos[(m+n)\alpha]}e^{\cos[|m-n|\alpha]} = (\mathcal{T}_*)_{m+n}(t)(\mathcal{T}_*)_{|m-n|}(t)$$

6. We have

$$\mathrm{s}\,(\mathcal{T}_*)_r(t) = e^{\cos(r\alpha)} = 1$$

where $\alpha = \arccos(t)$ for $-1 \le t \le 1$. Therefore:

$$e^{\cos(r\alpha)} = 1$$

$$\cos(r\alpha) = 0$$

$$r\alpha = \frac{2k-1}{2}\pi, \quad k \in \mathbb{Z}$$

$$\alpha = \frac{2k-1}{2r}\pi$$

Since $t = \cos\alpha$, we can write:

$$t = \cos\left(\frac{2k-1}{2r}\pi\right)$$

The distinct values of t are $t_k = \cos\left[\dfrac{(2k-1)\pi}{2r}\right]$ for $k = 1, 2, \cdots, r$ as shown:

	1	2	…	r
t	$t_1 = \cos\left(\dfrac{\pi}{2r}\right)$	$t_2 = \cos\left(\dfrac{3\pi}{2r}\right)$	…	$t_r = \cos\left[\dfrac{(2r-1)\pi}{2r}\right]$

For other values of k we get one of the previous t values. For example:

k	$r + 1$	$r + 2$	…	$2r$
t	$\cos\left[\dfrac{(2r+1)\pi}{2r}\right] = t_r$	$\cos\left[\dfrac{(2r+3)\pi}{2r}\right] = t_{r-1}$	…	$\cos\left[\dfrac{(4r-1)\pi}{2r}\right] = t_1$

Thus, there are r distinct roots of the given equation.

The 2nd type multiplicative Chebyshev differential equations and multiplicative Chebyshev polynomials

Consider the following second type multiplicative Chebyshev differential equation:

$$(y^{**})^{1-t^2}(y^*)^{-3t}(y)^{r(r+2)} = 1 \tag{18}$$

Here r is a constant. The $t = 0$ is a multiplicative ordinary point of the eq. (18). Let us search for the solutions of eq. (18) as multiplicative power series written:

$$y(t) = \prod_{i=0}^{\infty}(a_i)^{t^i}$$

Then taking the multiplicative derivatives of $y(t)$:

$$y^*(t) = \prod_{i=1}^{\infty}(a_i)^{it^{i-1}}, \quad y^{**}(t) = \prod_{i=2}^{\infty}(a_i)^{i(i-1)t^{i-2}}$$

and replacing the power series equivalents of $y(t)$ and its derivatives in eq. (18) we have:

$$\left[\prod_{i=2}^{\infty}(a_i)^{i(i-1)t^{i-2}}\right]^{1-t^2}\left[\prod_{i=1}^{\infty}(a_i)^{it^{i-1}}\right]^{-3t}\left[\prod_{i=0}^{\infty}(a_i)^{t^i}\right]^{r(r+2)} = 1$$

$$\left[\prod_{i=2}^{\infty}(a_i)^{i(i-1)t^{i-2}-i(i-1)t^i}\right]\left[\prod_{i=1}^{\infty}(a_i)^{-3it^i}\right]\left[\prod_{i=0}^{\infty}(a_i)^{r(r+2)t^i}\right] = 1$$

$$(a_0)^{r(r+2)}(a_1)^{r(r+2)t}(a_1)^{-3t}\prod_{i=2}^{\infty}(a_1)^{i(i-1)t^{i-2}+(-i^2+i-3i+r^2+2r)t^i} = 1$$

Arranging the last equation we can write:

$$(a_0)^{r(r+2)}(a_1)^{r(r+2)t}(a_1)^{-3t}(a_2)^2(a3)^{6t}\prod_{i=2}^{\infty}\left[(a_{i+2})^{(i+1)(i+2)}(a_1)^{-i^2-2i+r^2+2r}\right]^{t^i} = 1$$

