A COMPUTATIONAL APPROACH FOR THE CLASSIFICATIONS OF ALL POSSIBLE DERIVATIONS OF NILSOLITONS IN DIMENSION 9

by

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In mathematics and engineering, a manifold is a topological space that locally resembles Euclidean space near each point. Defining the best metric for these manifolds have several engineering and science implications from controls to optimization for generalized inner product applications of Gram Matrices that appear in these applications. These smooth geometric manifold applications can be formalized by Lie Groups and their Lie Algebras on its infinitesimal elements. Nilpotent matrices that are matrices with zero power with left-invariant metric on Lie group with non-commutative properties namely non-abelian nilsoliton metric Lie algebras will be the focus of this article. In this study, we present an algorithm to classify eigenvalues of nilsoliton derivations for 9-D non-abelian nilsoliton metric Lie algebras with non-singular Gram matrices.

Keywords: *nilsoliton metrics, nilradical, solvable lie algebra*

Introduction

In analyzing and solving differential equations, Lie group techniques provide a useful and significant tool. Since they are not based on linear operators, superposition, or any other aspects of linear solution techniques, group-theoretical methods provide a powerful tool for analyzing non-linear differential models. It is well known that the notion of conservation law is the mathematical formulation of well known physical laws of conservation of energy, momentum *etc.* Oliver [1]. Therefore for the study of systems of differential equations, the concept of conservation law plays an important role in analysis of basic properties of solutions. It is proven by Emmy Noether that every conservation law of a system which is arising from a variational principle comes from a corresponding symmetry property. For example, for the solutions of associated Euler-Lagrange equations, the conservation of energy is implied by the invariance of variational principle under a group of time translations whereas conservation of momentum is implied by invariance under a group of spatial translations. This basic principle constitutes the first fundamental result in the study of classical old quantum –mechanical systems with prescribed groups of symmetries. Moreover, Noether's method is the only really systematic procedure for constructing conservation laws for complicated systems of partial differential equations Oliver [1].

In mathematical point of view, classifications of Lie algebras are of great importance. As the computational methods became prevalent, the symbolic computation gave

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hand to make it easier to classify more accurately and easily. Namely symbolic computation is an area of mathematics which deals with developing, executing and applying the algorithms to manipulate and analyze the mathematical expressions or other mathematical objects. It concerns with the formulation of algorithms to find mostly exact solutions of symbolic mathematical problems and also concerns with the implementation of these algorithms in terms of the operations and control structures available in computer algebra programming languages such as MATLAB, GAP, FORTRAN *etc.* It is a useful tool since it does the computations more productively and accurately than doing by hand, or does the computations that are almost impossible to carry out by hand.

Recently, symbolic computation methods have been used for Lie algebra theory which is mostly result highly complex symbolic expressions that are very difficult to carry out without the aid of computer algorithms. But introduction and implementation of new algorithms make it possible to work with Lie algebras that are too big to deal with by hand. These days, most of the fundamental objects occurring structure theory of Lie algebras can be constructed by computer algorithms. For example, for a given algebra one can develop algorithms for constructing quotient algebras, its centralizer, normalizer or minimal matrix representations *etc.* Ceballos *et al.* [2], Beck *et al.* [3], De Graaf [4, 5], and Ronyai [6].

There are three methods to represent a Lie algebra and its related structures: Representing a Lie algebra as a linear Lie algebra, *i.e.* subalgebra of $gl(n)$, using table of its structure constants or using generators and relations De Graaf [4]. In this paper, we use the table of structure constants related to a Lie algebra. We vary the Lie algebra structure by finding structure constants. Namely we determine a Lie algebra η with a fixed basis $\{X_i : 1 \leq i \leq n\}$ explicitly by given multiplication table, consisting of structure constants α_{ij}^{k} which are defined by the relations:

$$[X_i, X_j] = \sum \alpha_{ij}^{k} X_k \tag{1}$$

It is possible to define several different Riemannian metrics on Lie groups. Considering any Riemannian metrics, Einstein metrics are the most preferable metric, as the Ricci tensor complies the Einstein metric: $Ric = cg$ for some constant $c \in \mathbb{R}$. But, it is not possible to define Einstein metrics on non-abelian nilpotent Lie algebras, therefore we consider the following weaker condition on a left invariant metric g on a nilpotent Lie group G:

$$Ric_g = \beta I + D \tag{2}$$

for some $\beta \in \mathbb{R}$ and $D \in Der(\eta)$, where Ric_g denotes the Ricci operator of (η, g), η is the Lie algebra of G and $Der(\eta)$ denotes the Lie algebra of derivations of η. Equation (2) is called nilsoliton condition, D is the called nilsoliton derivation, and β is the called nilsoliton constant.

Nilsolitons are an important topic in mathematics for several reasons. First, nilsoliton metric Lie algebras are unique up to isometry and scaling. In Lauret [7], *Theorem 2.11* states that a nilpotent Lie algebra η is an Einstein nilradical if and only if η admits a nilsoliton metric. Therefore it indicates that classification of nilsoliton metrics on a nilpotent Lie algebra is equivalent to the same of Einstein nilradicals. On the other hand, an Einstein solvmanifold δ can completely be determined by the Lie algebra $\eta = [\delta, \delta]$. Therefore the study of solvmanifolds are actually the study of nilsolitons. See Lauret [7] and Lauret [8] for a survey on nilsoliton metric Lie algebras.

Nilsoliton metrics are classified in different dimensions by several approaches, Arroyo [9], Culma [10], De Graaf [11], Lauret [9, 12], Payne [13], Will [14], and Nikolayevski

[15]. In dimension 7 and 8 nilsolitons with non-singular Gram matrix are classified in Kadioglu [16] and also dimension 7 in Culma [10, 17]. Also in Kadioglu [18] the ordered type of nilsoliton metrics in higher dimensions were classified. Nilsoliton metrix with the corresponding Gram matrix being singular in dimension 8 is classified in Kadioglu [19]. In this paper, we classify eigenvalues of the nilsoliton derivations of 9-D nilsoliton metric Lie algebras with simple nilsoliton derivation *D* corresponding to a non-singular Gram matrix. In this paper we are studying indecomposable nilsolitons. The reason behind this is that any nilpotent Lie algebra of dimension less than or equal to 6 is an Einstein nil-radical, and that all nilsoliton metric Lie algebras are classified in dimension 7. Also, *Theorem 7* in Nikolayevski [20] states that direct sums of nilradicals is also a nilradical. Therefore one obtains that any decomposable 9-D nilpotent Lie algebra is an Einstein nilradical and it is easy to give a nilsoliton metric in each case.

This paper can be considered as a continuation paper to our last paper. In our last paper, we defined an algorithm which prunes algebras with non-simple derivation and with non-singular Gram matrix Kadioglu [21]. In dimension 9, the classifications of such nilsoliton metric Lie algebras corresponding to a non- singular Gram matrix can be found in the following theorem:

Theorem 1. Let (η, Q) be a 9-D nilsoliton metric Lie algebra with nilsoliton derivation D having distinct positive eigenvalues. Suppose that the canonical Gram matrix U is non-singular. Then the eigenvalues of the derivations of each nilsoliton is one of the 490 eigenvalue 9-tuples given in tabs. 1-10.

Proof. The proof of *Theorem 1* relies on a computational procedure implemented in the computer algebra system MATLAB.

The computational procedure was implemented using MATLAB R2022b with symbolic computation package on Intel(R)Core(TM) i3-5015U CPU at 2.10 GHz processor and 4 GB of RAM.

Preliminaries

In order to use computer algebra systems, we encode non-zero structure constants by using index set given by the following definition.

Definition 2. Let η be a Lie algebra with a fixed basis $\{X_i : 1 \leq i \leq n\}$ such that α_{ij}^k are the structure constants which are defined by the relations:

$$[X_i, X_j] = \sum \alpha_{ij}^k X_k \tag{3}$$

The set of all triples (i, j, k) indexing the non zero structure constants of η is defined by:

$$\Lambda = \left\{(i,j,k) \middle| \alpha_{ij}^k \neq 0, \quad i < j < k\right\}$$

where Λ is called the index set corresponding to the Lie algebra η. The triples $(i, j, k) \in \Lambda$ such that $i < j < k$, and if $(i, j, k), (i, j, m) \in \Lambda$ then $k = m$ and $(i, j_1, k), (i, j_2, k) \in \Lambda$ then $j_1 = j_2$. Using the triples described previously, we fix basis $\{X_1, ..., X_n\}$ for a nilpotent Lie algebra η with $[X_i, X_j] = \Sigma\, \alpha_{ij}^k X_k \neq 0$, such that for every i, j, $\#\{j : \alpha_{ij}^k \neq 0\} \leq 1$, and for every i, k, $\#\{k : \alpha_{ij}^k \neq 0\} \leq 1$. Such basis $\{X_j\}$ is called nice basis (Nikolayevski [15]). Throughout this paper, we call such basis as Nikaloyevski basis.

Definition 3. Let ($\eta\mu$, Q) be a metric algebra, where $\mu \in \Lambda^2\eta \otimes \eta^*$. Let $B = \{X_i\}_{i=1}^n$ be a Q-orthonormal (ordered) basis of $\eta\mu$. The nil-Ricci endomorphism Ric_μ is defined as $< Ric_\mu X, Y >= ric_\mu(X, Y)$, where:

$$ric_\mu(X,Y) = -\frac{1}{2}\sum_{i=1}^{n} <[X,X_i],[Y,X_i]> + \frac{1}{4}\sum_{i=1}^{n} <[X_i,X_j],X><[X_i,X_j],Y>, \quad \text{for } X,Y \in \eta \quad (4)$$

Remark 4. In this paper, we denote an inner product $Q(.,.)$ as $< ., . >$. When η is a nilpotent Lie algebra, the nil-Ricci endomorphism is the Ricci endomorphism. If all elements of the basis are eigenvectors for the nil-Ricci endomorphism Ric_μ, we call the orthonormal basis a Ricci eigenvector basis.

Now we define some combinatorial objects related to a given index triples $(i, j, k) \in \Lambda$, where Λ is the subset of $\{(i, j, k)|1 \le i < j < k \le n\}$.

Definition 5. Let Λ be a subset of the finite set $\{(i, j, k)|1 \le i < j < k \le n\}$ which indexes the set of non-zero structure constants corresponding to a Lie algebra η (which ignores repetitions due to skew-symmetry). Then:

– For $1 \le i, j, k \le n$, define $1 \times n$ row vector $y_{i,j}^k$ to be $\in_i^T + \in_j^T - \in_k^t$ where $\{\in\}_{i=1}^n$ is the standard orthonormal basis for $\mathbb{R}^n$. We call the vectors in $\{y_{ij}^k \,|\, (i,j,k) \in \Lambda\}$ root vectors for Λ.
– Let $y_1, y_2, ..., y_m$ (where $m = |\Lambda|$) be an enumeration of the root vectors in dictionary order. We define root matrix Y_Λ for Λ to be the $m \times n$ matrix whose rows are the root vectors $y_1, y_2, ..., y_m$.
– The Gram matrix U_Λ for Λ is the $m \times m$ matrix defined by $U_\Lambda = Y_\Lambda Y_\Lambda^T$; the (i, j) entry of U_Λ is the inner product of the i^{th} and j^{th} root vectors.

In the following, we define root vector, root matrix and Gram matrix corresponding to a Lie algebra.

Remark 6. By definition (i, j) entry of U_Λ is the inner product of the i^{th} and j^{th} root vectors. From *Theorem 5* in Payne [22] we know that U is a symmetric and semi-definite matrix where its all diagonal entries are 3 and its off-diagonal entries are in the set $\{-2, -1, 0, 1, 2\}$. Also, the rank of the Gram matrix U_Λ is the same as the rank of the root matrix Y_Λ. Nikolayevski showed that every Lie algebra admitting a derivation with all the eigenvalues of multiplicity one has a nice basis Nikolayevski [15], we use this type of basis in our classifications. Since we are classifying nilsolitons with simple Nilsoliton derivation, our Gram matrices corresponding to metric nilpotent Lie algebras does not have a 2 as an entry (*Lemma 2* in Kadioglu [18]).

