ON OPTIMAL CONTROL OF THE INITIAL VELOCITY OF AN EULER-BERNOULLI BEAM SYSTEM

by

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In this study, we consider an optimal control problem for an Euler-Bernoulli beam equation. The initial velocity of the system is given by the control function. We give sufficient conditions for the existence of a unique solution of the hyperbolic system and prove that the optimal solution for the considered optimal control problem is exists and unique. After obtaining the Frechet derivative of the cost functional via an adjoint problem, we also give an iteration algorithm for the numerical solution of the problem by using the Gradient method. Finally, we furnish some numerical examples to demonstrate the effectiveness of the result obtained.

Key words: *hyperbolic system, beam equation, frechet derivative, optimal control*

Introduction

Vibration theory has many research applications in the area of applied science, especially, in fields of building, mechanical and aircraft engineering [1]. So it should be profitable to study the control problems associated with the beam systems. Various optimal control problems for the beam have been considered recently in the literature. The problems of controlling the coefficient function in the beam equation have been investigated in [2-5]. The boundary control problems for the beam system have been studied in [6-11]. When the control function is the source term, there have been some control problems [12-16].

In PDE, the problems of optimal control with the initial condition are studied for the different cost functional. There are some studies about the initial control for parabolic problem [17-19] and for hyperbolic problem [20-22]. Sarac [22] has controlled the initial velocity for wave equation $u_{tt} + a^2 u_{xx} = f(x,t)$ with homogeneous Neumann boundary conditions by using the following cost functional:

$$J_\alpha(v) = \int_0^l \left[u(x,T;v) - y(x)\right]^2 \mathrm{d}x + \alpha \int_0^l v^2(x)\mathrm{d}x$$

Kowalewski [20] has studied the control problem with the initial condition for the hyperbolic problem:

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$$\begin{aligned}
&u_{tt} + \left[\sum_{|\alpha|=0}^{\infty} (-1)^{|\alpha|} a_\alpha D^{2\alpha} + 1\right] u + u(x,t-h) = f(x,t),\ (x,t) \in \Omega(0,\mathrm{T}) \\
&u(x,t') = \Phi_0 u(x,t'), \quad x \in \Omega, \quad t' \in [-h,0) \\
&u(x,0) = 0, \quad u_t(x,0) = v, \quad x \in \Omega \\
&u(x,t) = 0, \quad x \in \Gamma, \quad t \in (0,\mathrm{T})
\end{aligned}$$

by minimizing the performance functional:

$$J(v) = \lambda_1 \int_\Omega |u(x,T;v) - z_\mathrm{d}|^2 \mathrm{d}x + \lambda_2 \int_\Omega N(v) v \mathrm{d}x$$

where $\lambda_1, \lambda_2 \ge 0$ with $\lambda_1 + \lambda_2 > 0;$ $z_d \in H^0(\Omega)$ is a given element; $N : H^0(\Omega) \to H^0(\Omega)$ is a positive linear operator. Kowalewski [21] has controlled the initial conditions for a linear hyperbolic system in which multiple time delays appear in the state equation.

In this paper, we consider a beam system given by:

$$\begin{aligned}
&u_{tt} + u_{xxxx} = F(x,t), \quad (x,t) \in \Omega{:=}(0,l) \times (0,T] \\
&u(x,0) = w(x), \quad u_t(x,0) = v(x), \quad x \in (0,l) \\
&u(0,t) = 0, \quad u_{xx}(0,t) = 0, \quad t \in (0,T] \\
&u(l,t) = 0, \quad u_{xx}(l,t) = 0, \quad t \in (0,T]
\end{aligned} \tag{1}$$

where the function $F(x, t)$ is the external load, $w(x)$ is the initial displacement, $v(x)$ is the initial velocity and l is the length of the beam. The deflection of the beam is denoted by $u(x,t)$ in the position x along beam and time t. We assume that $F \in L^2[0,T;L^2(0,l)],$ $w \in H^2(0,l)$ are given functions, $v \in L^2(0,l)$ is the control function and $u = u(x,t;v)$ is the solution of the problem (1) at (x, t) corresponding to a given control v.

Now, we recall an admissible controls set $V_{ad} := \{v \in L^2(0,l) : \|v\|_{L^2(0,l)} \le v_c\}$ as a closed and convex subset of Hilbert space $L^2(0,l)$, where v_c is a constant. The inner product and norm in this set will be defined in the same way as on $L^2(0,l)$.