Consequently, we have:

$$(a_0)^{r(r+2)}(a_2)^2 = 1$$

$$(a_1)^{r^2+2r-3}(a_3)^6 = 1$$

$$(a_{i+2})^{(i+1)(i+2)}(a_i)^{(-i^2-2i+r^2+2r)} = 1$$

With the help of these equalities we write:

$$a_2 = (a_0)^{\frac{-r(r+2)}{2!}}$$

$$a_4 = (a_0)^{\frac{-r(r+2)[2\cdot 4-r(r+2)]}{4!}}$$

$$a_6 = (a_0)^{\frac{-r(r+2)[2\cdot 4-r(r+2)][4\cdot 6-r(r+2)]}{6!}}$$

$$\vdots$$

$$a_{2i} = (a_0)^{\frac{-r(r+2)[2\cdot 4-r(r+2)]\cdots[(2i-2)2i-r(r+2)]}{(2i)!}}$$

When i is odd, then:

$$a_3 = (a_1)^{\frac{[1\cdot 3-r(r+2)]}{3!}}$$

$$a_5 = (a_1)^{\frac{[1\cdot 3-r(r+2)][3\cdot 5-r(r+2)]}{5!}}$$

$$a_7 = (a_1)^{\frac{[1\cdot 3-r(r+2)][3\cdot 5-r(r+2)][5\cdot 7-r(r+2)]}{7!}}$$

$$\vdots$$

$$a_{2i+1} = (a_1)^{\frac{[1\cdot 3-r(r+2)][3\cdot 5-r(r+2)]\cdots[(2i-1)(2i+1)-r(r+2)]}{(2i+1)!}}.$$

So the solution of eq. (18) is found:

$$y(t) = a_0\prod_{i=1}^{\infty}(a_0)^{\frac{-r(r+2)[2\cdot 4-r(r+2)]\cdots[(2i-2)2i-r(r+2)]}{(2i)!}t^{2i}}(a_1)^t\cdot$$
$$\cdot\prod_{i=1}^{\infty}(a_1)^{\frac{[1\cdot 3-r(r+2)]\cdot[3\cdot 5-r(r+2)]\cdots[(2i-2)\cdot(2i+1)-r(r+2)]}{(2i+1)!}t^{2i+1}}$$

Here a_0 and a_1 are arbitrary numbers.
For $a_0 = e$ and $a_{1=1}$ we have the solution of eq. (18) as $F_*(t)$ shown:

$$F_*(t) = e \cdot e^{\frac{-r(r+2)}{2!}t^2} e^{\frac{-r(r+2)[2\cdot 4-r(r+2)]}{4!}t^4} e^{\frac{-r(r+2)[2\cdot 4-r(r+2)][4\cdot 6-r(r+2)]}{6!}t^6} \cdots$$

Similarly, for $a_0 = 1$ ve $a_1 = e$ we have the solution of eq. (18) as $G_*(t)$:

$$G_*(t) = e^{2t} \cdot e^{\frac{2[1\cdot 3-r(r+2)]}{3!}t^3} e^{\frac{2[1\cdot 3-r(r+2)][3\cdot 5-r(r+2)]}{5!}t^5} \cdots$$

When r is an integer, then one of the two functions $F_*(t)$ or $G_*(t)$ has a finite number of terms. If r is an even integer then $F_*(t)$ has a finite number of terms. On the contrary, if r is an odd integer then $G_*(t)$ has a finite number of terms. One of the functions $F_*(t)$ or $G_*(t)$ which we get by this way is a r^{th} degree multiplicative polynomial. By using these polynomials, r^{th} degree second type multiplicative Chebyshev polynomial is defined:

$$(U_*)_r(t) = [F_*(t)]^{(-1)^{r/2}} \text{ when } r \text{ is even and}$$

$$(U_*)_r(t) = [G_*(t)]^{(-1)^{(r-1)/2}\left(\frac{r+1}{2}\right)}$$

when r is odd. A few second types of multiplicative Chebyshev polynomials are:

$$(U_*)_0(t) = e$$
$$(U_*)_1(t) = e^{2t}$$
$$(U_*)_2(t) = e^{-1+4t^2}$$
$$(U_*)_3(t) = e^{-4t+8t^3}$$
$$(U_*)_4(t) = e^{1-12t^2+16t^4}$$
$$(U_*)_5(t) = e^{6t-32t^3+32t^5}$$
$$(U_*)_6(t) = e^{1-24t^2-80t^4+64t^6}$$