Now, suppose that $|\Lambda| = m$ and $[1]_m$ represents a column vector $[1\ 1\ 1\ ..\ 1]_T$ in $\mathbb{R}_m$.

Lemma 7. (*Lemma 2.10* in Kadioglu [18]). Let η be a non-abelian nilpotent Lie algebra where η admits a derivation with distinct positive eigenvalues. Let $B = \{X_i\}_{i=1}^n$ be a basis for η. Let Λ indexes the non-zero structure constants with respect to B, and let U_Λ be the Gram matrix associated to Λ. For $s = 1, ..., n$, let $B_s = \{X_i\}_{i=s}^n$ and define the subspace η_s of η by $\eta_s = span\{X_i\}_{i=s}^n$. Let $\Lambda_s = \{(i, j, k) \in \Lambda : s \le i, j, k\}$ be the subset of Λ_s consisting of triples with entries from the set $\{s, s + 1, ..., n\}$, and let U_s be the Gram matrix for Λ_s. Then η_s is an ideal.

Corollary 8. Let η be a non-abelian nilpotent Lie algebra that admits a derivation with distinct positive eigenvalues. Let $B = \{X_i\}_{i=1}^n$, Λ, U_Λ be the structural elements for η, and $B_s = \{X_i\}_{i=s}^n$, Λ_s, and U_s defined as in *Lemma 7* all $s = 1, ...n$ for η_s. Then if the Gram matrix

U_Λ does not have any entry in the set $\{-1, 2\}$ and η_s is non-abelian, then the Gram matrix U_s for Λ_s does not have any enree from the set $\{-1, 2\}$.

Proof. From *Lemma 7*, η_s is an ideal. Therefore if η_s is non-abelian, then U_s is a minor of U_Λ. Therefore if U_Λ does not have any enree from $\{-1, 2\}$, then so is U_s.

Theorem 9. (*Theorem 1* in Payne [22]) Let η be a non-abelian metric algebra with Ricci eigenvector basis B. Let U and $[\alpha^2]$ be the Gram matrix and the structure vector for η with respect to B. Then η satisfies the nilsoliton condition with nilsoliton constant β if and only if $U[\alpha_2] = -2\beta[1]_m$.

Remark 10. Above theorem indicates a Lie algebra η admits a nilsoliton metric if there exists a solution $v \in \mathbb{R}_m$ of the linear system $U\Lambda v = [1]_m$ where all entries are positive real numbers.

Theorem 11. (Payne [22]) Let (η, Q) be a metric algebra and $B=\{X_i\}_{i=1}^n$ be a Ricci eigenvector basis for η. Let Y be the root matrix for η. Then the eigenvalues of the nil-Ricci endomorphism are given by:

$$Ric^B = -\frac{1}{2} Y^T v \tag{5}$$

where $v = [\alpha^2]$.

Theorem 12. (Payne [22], Kadioglu [18]) Let (η, Q) be a non-abelian metric algebra with Ricci eigen-vector basis B. The following are equivalent:

1. (η, Q) satisfies the nilsoliton condition with nilsoliton constant β.
2. The eigenvalue vector V_D for $D = Ric - \beta Id$ with respect to B lies in the kernel of the root matrix for (η, Q) with respect to B.
3. For non-commuting eigenvectors X and Y for the nil-Ricci endomorphism with eigenvalues κ_X and κ_Y, the bracket $[X, Y]$ is an eigenvector for the nil-Ricci endomorphism with eigenvalue $\kappa_X + \kappa_Y - \beta$.
4. $$\beta = y_{ij}^k Ric \qquad \forall (i,j,k) \in \Lambda(\eta, B) \tag{6}$$

Theorem 13. (Payne [22]) Let η be an n-D vector space, $B=\{X_i\}_{i=1}^n$ a basis for η. Suppose that a set of non-zero structure constants $\alpha_{i,j}^k$ relative to B, indexed by Λ, defines a skew symmetric product on η. Assume that if $(i, j, k) \in \Lambda$, then $i < j < k$. Then η is a Lie algebra if and only if whenever there exists m so that the inner product of root vectors $< y_l, y_m >= -1$ for triples (i, j, l) and (l, k, m), or (k, l, m) in Λ, the equation:

$$\sum_{s<m} \alpha_{i,j}^s \alpha_{s,k}^m + \alpha_{j,k}^s \alpha_{s,i}^m + \alpha_{k,i}^s \alpha_{s,j}^m = 0 \tag{7}$$

holds. Furthermore, a term of form $\alpha_{i,j}^i \alpha_{i,k}^m$ is non-zero if and only if $< y_{i,j}^i y_{i,k}^m >= -1$.

Theorem 14. (see Kadioglu [18]) Let η be an n-D non-abelian nilpotent Lie algebra which admits a derivation D having distinct real positive eigenvalues, B a basis consisting of eigenvectors for the derivation D, Λ indexes the non-zero structure constants with respect to B, and U be the $m \times m$ Gram matrix. If U is invertible, then the following hold:

$$|\Lambda| \le n-1$$

$$\text{If } (i_1, j_1, k_1) \in \Lambda \quad \text{and} \quad (i_2, j_2, k_2) \in \Lambda, \quad \text{then} \quad < y_{i_1,j_1}^{k_1}, y_{i_2,j_2}^{k_2} > \neq -1$$

Remark 15. It follows from *Theorem 14* that if η be an n-D non-abelian nilpotent Lie algebra which admits a derivation D having distinct real positive eigenvalues with non-singular Gram matrix, then it does not have any –1 entry in its Gram matrix. Then from *Theorem 13,* since it does not have any –1 entry, then there will be no $\alpha_{i,j}^{l}\alpha_{l,k}^{m}$ term to appear, which means that the Jacobi identity is automatically satisfied. Therefore, we do not need to consider the Jacobi identity while classifying nilsolitons with non-singular Gram matrix.

Definition 16. Let η be a nilpotent Lie algebra and $Der(\eta)$ denote the derivation algebra of η. A maximal abelian subalgebra of $Der(\eta)$ comprised of semisimple elements is called "a maximal torus". The dimension of a maximal torus is called the "rank of η".

Corollary 17. (*Corollary 1* in Kadioglu [23]) Let η be a non-abelian n-D Lie algebra that admits a simple derivation. Let B be an eigenvector basis with index set Λ, and let Y be the root matrix associated to Λ. Then:

$$rank(\eta) = n + nullity(YY^t) - |\Lambda| \tag{8}$$

Remark 18. Let η be a non-abelian n-D Lie algebra that admits a simple derivation. Let B be an eigenvector basis with index set Λ. If the corresponding Gram matrix U_Λ of η is non-singular then the rank of η is:

$$rank(\eta) = n - |\Lambda| \tag{9}$$

Proof. Let η be a non-abelian n-D Lie algebra that admits a simple derivation. Let B be an eigenvector basis with index set Λ. If the corresponding Gram matrix U_Λ is non-singular, then its nullity is zero. Using *Corollary 17,* we have rank of η is $n - |\Lambda|$.

Since we are classifiying non-singular nilsolitons of dimension 9, eq. (9) from previous remark implies that the rank of a non-abelian 9-D nilsoliton η is $rank(\eta) = 9 - |\Lambda|$.

In the next subsection, we provided all fundamental definitions and theorems regarding to our structural elements, we talk about the algorithms regarding to the computations of structural elements of a Lie algebra.

The algorithm

In this section, we present algorithms regarding to creating all possible index subsets corresponding nilsolitons with non-singular Gram matrix, rank and index of a nilsolitons in dimension 9.

Now we present computer algorithm for the classifications. Before doing that, we give some fundamental definitions/properties we use in our algorithm:

– *Z Matrix:* We define the index set $\Lambda = \{(i,j,k) : \alpha_{ij}^{k} \neq 0, i < j < k\}$ is the set of all index triples such that $i < j < k$ where $1 \leq i, j, k \leq 9$. All index triples are encoded as rows of a matrix Z listed in dictionary order. We note that the kardinality of Λ is $\binom{9}{3}$. Therefore Z matrix is of type 84×3.
– *W matrix:* In order to list all subsets of the index set Λ, we create a row vector W_A, which is a 0-1 matrix where 1 stands for the index triple belongs to $A \in \Lambda$. All of these row vectors create W matrix, such that each row corresponds to a different subset of Λ. Therefore, W is a $2^{84} \times 84$ type of matrix. Since the matrix is too big to contain in the memory of a usual computer, we need to create the matrix by merging submatrices. To merge those submatrices, we use an algorithm for a function mergefn defined in *Algorithm 1.*
– *Submatrices:* We define 6 disjoint subsets of Λ, each having 14 elements respectively, namely $Z_1, ..., Z_6$.
– *Badpairs:* We say that two elements (i_1, j_1, k_1) and (i_2, j_2, k_2) of the sub index set are a bad pair if the inner product $< y_{i_1 j_1}^{k_1}, y_{i_2 j_2}^{k_2} > \in \{-1, 2\}$. We create these pair to delete correspond-

ing rows in 1/0 matrix. This follows from the *Remark 6* and *Theorem 14.* Also, from *Corollary 8,* the Gram matrix corresponding to each subset of the index set Λ should not contain any badpairs. Therefore we create and delete badpairs for the subsets of Λ in *Algorithm 2* and *Algorithm 3*.

– *Eigenvalues of Derivation:* We compute eigenvalues of the derivation by using the equations 5 and 6. Since we are classifying nilsolitons with simple derivations, the eigenvalues of the derivation of each nilsoliton has to be distinct.

The following algorithm of the function Mergefn helps us to merge the columns of two matrices of a 1/0 matrix.

Algorithm 1 Algorithm of the Function Mergefn

Input: W_1, W_2; matrices consisting of 1/0's where r and s number of columns appear in W_1 and W_2, respectively.
Output: 1/0 matrix W12 with (s1+s2) columns, where first s1 columns are coming from the W1 and the rest comes from the columns of W2 on its each row.
1: Define W1o and W12 as the empty matrices
2: Define r1 and r2 as the number of rows appear on W_1 and W_2, respectively
3: Define s1 and s2 as the number of columns appear on W_1 and W_2, respectively
4: Create W1 and W2 by adding a zero row to each of W_1, W_2, respectively.
5: **for** j=1 to r2+1 **do**
6: Define Wo as $(r1 + 1) \times s2$ matrix, such that all of rows of Wo is *jth* row of W2.
7: Define $W1o = [W1\ Wo]$
8: Define $W12 = [W12\ W1o]$
9: Define W1o as empty matrix
10: **end for**

To show the previous algorithm, we have following example:

Example 19.

$$\text{Let } W_1 = \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad W_2 = (1 \quad 0 \quad 0 \quad 0 \quad 0 \quad 1)$$

Then, $r1 = 2$ and $r2 = 1$ whereas $s1 = s2 = 6$. In the fourth line of the algorithm, we create $W1$ and $W2$ by adding zero rows to W_1, W_2, respectively. Therefore:

$$W1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad W2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Then:

$$W12 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Main algorithm

Now, we present our main algorithm to create a 1/0 matrix W, such that each row of the matrix W represents possible characteristic vectors for canonical index sets Λ for nilpotent Lie algebras of dimension *n* with non-singular Gram matrix.

