H_∞-NORM EVALUATION FOR A TRANSFER MATRIX VIA BISECTION ALGORITHM

by

Hasan GUNDUZ [a*] ***and Ercan CELIK*** [b]

[a] Department of Mathematics, Faculty of Science and Literature, Bingol University, Bingol, Turkey
[b] Department of Applied Mathematics and Informatics, Faculty of Science, Kyrgyz-Turkish Manas University, Bishkek, Kyrgyzstan

Original scientific paper
https://doi.org/10.2298/TSCI22S2745G

In this paper, we compute H_∞-norm of a transfer matrix, via bisection algorithm. The algorithm is given and applied some problems. The problems are choosen from various areas of control theory such as aircraft models and decentralized interconnected systems.

Key words: *H_∞-infinity control, bisection algorithm, Hankel singular values, Hamiltonian*

Introduction

In control theory, one of the most common handicaps is designing controllers for systems that can perform effectively even when faced with different kinds of variables and irregularities. The works performed to overcome this handicap breed many number of control methods. The H_∞ control is one of the most powerful techniques in control theory which was created to alleviate modelling errors and undetermined disturbances while getting measurable optimization for large-scale multivariable problems. Here H_∞ is *the space of all bounded analytic matrix valued functions in the open right-half complex plane*. The theory was introduced by Zames [1]. The main principle of the theory is based on formulating the problem of sensitivity reduction as an optimization problem with an operator norm that is H_∞-norm. To put it more clearly, to make effort to find the *the best* controller among others which minimizes the H_∞-norm of some kind of the transfer function of the system. In 1980's it was applied in many works and extended to associated various concepts of control theory such as gain matrix, algebraic Riccati equation, state-space solutions *etc.* [2-6]. Recently, it is studied by Jiang *et al.* [7].

This paper is organized in 5 sections. Some basic definitons, notations and propositions which will be used through the paper are given in Section 2. In Section 3, some theorems which our algorithm derived from them are told, the algortihm is explained in detail and a couple formulas and equations, which are essential to remove a large number of undesirable operations and to execute the algortihm in a simple way, are given. Two numerical examples and related tables of iterations and values are given in Section 4. Finally, Section 5 is about efficiency of the algorithm and the reason of an insignificiant amount of deviation of the error tolerance.

* Corresponding author, e-mail: hgunduz@bingol.edu.tr

Preliminaries

Consider the linear dynamic system:

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx + Du \end{aligned} \tag{1}$$

where $A\in\mathbb{R}^{n\times n}, B\in\mathbb{R}^{n\times m}, C\in\mathbb{R}^{p\times n}, D\in\mathbb{R}^{p\times m}$. Transfer matrix of the system (1) is defined:

$$G(s) = C(sI - A)^{-1}B + D \tag{2}$$

Let $\lambda_j(M), \sigma_j(M)$ denote the j^{th} eigenvalue and j^{th} singular value of a matrix M respectively, where $\sigma_j(M) = [\lambda_j(MM^T)]^{1/2}$. A is stable if $\mathrm{Re}[\lambda_j(A)] < 0$ for all j. If A is stable H_∞-*norm* of the transfer matrix $G(s)$ is given:

$$G_\infty = \sup_{\mathrm{Re}\, s>0} \sigma_{max}[G(s)] = \sup_{\omega\in\mathbb{R}} \sigma_{max}[G(j\omega)] \tag{3}$$

where $\sup_{\omega\in\mathbb{R}}$ denotes least upper bound for all frequencies ω which are real.

Let:

$$J_{2n\times 2n} = \begin{bmatrix} 0_n & I_n \\ -I_n & 0_n \end{bmatrix}$$

be a skew-symmetric matrix where 0_n, I_n are n-dimensional zero and identity matrices, respectively. The $H_{2n\times 2n}$ is called a Hamiltonian matrix, if HJ is symmetric, such that $(HJ)^T = HJ$. It can be verified from the definition that Hamiltonian matrices exhibit a characteristic block structure form [8]:

$$H = \begin{bmatrix} H_{11} & H_{12} \\ H_{21} & -H_{11}^T \end{bmatrix}, \text{ where } H_{12} \text{ and } H_{21} \text{ are symmetric.}$$