Corollary. Let $(U)_r(t)$ be the classical second type Chebyshev polynomials. Then the following equation holds:

$$(U_*)_r(t) = e^{(U)_r(t)} \tag{19}$$

The properties of the second type multiplicative Chebyshev polynomials

1) For $r \geq 2,$, the following equality holds:

$$(U_*)_r(t) = [(U_*)_{r-1}(t)]^{2t}[(U_*)_{r-2}(t)]^{-1}$$

2) For $-1 \leq t \leq 1,\ \alpha = \arccos t,$ the following equality holds:

$$(U_*)_r(t) = e^{\frac{\sin[(r+1)\alpha]}{\sin\alpha}}$$

3) For $r \geq 1$, the following equality holds:

$$(\mathcal{B}_*)_{r+1}(t) = [(\mathcal{B}_*)_r(t)]^t [(U_*)_{r-1}(t)]^{-(1-t^2)}$$

4) For $r \geq 1$, the following equality holds:

$$[D^*(\mathcal{B}_*)_r(t)] = [(U_*)_{r-1}(t)]^r$$

5) For $r \geq 1$, the following equality holds:

$$(\mathcal{B}_*)_r(t) = (U_*)_r(t)[(U_*)_{r-1}(t)]^{-t}$$

6) For $r \geq 2$, the following equality holds:

$$(\mathcal{B}_*)_r(t) = [(U_*)_r(t)]^{\frac{1}{2}}[(U_*)_{r-2}(t)]^{\frac{-1}{2}}$$

7) For $r \geq 1$, there are r distinct multiplicative roots of the equation $(U_*)_r(t) = 1$ in the interval $-1 \leq t \leq 1$ and these roots are $t_k = \cos[(k\pi)/(r+1)]$ for $k = 1, 2, \cdots, r$.

Proof.

1) It can be proved similarly to what was done in the proof in section *The properties of the first type multiplicative Chebyshev polynomials*.
2) We will use mathematical induction to prove the property.

First two-second type multiplicative Chebyshev polynomials are:

$$(U_*)_0(t) = e^{\frac{\sin\alpha}{\sin\alpha}} = e$$

$$(U_*)_1(t) = e^{\frac{\sin 2\alpha}{\sin\alpha}} = e^{2t}$$

and they satisfy the property. Now let us assume:

$$(U_*)_k(t) = e^{\frac{\sin[(k+1)\alpha]}{\sin\alpha}}$$

holds for $k = 2, 3, \cdots, r-1$ we have:

$$(U_*)_r(t) = [(U_*)_{r-1}(t)]^{2t}[(U_*)_{r-2}(t)]^{-1}$$

$$(U_*)_r(t) = e^{2t\frac{\sin(r\alpha)}{\sin\alpha}} e^{-\frac{\sin[(r-1)\alpha]}{\sin\alpha}}$$

$$(U_*)_r(t) = e^{\frac{[2\cos\alpha\cdot\sin(r\alpha)-\sin(r-1)\alpha]}{\sin\alpha}}$$

$$(U_*)_r(t) = e^{\frac{[2\cos\alpha\cdot\sin(r\alpha)-(\sin r\alpha\cdot\cos\alpha-\cos r\alpha\cdot\sin\alpha)]}{\sin\alpha}}$$

$$(U_*)_r(t) = e^{\frac{[\sin r\alpha\cdot\cos\alpha+\cos r\alpha\cdot\sin\alpha]}{\sin\alpha}}$$

$$(U_*)_r(t) = e^{\frac{\sin[(r+1)\alpha]}{\sin\alpha}}$$

3) For $\theta = \arccos t$:

$$(T_*)_r(t) = e^{\cos(r\alpha)}$$

$$(U_*)_{r-1}(t) = e^{\frac{\sin(r\alpha)}{\sin\alpha}}$$

$$1 - t^2 = \sin^2\alpha$$

For these results we get:

$$[(T_*)_r(t)]^t[(U_*)_{r-1}(t)]^{-(1-t^2)} = e^{\cos\alpha\cdot\cos(r\alpha)-\sin\alpha\cdot\sin(r\alpha)}$$

$$[(T_*)_r(t)]^t[(U_*)_{r-1}(t)]^{-(1-t^2)} = e^{\cos[(r+1)\alpha]}$$

$$[(T_*)_r(t)]^t[(U_*)_{r-1}(t)]^{-(1-t^2)} = (T_*)_{r+1}(t)$$

4) For $\alpha = \arccos t$ we have the following equalities:

$$[D^*(T_*)_r(t)] = D^*(e^{\cos r\alpha}) = e^{\frac{\mathrm{d}}{\mathrm{d}t}(\cos r\alpha)}$$