Algorithm 2 Computing possible subsets of Λ

Input: Integer n which represents the dimension of the Lie algebra.
Output A Matrix Wallpossible consisting of only ones and 0 s as entrys.
1: Enter the dimension:
2: Compute the index set Λ ⊳ Each triple is written as rows of a matrix Z.
3: Compute root matrix Y_Λ
4: $U_\Lambda = Y_\Lambda * Y_\Lambda^T$ ⊳ U_Λ is the Gram matrix
5: Compute Badpairs
6: Compute each Z_i matrix for $i = 1, ..., 6$.
7: Compute W_i matrix for $i = 1, ..., 6$.
8: Compute $(Badpairs)_i$ for each subset of Λ
9: Delete the rows of W_i contain $(Badpairs)_i$
10: **function** MERGEFN(W_1, W_2) ⊳ Create W12 by merging W_1 and W_2
11: Define a new matrix $W12$ as by adding columns of W2 to W1.
12: **end function**
13: $Z12 = \begin{bmatrix} Z_1 \\ Z_2 \end{bmatrix}$
14: Create $(Badpairs)_{12}$ for Z12
15: Delete the rows of $W12$ contain $(Badpairs)_{12}$
16: **function** MERGEFN($W12$, W_3) ⊳ Create W123 by merging W_{12} and W_3
17: Create W21 and W3 by adding a zero row to each of $W12$, W_3, respectively.
18: Define a new matrix $W123$ as by adding columns of W_3 to W12.
19: **end function**
20: Create Z123 same as line 13
21: Create $(Badpairs)_{123}$ same as line 14
22: Define Whalf1 by deleting the rows of $W123$ containing $(Badpairs)_{123}$
23: Create W456 by merging W_4, W_5 and W_6 same as line 16
24: Create Z456 same as line 13
25: Create $(Badpairs)_{456}$ same as line 14
26: Define Whalf2 by deleting rows of $W456$ containing $(Badpairs)_{456}$
27: Define rofWhalf2 as the number of rows of Whalf2

So far, we have created two matrices Whalf1 and Whalf2 both having 42 columns representing the first 42 rows of *Z* matrix and the second 42 rows of *Z* matrix respectively. Whalf1 and Whalf2 also does not contain any badpairs.

Algorithm 3 Computing possible subsets of Λ (continued)

```
Define Wpossible as the empty matrix
for i=1 to rofWhalf2 do
    Define Whalf2i as the i^th row of Whalf2        ⊳ We create Wpossible matrix
    W0=mergefn(Whalf1, Whalf2i)
    Delete the rows of W0 containing Badpairs
end for
```

After running previous algoritm, we have a 1/0 matrix *W possible* that has 84 columns, and is indexing each possible nilsoliton in its row. We note that we have eliminated Badpairs for created for the Λ. Thus, non of the rows in *W possible* corresponds to a Gram matrix with –1 or 2 as an entry.

Algorithm 4 Computing possible subsets of Λ (continued)

```
Delete all rows of Wpossible containing more than 8 number of 1s.        ⊳ Follows from the theorem 14
Delete the rows that corresponds singular Gram matrix
Delete the rows that corresponds to decomposable Lie algebras
Delete the rows that corresponds to Lie algebras with the nilsoliton derivation being non-positive or non-distinct eigenvalues
Delete the rows that corresponds to algebras with non-positive solutions to UΛv = [1]  ⊳ Fol lows from Remark 10
Remaining matrix is Wallpossible matrix.
```

After running the previous three algorithms, we have a 0/1 matrix Wallpossible (with 84 columns) where each row corresponds to a nilsoliton with simple derivation. If there is 1 in j^{th} column, then the corresponding nilsoliton's index set Λ_η includes j^{th} row of Z matrix. Also corresponding Gram matrix U_Λ is non-singular. Before classifying all possible derivations, we first create Rank1W, Rank2W, Rank3W and Rank4W matrices from the rows of Wallpossible. Then we compute all possible eigenvalues of derivations with respect to the ranks of each nilsoliton using *Theorems 11* and *12*.

Classifications

Remark 20. After running previous algorithms, we have following findings on the nilsolitons with nonsingular Gram matrix:

- In dimension 9, there is no nilsolitons of Ordered type which is defined as the nilsolitons with derivation type $1 < 2 < ... < 8 < 9$.
- Rank of each nilsoliton in dimension 9 ranges between 1 to 4.
- Each of Rank 1 nilsolitons in dimension 9 has one of the 58 eigenvalue set of derivations given in tab. 1.
- Each of Rank 2 nilsolitons in dimension 9 has one of the 271 eigenvalue set of derivations given in tabs. 2-5.
- Each of Rank 3 nilsolitons in dimension 9 has one of the 191 eigenvalue set of derivations given in tabs. 6-9.

– Each of Rank 4 nilsolitons in dimension 9 has one of the 24 eigenvalue set of derivations given in tab 10.

Table 1. Eigenvalues of derivations for Rank 1 Nilsolitons in dimension 9

	Derivation
1	522/6829 < 820/4291 < 207/677 < 4904/12831 < 1941/4232 < 635/1278 < 648/1211 < 647/1058 < 893/1298
2	199/2603 < 271/1418 < 196/641 < 1191/3116 < 749/1633 < 399/803 < 602/1125 < 1557/2546 < 2591/3766
3	442/5647 < 98/313 < 2481/7924 < 1461/3733 < 445/1137 < 147/313 < 343/626 < 196/313 < 851/1208
4	49/626 < 459/1466 < 552/1763 < 699/1786 < 1144/2923 < 472/1005 < 1229/2243 < 521/832 < 2391/3394
5	49/626 < 325/1038 < 552/1763 < 699/1786 < 427/1091 < 472/1005 < 1229/2243 < 521/832 < 441/626
6	169/2159 < 98/313 < 356/1137 < 227/580 < 445/1137 < 503/1071 < 463/845 < 325/519 < 739/1049
7	195/2491 < 1037/3312 < 289/923 < 427/1091 < 209/534 < 565/1203 < 663/1210 < 583/931 < 267/379
8	195/2491 < 289/923 < 289/923 < 427/1091 < 591/1510 < 565/1203 < 663/1210 < 583/931 < 739/1049
9	28/283 < 112/849 < 84/283 < 112/283 < 364/849 < 140/283 < 476/849 < 168/283 < 196/283
10	470/4621 < 609 < 424/1191 < 4043/9937 < 433/946 < 2017/3966 < 433/774 < 83/136 < 679/1027
11	470/4621 < 282/1109 < 424/1191 < 474/1165 < 877/1916 < 593/1166 < 753/1346 < 1209/1981 < 1159/1753
12	470/4621 < 608/2391 < 403/1132 < 474/1165 < 877/1916 < 593/1166 < 753/1346 < 83/136 < 1159/1753
13	338/3323 < 2062/8109 < 157/441 < 308/757 < 406/887 < 415/816 < 1153/2061 < 794/1301 < 1959/2963
14	136/1337 < 429/1687 < 633/1778 < 1621/3984 < 287/627 < 918/1805 < 767/1371 < 841/1378 < 921/1393
15	773/7599 < 533/2096 < 633/1778 < 1621/3984 < 1711/3738 < 918/1805 < 767/1371 < 1563/2561 < 921/1393
16	773/7599 < 429/1687 < 3515/9873 < 1621/3984 < 1711/3738 < 918/1805 < 767/1371 < 841/1378 < 921/1393
17	63/598 < 42/299 < 84/299 < 231/598 < 126/299 < 147/299 < 168/299 < 357/598 < 210/299
18	35/312 < 175/624 < 35/104 < 245/624 < 35/78 < 105/208 < 175/312 < 385/624 < 35/52
19	431/3293 < 1142/5235 < 528/2017 < 439/1118 < 387/887 < 562/1171 < 100/191 < 2964/4529 < 437/626
20	631/4821 < 1142/5235 < 439/1677 < 889/2264 < 798/1829 < 562/1171 < 1411/2695 < 4036/6167 < 985/1411
21	1114/8511 < 137/628 < 2761/10547 < 1436/3657 < 709/1625 < 921/1919 < 511/976 < 786/1201 < 1059/1517
22	761/5814 < 673/3085 < 661/2525 < 686/1747 < 435/997 < 610/1271 < 411/785 < 536/819 < 622/891
23	761/5814 < 435/1994 < 661/2525 < 686/1747 < 435/997 < 945/1969 < 411/785 < 536/819 < 1133/1623
24	486/3713 < 137/628 < 561/2143 < 611/1556 < 1031/2363 < 921/1919 < 722/1379 < 536/819 < 622/891
25	397/3033 < 1180/5409 < 311/1188 < 386/983 < 459/1052 < 634/1321 < 833/1591 < 411/628 < 696/997
26	397/3033 < 1276/5849 < 311/1188 < 386/983 < 459/1052 < 658/1371 < 833/1591 < 411/628 < 622/891
27	391/2901 < 355/1756 < 551/1817 < 653/1938 < 827/1888 < 609/1291 < 1027/1905 < 803/1324 < 659/889
28	391/2901 < 355/1756 < 1213/4000 < 653/1938 < 668/1525 < 609/1291 < 1027/1905 < 803/1324 < 255/344
29	31/230 < 1531/7573 < 457/1507 < 373/1107 < 205/468 < 292/619 < 448/831 < 1643/2709 < 1169/1577
30	32/203 < 40/203 < 8/29 < 72/203 < 88/203 < 96/203 < 16/29 < 120/203 < 152/203

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31	519/3289 < 89/376 < 520/1883 < 518/1313 < 2908/6701 < 383/809 < 95/172 < 469/743 < 1073/1511
32	519/3289 < 463/1956 < 351/1271 < 821/2081 < 1022/2355 < 383/809 < 628/1137 < 291/461 < 757/1066
33	982/6223 < 819/3460 < 161/583 < 1195/3029 < 1413/3256 < 677/1430 < 1103/1997 < 2179/3452 < 659/928
34	982/6223 < 757/3198 < 1549/5609 < 245/621 < 746/1719 < 1283/2710 < 1103/1997 < 481/762 < 1835/2584
35	98/621 < 303/1280 < 277/1003 < 765/1939 < 424/977 < 499/1054 < 401/726 < 1402/2221 < 3991/5620
36	98/621 < 775/3274 < 219/793 < 447/1133 < 585/1348 < 499/1054 < 591/1070 < 291/461 < 1296/1825
37	98/621 < 303/1280 < 277/1003 < 765/1939 < 424/977 < 1123/2372 < 591/1070 < 493/781 < 3849/5420
38	98/621 < 370/1563 < 575/2082 < 419/1062 < 332/765 < 1123/2372 < 401/726 < 1628/2579 < 1301/1832
39	517/3276 < 419/1770 < 182/659 < 1459/3698 < 687/1583 < 508/1073 < 153/277 < 2537/4019 < 5564/7835
40	517/3276 < 419/1770 < 182/659 < 1459/3698 < 687/1583 < 312/659 < 401/726 < 998/1581 < 1448/2039
41	517/3276 < 401/1694 < 401/1452 < 1472/3731 < 447/1030 < 508/1073 < 401/726 < 885/1402 < 3011/4240
42	111/698 < 185/698 < 111/349 < 259/698 < 148/349 < 333/698 < 407/698 < 222/349 < 481/698
43	105/634 < 70/317 < 175/634 < 245/634 < 140/317 < 315/634 < 175/317 < 385/634 < 455/634
44	14/81 < 7/27 < 49/162 < 7/18 < 35/81 < 77/162 < 91/162 < 35/54 < 56/81
45	86/479 < 215/958 < 136/505 < 172/479 < 487/1085 < 473/958 < 258/479 < 301/479 < 423/589
46	86/479 < 215/958 < 129/479 < 172/479 < 215/479 < 473/958 < 258/479 < 301/479 < 344/479
47	481/2679 < 305/1359 < 129/479 < 581/1618 < 215/479 < 355/719 < 2015/3741 < 717/1141 < 581/809
48	63/334 < 42/167 < 105/334 < 63/167 < 147/334 < 84/167 < 189/334 < 105/167 < 231/334
49	264/1325 < 653/2622 < 2317/7753 < 614/1761 < 724/1615 < 2490/4999 < 223/407 < 485/749 < 288/413
50	1324/6645 < 589/2365 < 965/3229 < 1387/3978 < 659/1470 < 656/1317 < 932/1701 < 643/993 < 1205/1728
51	1324/6645 < 197/791 < 497/1663 < 1387/3978 < 581/1296 < 920/1847 < 1475/2692 < 1117/1725 < 341/489
52	585/2936 < 197/791 < 497/1663 < 341/978 < 581/1296 < 920/1847 < 263/480 < 1038/1603 < 341/489
53	585/2936 < 659/2646 < 341/1141 < 341/978 < 516/1151 < 395/793 < 343/626 < 3092/4775 < 394/565
54	373/1872 < 66/265 < 893/2988 < 197/565 < 490/1093 < 527/1058 < 726/1325 < 79/122 < 1235/1771
55	698/3503 < 397/1594 < 500/1673 < 356/1021 < 915/2041 < 1583/3178 < 343/626 < 2596/4009 < 447/641
56	698/3503 < 332/1333 < 396/1325 < 356/1021 < 412/919 < 925/1857 < 503/918 < 1016/1569 < 553/793
57	215/1079 < 331/1329 < 237/793 < 409/1173 < 451/1006 < 3431/6888 < 1069/1951 < 1964/3033 < 947/1358
58	215/1079 < 598/2401 < 766/2563 < 409/1173 < 412/919 < 661/1327 < 503/918 < 858/1325 < 553/793