Bisection algorithm

In control area we know that the eigenvalues express *stability of a system* where Hankel singular values describes *the energy of each state of the system*. The governing goal of the bisection algorithm is constructed on using the connection between stability of the system and the energy of the states such that to interrelate singular values of the transfer matrix evaluated along the imaginary axis and imaginary eigenvalues of related Hamiltonian matrix M_γ, for given system (1), which is defined:

$$\text{let } \gamma > 0,\ M_\gamma = \begin{bmatrix} A & 0 \\ 0 & -A^T \end{bmatrix} + \begin{bmatrix} B & 0 \\ 0 & -C^T \end{bmatrix} \begin{bmatrix} -D & \gamma I \\ \gamma I & -D^T \end{bmatrix}^{-1} \begin{bmatrix} C & 0 \\ 0 & B^T \end{bmatrix} =$$

$$= \begin{bmatrix} A - BR^{-1}D^TC & -\gamma BR^{-1}B^T \\ \gamma C^T S^{-1} C & -A^T + C^T DR^{-1}B^T \end{bmatrix} \tag{4}$$

where $R = D^T D - \gamma^2 I$ and $S = DD^T - \gamma^2 I$.

For special case:

$$D=0\,,\; M_\gamma = \begin{bmatrix} A & \dfrac{1}{\gamma}BB^T \\ -\dfrac{1}{\gamma}C^TC & -A^T \end{bmatrix}.$$

The related Hamiltonian matrix M_γ comes from the proof of the following *Theorem*.

Theorem 1. Suppose *A* has no imaginary eigenvalues, $\gamma > 0$ is not a singular value of *D* and $\omega_0 \in \mathbb{R}$. Then, γ is a singular value of $G(j\omega_0)$ if and only if $(M_\gamma - j\omega_0 I)$ is singular.

Here the system (1) need not be observable, controllable or stable. For proof and more details see [9]. From *Theorem 1* we can also get the following theorem.

Theorem 2. Let *A* be stable and $\gamma > \sigma_{max}(D)$. Then, $\|G\|_\infty \geq \gamma$ if and only if M_γ has purely imaginary eigenvalues (*i.e.* at least one). For proof see [9].

Bisection algorithm is based on *Theorem 2* that is, first we determine lower(γ_{lb}) and upper(γ_{ub}) bounds. We can choose $\gamma_{lb}=0$ and γ_{ub} is sufficiently large and maintain with the bisection protocol. But to get convenient bounds we must use Hankel singular values which derived by Enns [10] and Glover [11] are:

$$\gamma_{lb} = \max\left\{\sigma_{max}(D), \sigma H_1\right\} = \max\left\{\sigma_{\max}(D), \sqrt{Tr(W_{\mathcal{C}}W_{\mathcal{O}})/n}\right\}$$

$$\gamma_{ub} = \sigma_{\max}(D) + 2\sum_{j=1}^{n}\sigma H_j = \sigma_{\max}(D) + 2\sqrt{nTr(W_{\mathcal{C}}W_{\mathcal{O}})} \tag{5}$$

where σH_j s are the Hankel singular values and $W_{\mathcal{O}}$, $W_{\mathcal{C}}$ are observability and controllability Grammians of the system (1) respectively which can be evaluated by solving related Lyapunov equations (by using *lyap* command in MATLAB) are:

$$A^T W_{\mathcal{O}} + W_{\mathcal{O}}A + C^TC = 0$$

$$AW_{\mathcal{C}} + W_{\mathcal{C}}A^T + BB^T = 0 \tag{6}$$

Let *A* be stable and $\varepsilon > 0$ be error tolerance for system (1) then bisection algorithm is as follows;

Step 1. Calculate lower and upper bounds for bisection algorithm, where:

$$\gamma_{lb} = \max\{\sigma_{\max}(D), \sigma H_1\}$$

$$\gamma_{ub} = \sigma_{\max}(D) + 2\sum_{j=1}^{n}\sigma H_j$$

Step 2. Set $\gamma = (\gamma_{lb} + \gamma_{ub})/2$
If $\gamma_{ub} - \gamma_{lb} < \varepsilon/2$, end.
Step 3. Compute M_γ.
Step 4. Check eigenvalues of M_γ;
if there exists a purely imaginary eigenvalue set $\gamma_{lb} = \gamma$
else, set $\gamma_{ub} = \gamma$.