$$= e^{\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathrm{d}}{\mathrm{d}\alpha}\cos r\alpha\right)} = e^{\frac{-1}{\sin\alpha}[r(-\sin r\alpha)]}$$

$$= \left[e^{\frac{\sin(r\alpha)}{\sin\alpha}}\right]^r$$

$$[D^*(T_*)_r(t)] = [(U_*)_{r-1}(t)]^r$$

5) According to trigonometric formulas we have the following equalities:

$$\cos(r\alpha) = \sin[(r+1)\alpha]\sin\alpha + \cos[(r+1)\alpha]\cos\alpha$$

$$\cos[(r+1)\alpha] = \cos(r\alpha)\cos\alpha - \sin(r\alpha)\sin\alpha$$

$$\cos(r\alpha)\sin[(r+1)\alpha]\sin\alpha - \sin(r\alpha)\sin\alpha\cdot\cos\alpha + \cos(r\alpha)\cos^2\alpha$$

Then we have:

$$\cos(r\alpha) = \frac{\sin[(r+1)\alpha]}{\sin\alpha} - \frac{\sin(r\alpha)}{\sin\alpha}\cos\alpha$$

Finally, taking $\cos\alpha = t$ we get:

$$(T_*)_r(t) = [(U_*)_r(t)][(U_*)_{r-1}(t)]^{-t}$$

6)

$$[(U_*)_r(t)]^{\frac{1}{2}}[(U_*)_{r-2}(t)]^{\frac{-1}{2}} = e^{\frac{1}{2}\frac{\sin(r+1)\alpha}{\sin\alpha}}\, e^{-\frac{1}{2}\frac{\sin(r-1)\alpha}{\sin\alpha}}$$

$$= e^{\frac{1}{2}\frac{[\sin(r+1)\alpha-\sin(r-1)\alpha]}{\sin\alpha}}$$

$$= e^{\frac{1}{2}\frac{2\cos\alpha\cdot\sin\alpha}{\sin\alpha}}$$

$$= e^{\cos\alpha}$$
$$= (T_*)_r(t)$$

7) The given equation is:

$$(U_*)_r(t) = 1$$

Using the second property of the second type multiplicative Chebyshev polynomials, we have the following equation:

$$(U_*)_r(t) = e^{\frac{\sin[(r+1)\alpha]}{\sin\alpha}} = 1$$

where $\alpha = \arccos(t)$ for $-1 \le t \le 1$. Therefore:

$$\frac{\sin[(r+1)\alpha]}{\sin\alpha} = 0, \quad \sin[(r+1)\alpha] = 0, \quad \text{and} \quad \sin\alpha \neq 0$$

$$\alpha = \frac{k}{r+1}\pi \quad \text{and} \quad k \in \mathbb{Z} \setminus \{(r+1)\cdot z \mid z \in \mathbb{Z}\}$$

Since $t = \cos\alpha$, we can write:

$$t = \cos\left(\frac{k}{r+1}\pi\right), \quad k \in \mathbb{Z} \setminus \{(r+1)\cdot z \mid z \in \mathbb{Z}\}$$

The distinct values of t are $t_k = \cos[(k\pi)/(r+1)]$ for $k = 1, 2, \cdots, r$ as shown:

k	1	2	…	r
t	$t_1 = \cos\left(\frac{\pi}{r+1}\right)$	$t_2 = \cos\left(\frac{2}{r+1}\pi\right)$	…	$t_r = \cos\left(\frac{r}{r+1}\pi\right)$

For $k = (r+1)z, \quad z \in \mathbb{Z}$, t is undefined. For other values of k, we get one of the t values above. For example:

k	$r + 2$	$r + 3$	…	$2r + 1$
t	$\cos\left(\frac{r+2}{r+1}\pi\right) = t_r$	$\cos\left(\frac{r+3}{r+1}\pi\right) = t_{r-1}$	…	$\cos\left(\frac{2r+1}{r+1}\pi\right) = t_1$

Thus, there are r distinct roots of the given equation.

Conclusions

The In this study, different-type multiplicative Chebyshev polynomials are obtained by solving first and second-type Chebyshev differential equations in geometric (multiplicative) analysis. In addition, some properties of the first and second-type multiplicative Chebyshev polynomials are presented. In this study, it has been seen that the results obtained with the help of multiplicative derivatives are compatible with the results obtained with the help of classical derivatives.

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