Table 2. Eigenvalues of derivations for Rank 2 Nilsolitons in dimension 9 (Continues)

	Derivation
1	1/86 < 79/172 < 81/172 < 83/172 < 85/172 < 87/172 < 89/172 < 91/172 < 93/172
2	13/194 < 32/97 < 32/97 < 77/194 < 77/194 < 45/97 < 103/194 < 58/97 < 141/194
3	7/99 < 35/198 < 32/99 < 13/33 < 46/99 < 1/2 < 53/99 < 20/33 < 67/99
4	8/109 < 32/109 < 40/109 < 48/109 < 101/218 < 56/109 < 117/218 < 64/109 < 72/109

5	14/177 < 47/177 < 61/177 < 70/177 < 25/59 < 89/177 < 103/177 < 36/59 < 39/59
6	5/63 < 4/21 < 19/63 < 8/21 < 29/63 < 31/63 < 34/63 < 13/21 < 43/63
7	13/162 < 5/27 < 26/81 < 10/27 < 73/162 < 41/81 < 43/81 < 11/18 < 56/81
8	23/282 < 29/94 < 46/141 < 35/94 < 55/141 < 133/282 < 26/47 < 179/282 < 197/282
9	7/82 < 14/123 < 27/82 < 17/41 < 109/246 < 1/2 < 137/246 < 24/41 < 55/82
10	19/208 < 121/416 < 159/416 < 83/208 < 197/416 < 51/104 < 235/416 < 121/208 < 35/52
11	26/281 < 164/843 < 78/281 < 104/281 < 130/281 < 398/843 < 156/281 < 182/281 < 2/3
12	13/138 < 13/46 < 26/69 < 28/69 < 65/138 < 1/2 < 13/23 < 41/69 < 91/138
13	28/283 < 84/283 < 169/566 < 223/566 < 112/283 < 140/283 < 168/283 < 337/566 < 196/283
14	35/342 < 259/1026 < 182/513 < 70/171 < 469/1026 < 175/342 < 287/513< 623/1026 < 679/1026
15	11/107 < 62/321 < 33/107 < 1/3 < 44/107 < 55/107 < 169/321 < 66/107 < 77/107
16	19/184 < 57/184 < 127/368 < 19/46 < 165/368 < 95/184 < 203/368 < 57/92 < 241/368
17	11/106 < 33/106 < 17/53 < 43/106 < 22/53 < 45/106 < 28/53 < 67/106 < 77/106
18	16/151 < 40/151 < 44/151 < 56/151 < 60/151 < 72/151 < 88/151 < 96/151 < 104/151
19	1574/14131 < 175/597 < 544/1657 < 147/398 < 161/398 < 1123/2554 < 329/597 < 791/1194 < 833/1194
20	133/1194 < 615/2098 < 196/597 < 711/1925 < 1269/3137 < 1258/2861 < 933/1693 < 475/717 < 1043/1495
21	388/3483 < 124/423 < 349/1063 < 311/842 < 339/838 < 1601/3641 < 701/1272 < 899/1357 < 2153/3086
22	26/219 < 502/2807 < 1683/5656 < 997/2395 < 1021/2150 < 313/657 < 703/1314 < 623/953 < 593/905
23	26/219 < 235/1314 < 391/1314 < 1927/4629 < 851/1792 < 919/1929 < 703/1314 < 674/1031 < 555/847
24	26/219 < 208/1163 < 355/1193 < 1927/4629 < 851/1792 < 313/657 < 703/1314 < 674/1031 < 555/847
25	227/1912 < 722/4037 < 477/1603 < 1385/3327 < 435/916 < 101/212 < 1085/2028 < 1146/1753 < 2163/3301
26	42/353 < 140/1059 < 98/353 < 140/353 < 434/1059 < 182/353 < 574/1059 < 224/353 < 238/353
27	5/42 < 1/7 < 2/7 < 5/14 < 3/7 < 10/21 < 4/7 < 25/42 < 5/7
28	7/58 < 59/232 < 21/58 < 3/8 < 14/29 < 115/232 < 35/58 < 143/232 < 73/116
29	13/106 < 29/106 < 16/53 < 39/106 < 21/53 < 55/106 < 61/106 < 34/53 < 71/106
30	7/57 < 35/114 < 6/19 < 43/114 < 25/57 < 1/2 < 32/57 < 71/114 < 13/19
31	7/57 < 16/57 < 6/19 < 23/57 < 25/57 < 10/19 < 32/57 < 34/57 < 13/19
32	1/8 < 41/144 < 23/72 < 59/144 < 4/9 < 23/48 < 41/72 < 29/48 < 25/36
33	37/294 < 12/49 < 107/294 < 109/294 < 24/49 < 73/147 < 179/294 < 181/294 < 61/98
34	18/143 < 27/143 < 45/143 < 58/143 < 63/143 < 72/143 < 81/143 < 85/143 < 9/13
35	31/246 < 11/82 < 11/41 < 97/246 < 33/82 < 64/123 < 22/41 < 53/82 < 55/82
36	9/71 < 27/142 < 45/142 < 31/71 < 63/142 < 36/71 < 40/71 < 81/142 < 99/142
37	117/898 < 123/449 < 339/898 < 351/898 < 363/898 < 234/449 < 240/449 < 585/898 < 597/898
38	89/681 < 99/454 < 119/454 < 535/1362 < 99/227 < 109/227 < 713/1362 < 297/454 < 317/454

39	73/558 < 56/279 < 151/558 < 112/279 < 43/93 < 263/558 < 33/62 < 185/279 < 125/186
40	329/2474 < 179/673 < 166/579 < 843/2113 < 81/193 < 898/2039 < 308/579 < 897/1349 < 1239/1754
41	943/7091 < 154/579 < 125/436 < 845/2118 < 614/1463 < 85/193 < 1157/2175 < 385/579 < 1624/2299
42	943/7091 < 154/579 < 207/722 < 845/2118 < 844/2011 < 85/193 < 1157/2175 < 385/579 < 575/814
43	77/579 < 541/2034 < 457/1594 < 77/193 < 1748/4165 < 972/2207 < 541/1017 < 514/773 < 1778/2517
44	77/579 < 541/2034 < 289/1008 < 77/193 < 439/1046 < 972/2207 < 541/1017 < 514/773 < 409/579
45	206/1549 < 129/485 < 166/579 < 1696/4251 < 81/193 < 632/1435 < 1057/1987 < 643/967 < 806/1141
46	335/2519 < 179/673 < 166/579 < 1542/3865 < 81/193 < 898/2039 < 741/1393 < 897/1349 < 1239/1754
47	597/4489 < 154/579 < 125/436 < 541/1356 < 614/1463 < 85/193 < 691/1299 < 385/579 < 1624/2299
48	597/4489 < 154/579 < 207/722 < 541/1356 < 844/2011 < 85/193 < 691/1299 < 385/579 < 575/814
49	210/1579 < 129/485 < 166/579 < 543/1361 < 81/193 < 632/1435 < 308/579 < 643/967 < 806/1141
50	53/398 < 59/199 < 65/199 < 159/398 < 171/398 < 183/398 < 112/199 < 118/199 < 289/398
51	2/15 < 3/10 < 1/3 < 2/5 < 13/30 < 7/15 < 8/15 < 3/5 < 11/15
52	11/82 < 8/41 < 11/41 < 33/82 < 19/41 < 39/82 < 22/41 < 27/41 < 55/82
53	9/67 < 35/134 < 18/67 < 53/134 < 27/67 < 71/134 < 36/67 < 44/67 < 45/67
54	29/214 < 26/107 < 81/214 < 87/214 < 52/107 < 55/107 < 58/107 < 133/214 < 139/214
55	3/22 < 3/11 < 4/11 < 9/22 < 9/22 < 1/2 < 6/11 < 7/11 < 15/22
56	75/542 < 74/271 < 149/542 < 111/271 < 223/542 < 112/271 < 149/271 < 371/542 < 373/542
57	125/898 < 119/449 < 339/898 < 351/898 < 363/898 < 232/449 < 238/449 < 589/898 < 601/898
58	8/57 < 16/57 < 35/114 < 7/19 < 15/38 < 8/19 < 32/57 < 77/114 < 40/57
59	8/57 < 16/57 < 6/19 < 22/57 < 8/19 < 10/19 < 32/57 < 34/57 < 40/57
60	8/57 < 29/114 < 16/57 < 15/38 < 8/19 < 17/38 < 32/57 < 37/57 < 40/57

Table 3. Eigenvalues of derivations for Rank 2 Nilsolitons in dimension 9 (Continues)

	Derivation
61	10/71 < 14/71 < 21/71 < 24/71 < 31/71 < 34/71 < 38/71 < 44/71 < 52/71
62	40/283 < 103/566 < 183/566 < 229/566 < 263/566 < 143/283 < 309/566 < 343/566 < 389/566
63	40/283 < 71/283 < 151/566 < 111/283 < 231/566 < 151/283 < 311/566 < 182/283 < 191/283
64	16/113 < 24/113 < 32/113 < 89/226 < 48/113 < 56/113 < 64/113 < 137/226 < 80/113
65	1/7 < 2/7 < 9/28 < 11/28 < 3/7 < 13/28 < 4/7 < 17/28 < 5/7
66	1/7 < 4/21 < 11/42 < 2/7 < 3/7 < 19/42 < 4/7 < 9/14 < 5/7
67	12/83 < 50/249 < 26/83 < 86/249 < 36/83 < 38/83 < 136/249 < 50/83 < 62/83
68	34/233 < 50/233 < 52/233 < 84/233 < 102/233 < 118/233 < 134/233 < 152/233 < 154/233
69	34/233 < 52/233 < 66/233 < 68/233 < 102/233 < 118/233 < 134/233 < 136/233 < 170/233
70	16/109 < 24/109 < 40/109 < 85/218 < 48/109 < 56/109 < 64/109 < 133/218 < 72/109