We shall now formulate an optimal control problem whose solution gives unknown initial velocity v. The cost functional is given by:

$$\begin{aligned}
J_\alpha(v) = \lambda_1 &\int_0^l [u(x,T:v) - y_1(x)]^2 \mathrm{d}x \\
&+ \lambda_2 \int_0^l [u_t(x,T:v) - y_2(x)]^2 \mathrm{d}x + \alpha \int_0^l v^2(x) \mathrm{d}x
\end{aligned} \tag{2}$$

where $\lambda_1, \lambda_2 \ge 0$ with $\lambda_1 + \lambda_2 > 0;$ the functions $y_1, y_2 \in L^2(0,l)$ are given target functions; $\alpha > 0$ is a regularization parameter ensures the uniqueness of the solution.

The aim of this paper is to find the initial velocity from the set V_{ad} that minimizes the distances between the solutions $u(x,T), u_t(x,T)$ and desired target functions $y_1(x), y_2(x)$. In other words, our objective is solve the following optimal control problem:

$$J_\alpha(v_*) = \min_{v \in V_{ad}} J_\alpha(v) \tag{3}$$

The most commonly used beam models are based on Euler-Bernoulli beam theory. So the control of the initial conditions for Euler-Bernoulli beam systems is an important problem. This study makes an important contribution to the subject because we control one of the initial conditions of a beam problem and focus on numerical computations.

Solvability of the optimal control problem

Firstly, we give solvability of the problem (1) for given functions $F \in L^2[0,T;L^2(0,l)]$, $w \in H^2(0,l)$ and the control function $v \in V_{ad}$. The solution of the problem (1) is understood in the weak sense. The problem (1) has a unique weak solution $u \in L^2[0,T;V(0,l)]$, $u_t \in L^2[0,T;L^2(0,l)]$, $u_{tt} \in L^2[0,T;H^{-2}(0,l)]$ where:

$$V(0,l) := \{ f \in H^2(0,l) :\ f(0)=0,\ f(l)=0 \} \text{ [23-25].}$$

If we give an increment $\Delta v \in V_{ad}$ to the control function v such that $v+\Delta v \in V_{ad}$, the difference function $\Delta u = \Delta u(x,t;v)$ is the solution of the following difference problem:

$$\begin{aligned} &\Delta u_{tt} + \Delta u_{xxxx} = 0, \quad (x,t) \in \Omega \\ &\Delta u(x,0) = 0, \quad \Delta u_t(x,0) = \Delta v, \quad x \in (0,l) \\ &\Delta u(0,t) = 0, \quad \Delta u_{xx}(0,t) = 0, \quad t \in (0,T] \\ &\Delta u(l,t) = 0, \quad \Delta u_{xx}(l,t) = 0, \quad t \in (0,T] \end{aligned} \tag{4}$$

The following lemma will be used in the derivation of Gradient of the cost functional.

Lemma 1. Let Δu be the weak solution of the hyperbolic problem (4) and $F \in L^2[0,T;L^2(0,l)]$, $w \in H^2(0,l)$ and $v \in V_{ad}$. We have the following estimates:

$$\|\Delta u(x,T)\|^2_{L^2(0,l)} \le T^2 \|\Delta v\|^2_{L^2(0,l)}, \quad \forall v \in V_{ad} \tag{5}$$

and

$$\|\Delta u_t(x,T)\|^2_{L^2(0,l)} \le \|\Delta v\|^2_{L^2(0,l)}, \quad \forall v \in V_{ad} \tag{6}$$

Let's rewrite the cost functional:

$$J_\alpha(v) = J_0(v) + \alpha \|v\|^2_{L^2(0,l)}$$

where

$$J_0(v) = \lambda_1 \int_0^l [u(x,T:v) - y_1(x)]^2 \mathrm{d}x + \lambda_2 \int_0^l [u_t(x,T:v) - y_2(x)]^2 \mathrm{d}x$$