Applications to numerical examples

In this section, we demonstrated the effectiveness of the method on two examples.

Example 1. Lateral axis dynamic of an aircraft (L-1011) model. See [12] for more details.

Consider the system (1) with the appropriate parameter given:

$$A = \begin{bmatrix} -2.98 & 0.93 & 0 & -0.034 \\ -0.99 & -0.21 & 0.035 & -0.0011 \\ 0 & 0 & 0 & 1 \\ 0.39 & -5.555 & 0 & -1.89 \end{bmatrix}, \quad B = \begin{bmatrix} -0.032 \\ 0 \\ 0 \\ -1.6 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad D = 0$$

$$\dot{x} = \begin{bmatrix} -2.98 & 0.93 & 0 & -0.034 \\ -0.99 & -0.21 & 0.035 & -0.0011 \\ 0 & 0 & 0 & 1 \\ 0.39 & -5.555 & 0 & -1.89 \end{bmatrix} x + \begin{bmatrix} -0.032 \\ 0 \\ 0 \\ -1.6 \end{bmatrix} u$$

$$y = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} x$$

If the method given above is applied, we obtain the values in the tab. 1. In the table the last column is about checking for existence of purely imaginary eigenvalues. We write the title as Eig briefly.

Table 1. Related values of *Example 1*

Iteration	γ_{lb}	γ_{ub}	γ	Eig
1	4.6808	16.6071	10.6439	no
2	4.6808	10.6439	7.6624	no
3	4.6808	7.6624	6.1716	no
4	4.6808	6.1716	5.4262	no
5	4.6808	5.4262	5.0535	no
6	4.6808	5.0535	4.8671	no
7	4.6808	4.8671	4.7739	no
8	4.6808	4.7739	4.7273	no
9	4.6808	4.7273	4.7040	no
10	4.6808	4.7040	4.6924	no
11	4.6808	4.6924	4.6866	no
12	4.6808	4.6866	4.6837	no
13	4.6808	4.6837	4.6822	no
14	4.6808	4.6822	4.6815	yes
15	4.6815	4.6822	4.6818	no
16	4.6815	4.6818	4.6816	no

After 16 iterations γ_{lb} and γ_{ub} are so close but the error bound ε is not satisfied which was described in details in $[9]$, to get a valid ε we must continue iterating. On the other hand after 16 iterations all the γ values evaluated by MATLAB will be the same. Thus the number of iteration after the 16th will be unnecessary and we can terminate the process and say that $\|G(s)\|_\infty \approx 4.6816.$.

Example 2. A decentralized interconnected system. For additional details see [13].

Consider the system (1) with the appropriate parameter given:

$$A=\begin{bmatrix} -1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 1 & 0 & 0 & 0 \\ 1 & -2 & -1 & -1 & 1 & 1 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ -8 & 1 & -1 & -1 & -2 & 0 \\ 4 & -0.5 & 0.5 & 0 & 0 & -4 \end{bmatrix},\quad B=\begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},\quad C=\begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix},\quad D=0$$

$$\dot{x}=\begin{bmatrix} -1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 1 & 0 & 0 & 0 \\ 1 & -2 & -1 & -1 & 1 & 1 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ -8 & 1 & -1 & -1 & -2 & 0 \\ 4 & -0.5 & 0.5 & 0 & 0 & -4 \end{bmatrix}x+\begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}u$$

$$y=\begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}x$$

If the given method is applied, we obtain the values in the tab. 2. The last column of the table is also about checking for, *is there purely imaginary eigenvalue or not?*. The notation Eig is used for the same reason as in tab. 1.

After 20 iterations γ_{lb} and γ_{ub} are so close but the error bound ε is not satisfied which as told in *Example 1*. By the exactly same reason in the *Example 1* we can say that $\|G(s)\|_\infty \approx 29.6784$.

Conclusion

In this study, bisection method has been applied to a linear dynamic system for case $D = 0$ to find H_∞– norm of its transfer function. Two numerical examples has been solved and the results showed that the method works satisfactorily and error tolerance is sufficiently small after certain number of iterations. Sufficiency of this error tolerance stems from the number of decimal digits which MATLAB assigns automatically.