71	256/1743 < 473/2204 < 251/889 < 886/2451 < 285/664 < 733/1475 < 547/1076 < 937/1430 < 671/943
72	652/4439 < 47/219 < 515/1824 < 475/1314 < 94/219 < 163/328 < 334/657 < 287/438 < 861/1210
73	318/2165 < 426/1985 < 371/1314 < 813/2249 < 843/1964 < 1062/2137 < 941/1851 < 555/847 < 1199/1685
74	719/4895 < 47/219 < 515/1824 < 860/2379 < 373/869 < 327/658 < 789/1552 < 2163/3301 < 375/527
75	4/27 < 23/81 < 47/162 < 35/81 < 71/162 < 4/9 < 31/54 < 47/81 < 59/81
76	101/681 < 91/454 < 111/454 < 535/1362 < 101/227 < 111/227 < 737/1362 < 293/454 < 313/454
77	17/114 < 5/18 < 109/342 < 139/342 < 73/171 < 17/38 < 197/342 < 34/57 < 124/171
78	8/53 < 13/53 < 29/106 < 21/53 < 45/106 < 26/53 < 29/53 < 34/53 < 37/53
79	2575/16974 < 275/1114 < 381/1114 < 401/1114 < 277/695 < 275/557 < 405/736 < 719/1114 < 391/557
80	169/1114 < 569/2305 < 92/269 < 365/1014 < 222/557 < 746/1511 < 1259/2288 < 162/251 < 1816/2587
81	169/1114 < 1278/5177 < 197/576 < 437/1214 < 222/557 < 904/1831 < 1100/1999 < 395/612 < 497/708
82	1043/6875 < 275/1114 < 381/1114 < 401/1114 < 1224/3071 < 275/557 < 2381/4327 < 719/1114 < 391/557
83	12/79 < 18/79 < 39/158 < 63/158 < 33/79 < 75/158 < 87/158 < 51/79 < 111/158
84	1303/8578 < 447/2126 < 5046/13933 < 515/1296 < 211/463 < 968/1883 < 411/718 < 1067/1756 < 1302/1955
85	379/2495 < 447/2126 < 5046/13933 < 3944/9925 < 561/1231 < 493/959 < 245/428 < 971/1598 < 327/491
86	379/2495 < 1029/4894 < 649/1792 < 486/1223 < 561/1231 < 858/1669 < 1383/2416 < 1002/1649 < 329/494
87	211/1389 < 209/994 < 561/1549 < 213/536 < 839/1841 < 73/142 < 162/283 < 1685/2773 < 331/497
88	767/5049 < 209/994 < 561/1549 < 213/536 < 453/994 < 73/142 < 162/283 < 652/1073 < 331/497
89	767/5049 < 127/604 < 1867/5155 < 457/1150 < 453/994 < 1113/2165 < 565/987 < 429/706 < 1663/2497
90	151/994 < 660/3139 < 808/2231 < 395/994 < 453/994 < 73/142 < 162/283 < 302/497 < 331/497
91	151/994 < 209/994 < 180/497 < 213/536 < 453/994 < 1040/2023 < 565/987 < 1064/1751 < 333/500
92	151/994 < 209/994 < 180/497 < 395/994 < 453/994 < 73/142 < 569/994 < 302/497 < 331/497
93	151/994 < 127/604 < 406/1121 < 213/536 < 453/994 < 1040/2023 < 162/283 < 1064/1751 < 333/500
94	28/183 < 34/183 < 16/61 < 62/183 < 82/183 < 30/61 < 32/61 < 118/183 < 130/183
95	28/183 < 14/61 < 16/61 < 18/61 < 82/183 < 30/61 < 32/61 < 110/183 < 46/61
96	61/398 < 55/199 < 61/199 < 159/398 < 171/398 < 183/398 < 116/199 < 122/199 < 281/398
97	71/462 < 5/21 < 3/11 < 181/462 < 197/462 < 118/231 < 6/11 < 97/154 < 323/462
98	2/13 < 7/26 < 17/52 < 19/52 < 11/26 < 25/52 < 15/26 < 33/52 < 9/13
99	41/262 < 67/262 < 93/262 < 54/131 < 119/262 < 67/131 < 149/262 < 80/131 < 175/262
100	61/386 < 60/193 < 61/193 < 62/193 < 183/386 < 185/386 < 121/193 < 122/193 < 123/193
101	19/120 < 1/5 < 11/40 < 7/20 < 13/30 < 19/40 < 11/20 < 71/120 < 3/4
102	17/107 < 59/321 < 25/107 < 110/321 < 42/107 < 169/321 < 59/107 < 67/107 < 76/107
103	217/1362 < 315/1348 < 781/2828 < 1451/3692 < 449/1031 < 1987/4157 < 1583/2866 < 1709/2727 < 738/1037
104	76/477 < 111/475 < 781/2828 < 529/1346 < 449/1031 < 467/977 < 554/1003 < 737/1176 < 2456/3451

105	105/659 < 154/659 < 773/2799 < 259/659 < 763/1752 < 315/659 < 2084/3773 < 1793/2861 < 1249/1755
106	105/659 < 154/659 < 335/1213 < 259/659 < 601/1380 < 315/659 < 2084/3773 < 1793/2861 < 896/1259
107	1289/8090 < 154/659 < 335/1213 < 259/659 < 601/1380 < 315/659 < 2543/4604 < 1563/2494 < 1091/1533
108	1289/8090 < 154/659 < 649/2350 < 259/659 < 368/845 < 315/659 < 2543/4604 < 1563/2494 < 1249/1755
109	11/69 < 5/23 < 11/46 < 26/69 < 10/23 < 21/46 < 37/69 < 31/46 < 16/23
110	17/106 < 13/53 < 17/53 < 43/106 < 51/106 < 26/53 < 30/53 < 34/53 < 69/106
111	35/218 < 59/218 < 71/218 < 41/109 < 47/109 < 53/109 < 117/218 < 141/218 < 153/218
112	9/56 < 27/112 < 19/56 < 45/112 < 47/112 < 1/2 < 65/112 < 9/14 < 37/56
113	7/43 < 21/86 < 12/43 < 13/43 < 19/43 < 20/43 < 45/86 < 26/43 < 33/43
114	23/141 < 32/141 < 23/94 < 55/141 < 64/141 < 133/282 < 26/47 < 29/47 < 101/141
115	29/177 < 13/59 < 49/177 < 23/59 < 26/59 < 88/177 < 98/177 < 107/177 < 127/177
116	1/6 < 4/15 < 3/10 < 2/5 < 13/30 < 7/15 < 17/30 < 19/30 < 7/10
117	29/172 < 13/43 < 29/86 < 16/43 < 81/172 < 87/172 < 93/172 < 55/86 < 29/43
118	14/83 < 19/83 < 28/83 < 33/83 < 37/83 < 42/83 < 47/83 < 52/83 < 56/83
119	14/83 < 53/166 < 28/83 < 59/166 < 31/83 < 42/83 < 87/166 < 56/83 < 115/166
120	14/83 < 21/83 < 28/83 < 31/83 < 35/83 < 42/83 < 49/83 < 52/83 < 56/83

Table 4. Eigenvalues of derivations for Rank 2 Nilsolitons in dimension 9 (Continues)

	Derivation
121	7/41 < 8/41 < 21/82 < 15/41 < 37/82 < 39/82 < 22/41 < 53/82 < 29/41
122	7/41 < 8/41 < 11/41 < 15/41 < 18/41 < 19/41 < 22/41 < 27/41 < 29/41
123	7/41 < 21/82 < 14/41 < 17/41 < 35/82 < 21/41 < 24/41 < 49/82 < 28/41
124	1133/6573 < 95/337 < 781/2377 < 694/2013 < 575/1469 < 467/1028 < 791/1579 < 1517/2253 < 413/561
125	146/847 < 137/486 < 597/1817 < 614/1781 < 757/1934 < 457/1006 < 535/1068 < 371/551 < 759/1031
126	151/876 < 338/1199 < 1402/4267 < 181/525 < 420/1073 < 472/1039 < 263/525 < 775/1151 < 533/724
127	493/2860 < 148/525 < 1402/4267 < 181/525 < 137/350 < 159/350 < 262/523 < 101/150 < 773/1050
128	181/1050 < 656/2327 < 23/70 < 191/554 < 667/1704 < 1128/2483 < 1567/3128 < 1346/1999 < 1853/2517
129	181/1050 < 656/2327 < 1542/4693 < 201/583 < 667/1704 < 1128/2483 < 1567/3128 < 101/150 < 1853/2517
130	206/1195 < 148/525 < 1197/3643 < 181/525 < 137/350 < 159/350 < 264/527 < 101/150 < 773/1050
131	14/81 < 7/27 < 53/162 < 28/81 < 35/81 < 1/2 < 14/27 < 109/162 < 56/81
132	97/558 < 50/279 < 25/93 < 197/558 < 247/558 < 49/93 < 33/62 < 172/279 < 397/558
133	49/278 < 28/139 < 91/278 < 105/278 < 56/139 < 147/278 < 77/139 < 161/278 < 203/278
134	88/499 < 91/499 < 173/499 < 179/499 < 261/499 < 264/499 < 267/499 < 270/499 < 355/499
135	88/499 < 211/1157 < 173/499 < 617/1720 < 1021/1952 < 1511/2856 < 328/613 < 349/645 < 853/1199
136	101/566 < 62/283 < 78/283 < 101/283 < 257/566 < 140/283 < 303/566 < 179/283 < 202/283

137	7/39 < 5/26 < 9/26 < 29/78 < 5/13 < 7/13 < 43/78 < 15/26 < 19/26
138	165/914 < 239/914 < 313/914 < 165/457 < 387/914 < 202/457 < 276/457 < 569/914 < 643/914
139	23/127 < 29/127 < 32/127 < 52/127 < 55/127 < 58/127 < 75/127 < 81/127 < 87/127
140	7/38 < 41/171 < 107/342 < 7/19 < 145/342 < 85/171 < 21/38 < 227/342 < 233/342
141	28/151 < 32/151 < 44/151 < 52/151 < 60/151 < 72/151 < 84/151 < 88/151 < 116/151
142	19/102 < 9/34 < 9/34 < 19/51 < 23/51 < 23/51 < 19/34 < 65/102 < 73/102
143	41/218 < 55/218 < 31/109 < 89/218 < 48/109 < 103/218 < 117/218 < 72/109 < 151/218
144	259/1362 < 99/454 < 373/1362 < 137/454 < 99/227 < 316/681 < 118/227 < 297/454 < 335/454
145	5/26 < 1/4 < 4/13 < 5/13 < 23/52 < 1/2 < 29/52 < 33/52 < 9/13
146	5/26 < 1/4 < 4/13 < 17/52 < 23/52 < 1/2 < 15/26 < 33/52 < 9/13
147	8/41 < 21/82 < 13/41 < 16/41 < 37/82 < 21/41 < 47/82 < 24/41 < 29/41
148	191/978 < 87/326 < 52/163 < 331/978 < 401/978 < 226/489 < 87/163 < 643/978 < 713/978
149	9/46 < 45/184 < 117/368 < 135/368 < 81/184 < 189/368 < 9/16 < 117/184 < 63/92
150	9/46 < 93/368 < 57/184 < 135/368 < 165/368 < 93/184 < 9/16 < 57/92 < 129/184
151	9/46 < 19/69 < 1/3 < 49/138 < 19/46 < 65/138 < 38/69 < 14/23 < 103/138
152	11/56 < 2/7 < 33/112 < 3/8 < 43/112 < 27/56 < 65/112 < 75/112 < 19/28
153	25/127 < 28/127 < 31/127 < 50/127 < 56/127 < 59/127 < 75/127 < 81/127 < 87/127
154	43/218 < 55/218 < 67/218 < 37/109 < 49/109 < 55/109 < 117/218 < 141/218 < 153/218
155	43/218 < 55/218 < 37/109 < 79/218 < 43/109 < 49/109 < 129/218 < 141/218 < 153/218
156	46/233 < 50/233 < 58/233 < 66/233 < 108/233 < 112/233 < 116/233 < 158/233 < 166/233
157	46/233 < 54/233 < 58/233 < 62/233 < 108/233 < 112/233 < 116/233 < 154/233 < 170/233
158	21/106 < 13/53 < 17/53 < 39/106 < 47/106 < 26/53 < 30/53 < 34/53 < 73/106
159	21/106 < 14/53 < 16/53 < 39/106 < 23/53 < 1/2 < 30/53 < 67/106 < 37/53
160	159/802 < 169/802 < 154/401 < 159/401 < 164/401 < 467/802 < 477/802 < 487/802 < 497/802
161	41/204 < 4/17 < 31/102 < 41/102 < 89/204 < 8/17 < 103/204 < 137/204 < 12/17
162	9/44 < 95/352 < 59/176 < 141/352 < 9/22 < 167/352 < 95/176 < 213/352 < 131/176
163	17/82 < 25/82 < 13/41 < 33/82 < 17/41 < 21/41 < 43/82 < 51/82 < 59/82
164	61/294 < 12/49 < 83/294 < 37/98 < 19/42 < 24/49 < 155/294 < 97/147 < 205/294
165	11/53 < 23/106 < 16/53 < 17/53 < 22/53 < 55/106 < 28/53 < 33/53 < 39/53
166	187/898 < 123/449 < 269/898 < 164/449 < 351/898 < 433/898 < 515/898 < 597/898 < 310/449
167	14/67 < 29/134 < 15/67 < 28/67 < 57/134 < 29/67 < 85/134 < 43/67 < 44/67
168	9/43 < 37/172 < 71/172 < 18/43 < 73/172 < 37/86 < 27/43 < 109/172 < 55/86
169	9/43 < 49/172 < 59/172 < 31/86 < 18/43 < 85/172 < 49/86 < 27/43 < 121/172
170	447/2126 < 335/1332 < 916/3129 < 1013/3033 < 1636/3543 < 1171/2328 < 1113/2045 < 403/643 < 1102/1545

171	447/2126 < 376/1495 < 541/1848 < 168/503 < 483/1046 < 1171/2328 < 455/836 < 581/927 < 908/1273
172	1029/4894 < 376/1495 < 541/1848 < 167/500 < 634/1373 < 752/1495 < 707/1299 < 1026/1637 < 607/851
173	660/3139 < 125/497 < 291/994 < 166/497 < 459/994 < 584/1161 < 541/994 < 89/142 < 1423/1995
174	209/994 < 125/497 < 1022/3491 < 1827/5470 < 779/1687 < 751/1493 < 584/1073 < 89/142 < 1321/1852
175	209/994 < 125/497 < 291/994 < 991/2967 < 151/327 < 250/497 < 879/1615 < 1288/2055 < 102/143
176	209/994 < 291/1157 < 926/3163 < 1321/3955 < 3014/6527 < 250/497 < 793/1457 < 1199/1913 < 102/143
177	209/994 < 125/497 < 291/994 < 166/497 < 459/994 < 250/497 < 541/994 < 89/142 < 709/994
178	127/604 < 125/497 < 291/994 < 991/2967 < 151/327 < 333/662 < 879/1615 < 1288/2055 < 1321/1852
179	127/604 < 291/1157 < 291/994 < 826/2473 < 767/1661 < 250/497 < 1088/1999 < 1377/2197 < 102/143
180	127/604 < 291/1157 < 926/3163 < 493/1476 < 743/1609 < 333/662 < 2225/4088 < 665/1061 < 1433/2009
181	1483/7053 < 665/2644 < 291/994 < 991/2967 < 459/994 < 250/497 < 879/1615 < 1288/2055 < 102/143
182	7/33 < 5/22 < 13/33 < 9/22 < 14/33 < 29/66 < 41/66 < 7/11 < 43/66

Table 5. Eigenvalues of derivations for Rank 2 Nilsolitons in dimension 9

	Derivation
183	41/193 < 101/386 < 60/193 < 145/386 < 82/193 < 183/386 < 101/193 < 123/193 < 142/193
184	293/1368 < 45/191 < 384/1559 < 293/1140 < 851/1892 < 491/1042 < 413/857 < 2769/4040 < 619/850
185	1389/6485 < 700/2971 < 318/1291 < 631/2455 < 1111/2470 < 1637/3474 < 1133/2351 < 512/747 < 477/655
186	157/733 < 348/1477 < 653/2651 < 64/249 < 112/249 < 1015/2154 < 1653/3430 < 451/658 < 544/747
187	163/761 < 348/1477 < 653/2651 < 64/249 < 112/249 < 614/1303 < 1653/3430 < 573/836 < 544/747
188	3/14 < 11/49 < 12/49 < 22/49 < 45/98 < 23/49 < 47/98 < 33/49 < 34/49
189	3/14 < 1/4 < 2/7 < 11/28 < 3/7 < 13/28 < 1/2 < 19/28 < 5/7
190	47/218 < 59/218 < 35/109 < 71/218 < 41/109 < 53/109 < 129/218 < 141/218 < 153/218
191	23/106 < 14/53 < 16/53 < 37/106 < 23/53 < 51/106 < 30/53 < 69/106 < 37/53
192	7/32 < 9/32 < 21/64 < 25/64 < 7/16 < 1/2 < 35/64 < 39/64 < 23/32
193	35/159 < 77/318 < 35/106 < 56/159 < 49/106 < 175/318 < 91/159 < 63/106 < 217/318
194	24/109 < 55/218 < 31/109 < 41/109 < 48/109 < 103/218 < 55/109 < 72/109 < 79/109
195	13/59 < 27/118 < 14/59 < 53/118 < 27/59 < 55/118 < 28/59 < 40/59 < 41/59
196	13/59 < 31/118 < 18/59 < 39/118 < 49/118 < 57/118 < 31/59 < 75/118 < 44/59
197	19/86 < 128/473 < 303/946 < 175/473 < 371/946 < 465/946 < 256/473 < 13/22 < 721/946
198	19/86 < 21/86 < 23/86 < 27/86 < 21/43 < 22/43 < 23/43 < 24/43 < 65/86
199	99/446 < 58/223 < 133/446 < 157/446 < 87/223 < 215/446 < 116/223 < 145/223 < 331/446
200	2/9 < 25/108 < 13/54 < 4/9 < 49/108 < 25/54 < 17/36 < 73/108 < 19/27
201	2/9 < 3/13 < 55/234 < 53/117 < 107/234 < 6/13 < 55/117 < 80/117 < 9/13
202	37/166 < 19/83 < 28/83 < 57/166 < 37/83 < 75/166 < 47/83 < 56/83 < 113/166

203	67/298 < 67/298 < 38/149 < 125/298 < 67/149 < 143/298 < 143/298 < 201/298 < 105/149
204	12/53 < 73/318 < 37/159 < 24/53 < 73/159 < 49/106 < 74/159 < 109/159 < 110/159
205	53/234 < 3/13 < 55/234 < 35/78 < 107/234 < 6/13 < 109/234 < 80/117 < 163/234
206	5/22 < 8/33 < 23/66 < 4/11 < 31/66 < 16/33 < 19/33 < 13/22 < 47/66
207	25/109 < 55/218 < 30/109 < 35/109 < 50/109 < 55/109 < 115/218 < 60/109 < 85/109
208	401/1745 < 275/1114 < 147/557 < 166/557 < 275/557 < 569/1114 < 313/593 < 313/557 < 397/524
209	128/557 < 569/2305 < 683/2588 < 467/1567 < 746/1511 < 2087/4086 < 294/557 < 871/1550 < 969/1279
210	128/557 < 1278/5177 < 787/2982 < 529/1775 < 904/1831 < 1114/2181 < 294/557 < 1007/1792 < 1141/1506
211	569/2476 < 275/1114 < 147/557 < 166/557 < 275/557 < 569/1114 < 275/521 < 313/557 < 447/590
212	25/107 < 27/107 < 73/214 < 77/214 < 81/214 < 52/107 < 127/214 < 131/214 < 77/107
213	20/79 < 20/79 < 41/158 < 21/79 < 81/158 < 41/79 < 41/79 < 83/158 < 61/79
214	5/18 < 8/27 < 17/54 < 1/3 < 19/54 < 31/54 < 11/18 < 17/27 < 35/54
215	23/82 < 25/82 < 27/82 < 29/82 < 31/82 < 33/82 < 26/41 < 27/41 < 28/41
216	9/32 < 19/64 < 5/16 < 21/64 < 11/32 < 19/32 < 39/64 < 5/8 < 41/64
217	253/802 < 259/802 < 265/802 < 271/802 < 277/802 < 283/802 < 262/401 < 268/401 < 271/401

Table 6. Eigenvalues of derivations for Rank 3 Nilsolitons in dimension 9 (Continues)

	Derivation
1	1/23 < 43/138 < 49/138 < 19/46 < 21/46 < 1/2 < 25/46 < 27/46 < 2/3
2	7/94 < 49/282 < 15/47 < 37/94 < 22/47 < 139/282 < 51/94 < 29/47 < 2/3
3	7/94 < 49/282 < 1/3 < 18/47 < 43/94 < 143/282 < 25/47 < 57/94 < 32/47
4	8/99 < 167/594 < 215/594 < 263/594 < 91/198 < 311/594 < 107/198 < 359/594 < 191/297
5	7/82 < 23/82 < 27/82 < 16/41 < 17/41 < 1/2 < 24/41 < 25/41 < 55/82
6	25/282 < 17/94 < 15/47 < 17/47 < 127/282 < 1/2 < 76/141 < 59/94 < 32/47
7	1/11 < 7/22 < 4/11 < 9/22 < 5/11 < 1/2 < 6/11 < 13/22 < 15/22

Table 7. Eigenvalues of derivations for Rank 3 Nilsolitons in dimension 9 (Continues)

	Derivation
8	35/353 < 223/706 < 231/706 < 285/706 < 293/706 < 301/706 < 371/706 < 441/706 < 258/353
9	1/10 < 11/40 < 3/8 < 2/5 < 19/40 < 1/2 < 23/40 < 3/5 < 13/20
10	65/634 < 85/317 < 116/317 < 235/634 < 130/317 < 150/317 < 365/634 < 201/317 < 215/317
11	4/39 < 7/39 < 4/13 < 9/26 < 35/78 < 19/39 < 43/78 < 17/26 < 2/3
12	33/317 < 169/634 < 102/317 < 235/634 < 135/317 < 168/317 < 373/634 < 201/317 < 202/317
13	33/317 < 169/634 < 116/317 < 235/634 < 261/634 < 149/317 < 182/317 < 202/317 < 215/317
14	5/48 < 85/288 < 55/144 < 115/288 < 43/96 < 35/72 < 53/96 < 85/144 < 25/36

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15	7/66 < 13/66 < 8/33 < 4/11 < 29/66 < 31/66 < 19/33 < 7/11 < 15/22
16	3/28 < 4/21 < 17/56 < 1/3 < 23/56 < 29/56 < 11/21 < 5/8 < 5/7
17	49/454 < 137/454 < 78/227 < 93/227 < 205/454 < 235/454 < 127/227 < 142/227 < 293/454
18	49/454 < 137/454 < 161/454 < 93/227 < 100/227 < 235/454 < 249/454 < 142/227 < 149/227
19	49/454 < 78/227 < 161/454 < 85/227 < 205/454 < 219/454 < 127/227 < 134/227 < 317/454
20	8/69 < 13/46 < 7/23 < 8/23 < 29/69 < 37/69 < 27/46 < 29/46 < 15/23
21	8/69 < 7/23 < 8/23 < 17/46 < 19/46 < 29/69 < 37/69 < 15/23 < 33/46
22	13/111 < 23/74 < 12/37 < 27/74 < 14/37 < 49/111 < 62/111 < 25/37 < 51/74
23	2/17 < 35/136 < 14/51 < 3/8 < 20/51 < 67/136 < 83/136 < 43/68 < 2/3
24	2/17 < 11/34 < 6/17 < 13/34 < 15/34 < 1/2 < 19/34 < 21/34 < 23/34
25	42/353 < 98/353 < 211/706 < 265/706 < 140/353 < 182/353 < 407/706 < 224/353 < 238/353
26	10/83 < 23/83 < 63/166 < 33/83 < 36/83 < 1/2 < 46/83 < 103/166 < 56/83
27	4/31 < 25/93 < 19/62 < 12/31 < 37/93 < 27/62 < 35/62 < 2/3 < 43/62
28	7/54 < 2/9 < 7/27 < 7/18 < 4/9 < 13/27 < 14/27 < 35/54 < 19/27
29	7/54 < 49/162 < 25/81 < 32/81 < 71/162 < 13/27 < 46/81 < 11/18 < 113/162
30	7/54 < 2/9 < 8/27 < 19/54 < 4/9 < 13/27 < 14/27 < 11/18 < 20/27
31	3/23 < 6/23 < 17/46 < 9/23 < 10/23 < 12/23 < 13/23 < 29/46 < 15/23
32	9/68 < 4/17 < 25/68 < 13/34 < 8/17 < 1/2 < 41/68 < 21/34 < 43/68
33	11/83 < 22/83 < 61/166 < 33/83 < 36/83 < 1/2 < 47/83 < 105/166 < 55/83
34	2/15 < 1/5 < 3/10 < 2/5 < 13/30 < 1/2 < 17/30 < 3/5 < 7/10
35	11/82 < 11/41 < 12/41 < 29/82 < 33/82 < 22/41 < 23/41 « 53/82 < 55/82
36	13/96 < 19/72 < 53/144 < 115/288 < 13/32 < 77/144 < 13/24 < 91/144 < 193/288
37	35/257 < 58/257 < 93/257 < 195/514 < 128/257 < 265/514 < 151/257 < 311/514 < 163/257
38	35/257 < 58/257 < 93/257 < 203/514 < 249/514 < 128/257 < 151/257 < 319/514 < 163/257
39	3/22 < 1/6 < 3/11 < 1/3 < 9/22 < 1/2 < 6/11 < 2/3 < 15/22
40	5/36 < 2/9 < 2/9 < 13/36 < 4/9 < 1/2 < 7/12 < 23/36 < 2/3
41	11/79 < 43/158 < 32/79 < 65/158 < 34/79 < 43/79 < 87/158 < 45/79 < 54/79
42	1/7 < 2/7 < 9/28 < 11/28 < 3/7 < 13/28 < 4/7 < 17/28 < 5/7
43	12/83 < 21/83 < 61/166 < 33/83 < 36/83 < 85/166 < 45/83 < 103/166 < 57/83
44	12/83 < 21/83 < 63/166 < 33/83 < 36/83 < 81/166 < 45/83 < 105/166 < 57/83
45	12/83 < 26/83 < 61/166 < 63/166 < 36/83 < 38/83 < 85/166 < 50/83 < 62/83
46	11/76 < 6/19 < 7/19 < 15/38 < 35/76 < 39/76 < 41/76 < 23/38 < 13/19
47	7/48 < 5/24 < 5/16 < 1/3 < 7/16 < 11/24 < 13/24 < 29/48 < 3/4
48	7/48 < 73/288 < 49/144 < 115/288 < 43/96 < 35/72 < 19/32 < 91/144 < 47/72