Table 2. Related values of *Example 2*

Iteration	γ_{lb}	γ_{ub}	γ	Eig
1	8.4201	101.0417	54.7309	no
2	8.4201	54.7309	31.5755	no
3	8.4201	31.5755	19.9978	yes
4	19.9978	31.5755	25.7867	yes
5	25.7867	31.5755	28.6811	yes
6	28.6811	31.5755	30.1283	no
7	28.6811	30.1283	29.4047	yes
8	29.4047	30.1283	29.7665	no
9	29.4047	29.7665	29.5856	yes
10	29.5856	29.7665	29.6761	yes
11	29.6761	29.7665	29.7213	no
12	29.6761	29.7213	29.6987	no
13	29.6761	29.6987	29.6874	no
14	29.6761	29.6874	29.6818	no
15	29.6761	29.6818	29.6790	no
16	29.6761	29.6790	29.6776	yes
17	29.6776	29.6790	29.6783	yes
18	29.6783	29.6790	29.6786	no
19	29.6783	29.6786	29.6784	no
20	29.6783	29.6784	29.6784	yes

References

[1] Zames, G., Feedback and Optimal Sensitivity: Model Reference Transformations, Multiplicative Semi-norms, and Approximate Inverses, *IEEE Transactions on Automatic Control*, *26* (1981), 2, pp. 301-320
[2] Zhou, K., Doyle, J. C., *Essentials of Robust Control,* Prentice Hall, Upper Saddle River, NJ, USA, 1998
[3] Petersen, I., Disturbance Attenuation and H_∞ optimization: A Design Method Based on the Algebraic Riccati Equation, *IEEE Transactions on Automatic Control*, *32* (1987), 5, pp. 427-429
[4] Khargonekar, P. P., *et al.*, H_∞-optimal Control with State-Feedback, *Transactions on Automatic Control, IEEE*, *33*(1988), 8, pp. 786-788
[5] Doyle, J., *et al.*, State-Space Solutions to Standard H_2 and H_∞ Control Problems, *Proceedings,* IEEE American Control Conference, Atlanta, Geo., USA, pp. 1691-1696, 1988
[6] Khargonekar, P. P., *et al.*, Robust Stabilization of Uncertain Linear Systems: Quadratic Stabilizability and H_∞ Control Theory, *IEEE Transactions on Automatic Control*, *35* (1990), 3, pp. 356-361
[7] Jiang, B., *et al.*, A Distributed Dynamic Event-Triggered Mechanism to HMM-Based Observer Design for H_∞ Sliding Mode Control of Markov Jump Systems, *Automatica*, *142* (2022), 110357
[8] Losse, P., *et al.*, The Modified Optimal H_∞ Control Problem for Descriptor Systems, *SIAM Journal on Control and Optimization*, *47* (2009), 6, pp. 2795-811
[9] Boyd, S., *et al.*, A Bisection Method for Computing the H_∞-Norm of a Transfer Matrix and Related Problems, *Mathematics of Control, Signals and Systems*, *2* (1989), 3, pp. 207-219

[10] Enns, D. F., Model Reduction with Balanced Realizations: An Error Bound and a Frequency Weighted Generalization, *Proceedings,* 23rd IEEE Conference on Decision and Control, Las Vegas, Nev., USA, pp. 127-132, 1984
[11] Glover, K., All Optimal Hankel-Norm Approximations of Linear Multivariable Systems and their L_∞-Error Bounds, *International Journal of Control*, *39* (1984), 6, pp. 1115-1193
[12] Galimidi, A. R., Barmish, B. R., The Constrained Lyapunov Problem and Its Application to Robust Output Feedback Stabilization, *Proceedings,* IEEE American Control Conference, Seattle, Wash, USA, pp. 433-439, 1985
[13] Saif, M., Guan, Y., Decentralized State Estimation in Large-Scale Interconnected Dynamical Systems, *Automatica*, *28* (1992), 1, pp. 215-219

Paper submitted: August 12, 2022
Paper revised: September 10, 2022
Paper accepted: September 25, 2022