49	7/48 < 73/288 < 53/144 < 115/288 < 13/32 < 37/72 < 53/96 < 47/72 < 95/144
50	7/48 < 5/24 < 1/4 < 19/48 < 7/16 < 11/24 < 13/24 < 2/3 < 11/16
51	17/114 < 4/19 < 11/38 < 41/114 < 8/19 < 1/2 < 29/57 < 25/38 < 27/38
52	25/166 < 49/166 < 59/166 < 30/83 < 63/166 < 37/83 < 42/83 < 109/166 < 123/166
53	5/33 < 167/594 < 193/594 < 7/18 < 257/594 < 283/594 < 107/198 < 373/594 < 212/297
54	5/33 < 3/11 < 7/22 < 9/22 < 14/33 < 5/11 < 19/33 < 13/22 < 8/11
55	7/46 < 11/46 < 17/46 < 9/23 < 10/23 < 12/23 < 25/46 < 29/46 < 31/46
56	43/282 < 19/94 < 10/47 < 50/141 < 39/94 < 143/282 < 29/47 < 59/94 < 31/47
57	54/353 < 117/706 < 225/706 < 299/706 < 333/706 < 171/353 < 407/706 < 441/706 < 459/706
58	91/594 < 62/297 < 215/594 < 235/594 < 91/198 < 17/33 < 113/198 < 359/594 < 397/594
59	2/13 < 4/13 < 9/26 < 11/26 < 6/13 < 1/2 < 15/26 < 8/13 < 17/26
60	13/82 < 17/82 < 12/41 < 16/41 < 37/82 < 1/2 < 45/82 < 25/41 < 29/41
61	17/107 < 25/107 < 75/214 < 77/214 < 42/107 < 109/214 < 59/107 < 67/107 < 76/107
62	9/56 < 4/21 < 13/56 < 1/3 < 11/28 < 11/21 < 31/56 < 5/8 < 5/7
63	73/454 < 66/227 < 73/227 < 161/454 < 205/454 < 219/454 < 139/227 < 146/227 < 293/454
64	5/31 < 47/186 < 17/62 < 12/31 < 77/186 27/62 17/31 2/3 22/31
65	16/99 < 2/9 < 65/198 < 25/66 < 38/99 6/11 109/198 20/33 70/99
66	20/123 < 9/41 < 34/123 < 83/246 < 18/41 61/123 1/2 163/246 88/123
67	67/408 < 31/136 < 9/34 < 25/68 < 20/51 67/136 227/408 43/68 49/68
68	75/454 < 111/454 < 143/454 < 93/227 < 109/227 111/227 127/227 293/454 297/454

Table 8. Eigenvalues of derivations for Rank 3 Nilsolitons in dimension 9 (Continues)

	Derivation
69	75/454 < 111/454 < 161/454 < 93/227 < 100/227 111/227 118/227 297/454 311/454
70	48/281 < 139/562 < 88/281 < 99/281 < 235/562 < 136/281 < 337/562 < 184/281 < 187/281
71	35/204 < 15/68 < 35/136 < 53/136 < 20/51 < 65/136 < 115/204 < 83/136 < 25/34
72	17/99 < 28/99 < 65/198 < 34/99 < 13/33 < 5/11 < 1/2 < 133/198 < 73/99
73	39/227 < 137/454 < 141/454 < 78/227 < 100/227 < 219/454 < 139/227 < 293/454 < 297/454
74	39/227 < 137/454 < 78/227 < 157/454 < 88/227 < 235/454 < 127/227 < 293/454 < 313/454
75	39/227 < 141/454 < 78/227 < 161/454 < 88/227 < 219/454 < 127/227 < 297/454 < 317/454
76	17/98 < 13/49 < 16/49 < 20/49 < 43/98 < 1/2 < 29/49 < 30/49 < 33/49
77	4/23 < 11/46 < 17/69 < 19/46 < 29/69 < 11/23 < 27/46 < 15/23 < 2/3
78	4/23 < 25/138 < 49/138 < 17/46 < 19/46 < 37/69 < 25/46 < 27/46 < 33/46
79	45/257 < 139/514 < 90/257 < 94/257 < 106/257 < 229/514 < 151/257 < 319/514 < 184/257
80	10/57 < 7/38 < 11/38 < 41/114 < 8/19 < 9/19 < 61/114 < 25/38 < 27/38

81	3/17 < 9/34 < 23/68 < 25/68 < 29/68 < 15/34 < 41/68 < 21/34 < 12/17
82	3/17 < 9/34 < 6/17 < 13/34 < 15/34 < 8/17 < 9/17 < 11/17 < 12/17
83	56/317 < 123/634 < 213/634 < 235/634 < 261/634 < 168/317 < 347/634 < 373/634 < 459/634
84	50/281 < 137/562 < 87/281 < 97/281 < 237/562 < 137/281 < 337/562 < 184/281 < 187/281
85	5/28 < 25/112 < 87/224 < 45/112 < 97/224 < 127/224 < 65/112 < 137/224 < 5/8
86	57/317 < 145/634 < 78/317 < 259/634 < 135/317 < 156/317 < 373/634 < 202/317 < 213/317
87	2/11 < 5/22 < 7/22 < 4/11 < 9/22 < 1/2 < 13/22 < 7/11 < 15/22
88	2/11 < 5/22 < 7/22 < 9/22 5/11 < 1/2 < 6/11 7/11 15/22
89	2/11 < 8/33 < 7/22 < 4/11 14/33 < 1/2 < 6/11 2/3 15/22
90	2/11 < 21/88 < 3/11 < 25/88 < 5/11 < 41/88 < 45/88 < 7/11 < 3/4
91	2/11 < 155/594 < 83/297 < 14/33 < 263/594 < 137/297 < 107/198 < 191/297 < 19/27
92	2/11 < 3/11 < 7/22 < 4/11 < 9/22 < 1/2 < 13/22 < 7/11 < 15/22
93	15/82 < 19/82 < 10/41 < 17/41 35/82 < 18/41 < 25/41 < 53/82 < 55/82
94	9/49 < 25/98 < 16/49 < 20/49 < 43/98 < 25/49 < 57/98 < 29/49 < 34/49
95	7/38 < 23/114 < 29/114 < 13/38 < 26/57 < 9/19 < 10/19 < 25/38 < 27/38
96	7/38 < 23/114 < 5/19 < 13/38 < 17/38 < 53/114 < 10/19 < 2/3 < 27/38
97	117/634 < 64/317 < 85/317 < 128/317 < 287/634 < 149/317 < 373/634 < 202/317 < 213/317
98	117/634 < 64/317 < 104/317 < 115/317 < 128/317 < 168/317 < 347/634 < 373/634 < 232/317
99	5/27 < 2/9 < 13/54 < 1/3 < 4/9 < 25/54 < 14/27 < 37/54 < 19/27
100	5/27 < 2/9 < 5/18 < 8/27 < 4/9 < 25/54 < 14/27 < 35/54 < 20/27
101	23/123 < 8/41 < 65/246 < 40/123 < 37/82 < 21/41 < 64/123 < 157/246 < 88/123
102	3/16 < 5/18 < 91/288 < 53/144 < 13/32 < 67/144 < 19/32 < 47/72 < 197/288
103	3/16 < 7/32 < 1/4 < 13/32 < 7/16 < 15/32 < 19/32 < 5/8 < 11/16
104	3/16 < 9/32 < 5/16 < 21/64 < 23/64 < 1/2 < 39/64 < 41/64 < 11/16
105	55/293 < 159/586 < 88/293 < 203/586 < 110/293 < 143/293 < 181/293 < 379/586 < 198/293
106	55/293 < 95/293 < 203/586 < 207/586 < 110/293 < 112/293 < 150/293 < 205/293 < 427/586
107	11/58 < 13/58 < 21/58 < 12/29 < 13/29 < 16/29 < 17/29 < 35/58 < 37/58
108	11/58 < 9/29 < 19/58 < 21/58 < 13/29 < 1/2 < 16/29 < 37/58 < 20/29
109	15/79 < 35/158 < 30/79 < 32/79 < 65/158 < 45/79 < 47/79 < 95/158 < 50/79
110	4/21 < 2/7 < 5/14 < 8/21 < 17/42 < 10/21 < 23/42 < 4/7 < 16/21
111	5/26 < 7/26 < 9/26 < 5/13 < 6/13 < 7/13 < 15/26 < 8/13 < 17/26
112	7/36 < 2/9 < 1/4 < 5/18 < 17/36 < 17/36 < 1/2 < 2/3 < 13/18
113	9/46 < 14/69 < 6/23 < 9/23 < 21/46 < 32/69 < 27/46 < 15/23 < 2/3
114	9/46 < 14/69 < 1/3 < 8/23 < 9/23 < 37/69 < 25/46 < 27/46 < 17/23

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115	9/46 < 5/23 < 6/23 < 8/23 < 19/46 < 11/23 < 25/46 < 29/46 < 17/23
116	9/46 < 11/46 < 15/46 < 8/23 < 10/23 < 25/46 < 13/23 < 29/46 < 31/46
117	9/46 < 7/23 < 15/46 < 8/23 < 17/46 < 1/2 < 25/46 < 31/46 < 16/23
118	12/61 < 31/122 < 18/61 < 25/61 < 53/122 < 55/122 < 30/61 < 42/61 < 43/61
119	15/76 < 11/38 < 6/19 < 7/19 < 37/76 < 39/76 < 43/76 < 23/38 < 13/19
120	16/81 < 13/54 < 49/162 < 32/81 < 71/162 < 13/27 < 1/2 < 55/81 < 113/162
121	19/96 < 83/288 < 11/36 < 25/72 < 19/48 < 35/72 < 19/32 < 47/72 < 197/288
122	63/317 < 76/317 < 175/634 < 116/317 < 255/634 < 139/317 < 202/317 < 407/634 < 215/317
123	1/5 < 11/40 < 13/40 < 7/20 < 17/40 < 19/40 < 11/20 < 3/5 < 3/4
124	91/454 < 57/227 < 137/454 < 85/227 < 205/454 < 114/227 < 251/454 < 142/227 < 319/454
125	91/454 < 119/454 < 137/454 < 165/454 < 100/227 < 114/227 < 128/227 < 142/227 < 319/454
126	91/454 < 55/227 < 141/454 < 85/227 < 201/454 < 116/227 < 251/454 < 146/227 < 311/454
127	91/454 < 111/454 < 143/454 < 85/227 < 101/227 < 111/227 < 127/227 < 293/454 < 313/454
128	91/454 < 119/454 < 135/454 < 85/227 < 99/227 < 113/227 < 127/227 < 289/454 < 317/454
129	91/454 < 135/454 < 141/454 < 85/227 < 88/227 < 113/227 < 116/227 < 311/454 < 317/454
130	91/454 < 137/454 < 143/454 < 85/227 < 88/227 < 111/227 < 114/227 < 313/454 < 319/454

Table 9. Eigenvalues of derivations for Rank 3 Nilsolitons in dimension 9

	Derivation
131	64/317 < 151/634 < 87/317 < 115/317 < 128/317 < 279/634 < 202/317 < 407/634 < 215/317
132	143/706 < 76/353 < 106/353 < 115/353 < 299/706 < 182/353 < 373/706 < 221/353 < 258/353
133	15/74 < 8/37 < 25/111 < 31/74 < 16/37 < 49/111 < 23/37 < 47/74 < 2/3
134	12/59 < 16/59 < 20/59 < 47/118 < 24/59 < 28/59 < 32/59 < 71/118 < 44/59
135	11/54 < 37/162 < 49/162 < 11/27 < 35/81 < 13/27 < 41/81 < 107/162 < 115/162
136	11/54 < 37/162 < 47/162 < 11/27 < 35/81 < 40/81 < 14/27 < 107/162 < 113/162
137	9/44 < 3/11 < 4/11 < 9/22 < 19/44 < 21/44 < 25/44 < 7/11 < 15/22
138	17/83 < 26/83 < 53/166 < 28/83 < 71/166 < 36/83 < 45/83 < 105/166 < 62/83
139	17/83 < 21/83 < 53/166 < 61/166 < 36/83 < 38/83 < 95/166 < 55/83 < 57/83
140	7/34 < 91/408 < 39/136 < 1/3 < 25/68 < 67/136 < 227/408 < 39/68 < 53/68
141	7/34 < 4/17 < 13/34 < 7/17 < 15/34 < 8/17 < 10/17 < 21/34 < 23/34
142	13/63 < 29/126 < 5/14 < 8/21 < 55/126 < 71/126 < 37/63 < 11/18 < 9/14
143	6/29 < 13/58 < 23/58 < 12/29 < 25/58 < 13/29 < 35/58 < 37/58 < 19/29
144	17/82 < 9/41 < 19/82 < 16/41 < 18/41 < 37/82 < 25/41 < 53/82 < 55/82
145	5/24 < 1/4 < 13/48 < 5/12 < 7/16 < 11/24 < 23/48 < 11/16 < 17/24
146	5/24 < 1/4 < 5/16 < 17/48 < 19/48 < 11/24 < 9/16 < 2/3 < 17/24

147	62/297 < 25/99 < 8/27 < 197/594 < 137/297 < 50/99 < 107/198 < 373/594 < 212/297
148	62/297 < 25/99 < 8/27 < 221/594 < 91/198 < 137/297 < 50/99 < 397/594 < 212/297
149	4/19 < 1/4 < 29/76 < 15/38 < 8/19 < 35/76 < 23/38 < 12/19 < 51/76
150	125/586 < 68/293 < 69/293 < 263/586 < 136/293 < 137/293 < 285/586 < 194/293 < 205/293
151	125/586 < 68/293 < 88/293 < 99/293 < 136/293 < 301/586 < 156/293 < 323/586 < 224/293
152	3/14 < 2/9 < 7/18 < 53/126 < 3/7 < 55/126 < 11/18 < 9/14 < 41/63
153	3/14 < 1/4 < 2/7 < 9/28 < 13/28 < 1/2 < 4/7 < 17/28 < 5/7
154	3/14 < 1/4 < 2/7 < 11/28 < 3/7 < 13/28 < 1/2 < 19/28 < 5/7
155	3/14 < 13/49 < 16/49 < 18/49 < 41/98 < 47/98 < 29/49 < 31/49 < 34/49
156	31/144 < 19/72 < 83/288 < 55/144 < 13/32 < 23/48 < 53/96 < 193/288 < 25/36
157	5/23 < 11/46 < 15/46 < 17/46 < 10/23 < 21/46 < 25/46 < 31/46 < 16/23
158	5/23 < 6/23 < 7/23 < 17/46 < 10/23 < 11/23 < 12/23 < 29/46 < 17/23
159	56/257 < 72/257 < 79/257 < 161/514 < 207/514 < 128/257 < 151/257 < 319/514 < 184/257
160	99/454 < 119/454 < 135/454 < 81/227 < 99/227 < 109/227 < 127/227 < 297/454 < 317/454
161	99/454 < 135/454 < 137/454 < 81/227 < 99/227 < 100/227 < 118/227 < 297/454 < 335/454
162	50/227 < 119/454 < 137/454 < 78/227 < 100/227 < 219/454 < 128/227 < 293/454 < 319/454
163	50/227 < 135/454 < 137/454 < 161/454 < 99/227 < 100/227 < 235/454 < 149/227 < 335/454
164	35/158 < 19/79 < 26/79 < 28/79 < 73/158 < 87/158 < 45/79 < 47/79 < 54/79
165	13/58 < 15/58 < 10/29 < 21/58 < 13/29 < 14/29 < 17/29 < 35/58 < 41/58
166	13/58 < 17/58 < 10/29 < 23/58 < 12/29 < 13/29 < 15/29 < 37/58 < 43/58
167	58/257 < 127/514 < 69/257 < 94/257 < 243/514 < 127/257 < 265/514 < 163/257 < 185/257
168	127/562 < 64/281 < 66/281 < 127/281 < 259/562 < 130/281 < 261/562 < 193/281 < 194/281
169	127/562 < 64/281 < 87/281 < 88/281 < 127/281 < 301/562 < 151/281 < 303/562 < 215/281
170	5/22 < 3/11 < 7/22 < 4/11 < 9/22 < 5/11 < 1/2 < 15/22 < 8/11
171	37/162 < 13/54 < 49/162 < 32/81 < 11/27 < 38/81 < 13/27 < 113/162 < 115/162
172	37/162 < 19/81 < 47/162 < 65/162 < 11/27 < 25/54 < 14/27 < 56/81 < 113/162
173	27/118 < 29/118 < 20/59 < 45/118 < 47/118 < 28/59 < 67/118 < 37/59 < 85/118
174	68/297 < 76/297 < 95/297 < 193/594 < 7/18 < 16/33 < 19/33 < 383/594 < 212/297
175	68/297 < 85/297 < 95/297 < 211/594 < 7/18 < 14/33 < 17/33 < 401/594 < 221/297
176	49/214 < 25/107 < 30/107 < 31/107 < 109/214 < 55/107 < 111/214 < 61/107 < 80/107
177	14/61 < 16/61 < 18/61 < 39/122 < 53/122 < 30/61 < 32/61 < 75/122 < 46/61
178	3/13 < 7/26 < 4/13 < 11/26 < 6/13 < 1/2 < 7/13 < 15/26 < 19/26
179	39/166 < 20/83 < 23/83 < 24/83 < 40/83 < 43/83 < 87/166 < 47/83 < 63/83
180	4/17 < 1/4 < 9/34 < 25/68 < 8/17 < 33/68 < 1/2 < 43/68 < 25/34

181	21/88 < 1/4 < 15/44 < 31/88 < 17/44 < 43/88 < 13/22 < 53/88 < 8/11
182	29/118 < 33/118 < 20/59 < 22/59 < 47/118 < 51/118 < 31/59 < 73/118 < 91/118
183	1/4 < 7/24 < 1/3 < 3/8 < 5/12 < 13/24 < 7/12 < 5/8 < 2/3
184	9/34 < 5/17 < 11/34 < 6/17 < 13/34 < 19/34 < 10/17 < 21/34 < 23/34
185	13/48 < 7/24 < 5/16 < 1/3 < 3/8 < 7/12 < 29/48 < 5/8 < 31/48
186	3/11 < 13/44 < 7/22 < 15/44 < 4/11 < 25/44 < 13/22 < 7/11 < 29/44
187	11/38 < 23/76 < 6/19 < 7/19 < 29/76 < 15/38 < 47/76 < 51/76 < 13/19
188	17/58 < 9/29 < 19/58 < 10/29 < 21/58 < 23/58 < 37/58 < 19/29 < 20/29
189	37/126 < 19/63 < 43/126 < 22/63 < 5/14 < 7/18 < 9/14 < 41/63 < 29/42
190	33/112 < 9/28 < 73/224 < 39/112 < 79/224 < 3/8 < 145/224 < 75/112 < 151/224
191	49/158 < 25/79 < 26/79 < 53/158 < 28/79 < 29/79 < 51/79 < 105/158 < 54/79

Table 10. Eigenvalues of derivations for Rank 4 Nilsolitons in dimension 9

	Derivation
1	1/12 < 1/6 < 1/3 < 3/8 < 11/24 < 1/2 < 13/24 < 5/8 < 2/3
2	1/9 < 8/27 < 19/54 < 11/27 < 4/9 < 14/27 < 5/9 < 17/27 < 35/54
3	1/8 < 13/48 < 3/8 < 19/48 < 7/16 < 1/2 < 9/16 < 5/8 < 2/3
4	5/38 < 6/19 < 13/38 < 7/19 < 17/38 < 1/2 < 11/19 < 12/19 < 25/38
5	15/94 < 11/47 < 16/47 < 18/47 < 37/94 < 1/2 < 29/47 < 59/94 < 31/47
6	4/23 < 13/46 < 17/46 < 19/46 < 21/46 < 11/23 < 27/46 < 29/46 < 15/23
7	7/39 < 7/26 < 4/13 < 9/26 < 5/13 < 19/39 < 8/13 < 17/26 < 2/3
8	17/94 < 12/47 < 15/47 < 17/47 < 35/94 < 1/2 < 29/47 < 59/94 < 32/47
9	2/11 < 13/66 < 1/3 < 4/11 < 9/22 < 35/66 < 6/11 < 13/22 < 8/11
10	2/11 < 13/66 < 3/11 < 9/22 < 5/11 < 31/66 < 13/22 < 7/11 < 2/3
11	7/38 < 11/38 < 13/38 < 7/19 < 8/19 < 9/19 < 10/19 < 25/38 < 27/38
12	19/94 < 10/47 < 15/47 < 16/47 < 39/94 < 49/94 < 29/47 < 59/94 < 31/47
13	11/54 < 7/27 < 8/27 < 10/27 < 4/9 < 1/2 < 5/9 < 17/27 < 19/27
14	5/24 < 1/4 < 5/16 < 3/8 < 7/16 < 11/24 < 9/16 < 2/3 < 11/16
15	13/62 < 7/31 < 11/31 < 12/31 < 27/62 < 35/62 < 18/31 < 19/31 < 20/31
16	4/19 < 11/38 < 6/19 < 13/38 < 8/19 < 1/2 < 10/19 < 25/38 < 27/38
17	10/47 < 12/47 < 15/47 < 16/47 < 17/47 < 22/47 < 29/47 < 31/47 < 32/47
18	8/37 < 9/37 < 14/37 < 29/74 < 16/37 < 17/37 < 45/74 < 23/37 < 25/37
19	15/68 < 35/136 < 23/68 < 53/136 < 27/68 < 65/136 < 19/34 < 83/136 < 25/34
20	2/9 < 7/27 < 8/27 < 19/54 < 4/9 < 13/27 < 5/9 < 35/54 < 19/27
21	31/136 < 9/34 < 45/136 < 25/68 < 53/136 < 67/136 < 19/34 < 43/68 < 49/68

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22	1/4 < 5/16 < 11/32 < 3/8 < 13/32 < 7/16 < 19/32 < 21/32 < 11/16
23	21/74 < 23/74 < 12/37 < 27/74 < 14/37 < 29/74 < 45/74 < 25/37 < 51/74
24	9/31 < 19/62 < 21/62 < 11/31 < 11/31 < 12/31 < 20/31 < 20/31 < 43/62

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