Using the estimates (5) and (6) and applying the Cauchy-Schwarz inequality, for increment $\Delta J_0(v) = J_0(v+\Delta v) - J_0(v)$ of the functional $J_0(v)$, we get the following inequality:

$$\Delta J_0(v) \le c_1 [\|\Delta v\|_{L^2(0,l)} + \|\Delta v\|^2_{L^2(0,l)}] \tag{7}$$

where c_1 depends on the constants λ_1, λ_2 and the final time T. The inequality (7) implies that the functional $J_0(v)$ is continuous (so lower semi-continuous). Also, the functional $J_0(v)$ is bounded from below since $J_0(v) \ge 0$ for any $v \in V_{ad}$. The admissible control set V_{ad} is a non-empty closed, bounded and convex subset of the Sobolev space $L^2(0,l)$. In view of Weierstrass's existence theorem, the optimal control problem (1)-(2) has a minimum for

$\alpha = 0.$ Moreover, the regularization parameter $\alpha > 0$ on the functional (2) establishes the uniqueness and continuous dependence to the solution.

Using the Lagrange multipliers method, we obtain the following adjoint problem:

$$\begin{aligned}
&\psi_{tt} + \psi_{xxxx} = 0, \quad (x,t)\Omega \\
&\psi(x,t) = -2\lambda_2[u_t(x,T;v) - y_2(x)], \quad x \in (0,l) \\
&\psi_t(x,T) = 2\lambda_1[u(x,T;v) - y_1(x)], \quad x \in (0,l) \\
&\psi(0,t) = 0, \quad \psi_{xx}(0,t) = 0, \quad t \in (0,t] \\
&\psi(l,t) = 0, \quad \psi_{xx}(l,t) = 0, \quad t \in (0,t]
\end{aligned} \tag{8}$$

Now, one can get the Frechet derivative of the cost functional by using the adjoint approach. The first variation $\Delta J_\alpha(v) = J_\alpha(v+\Delta v) - J_\alpha(v)$ of the $J_\alpha(v)$ can easily written:

$$\begin{aligned}
\Delta J_\alpha(v) = {} & \lambda_1 \int_0^l 2\left[u(x,T;v) - y_1(x)\right]\Delta u(x,T;v)\mathrm{d}x + \lambda_1 \left\|\Delta u(x,T)\right\|^2_{L^2(0,l)} + \\
& + \lambda_2 \int_0^l 2\left[u_t(x,T;v) - y_1(x)\right]\Delta u_t(x,T;v)\mathrm{d}x + \lambda_2 \left\|\Delta u_t(x,T)\right\|^2_{L^2(0,l)} + \\
& + 2\alpha \int_0^l v(x)\Delta v(x)\mathrm{d}x + \alpha \left\|\Delta v\right\|^2_{L^2(0,l)}
\end{aligned} \tag{9}$$

The $\Delta J_\alpha(v)$ also rewritten in The terms of the solution of the adjoint problem:

$$\begin{aligned}
\Delta J_\alpha(v) = {} & \langle -\psi(x,0;v) + 2\alpha v(x), \Delta v(x)\rangle_{L^2(0,l)} + \\
& + \left\|\Delta u(x,T)\right\|^2_{L^2(0,l)} + \left\|\Delta u_t(x,T)\right\|^2_{L^2(0,l)} + \alpha \left\|\Delta v\right\|^2_{L^2(0,l)}
\end{aligned} \tag{10}$$

The *Lemma 1* implies that the second term and third term in the right-hand side of the eq. (10) is bounded by $o[\left\|\Delta v\right\|^2_{L^2(0,l)}]$. Taking into account the definition of Frechet differential at $v \in V_{ad}$, we have:

$$J_\alpha^{'}(v) = -\psi(x,0;v) + 2\alpha v(x) \tag{11}$$

Here one can point out that the Frechet derivative of the cost functional can be obtained via the solution of the adjoint problem.

Numerical examples and results

We consider the numerical schemes for optimal control problem (1)-(2) after the theoretical results.

The regularization parameter α has a main role in minimization process. We perform two numerical examples to show the efficiently of our algorithm for different α's values.

Let's state an iteration procedure based on the previous analysis for a numerical approximation of the optimal control. This procedure is described as:

Step 1. Choose the initial value $v_0 \in V_{ad}$

Step 2. Solve the state problem (1) in the weak sense and get the u_n

Step 3. Solve the adjoint problem (8) and find the ψ_n

Step 4. Calculate the gradient $J_\alpha^{'}(v_n)$ from the formula (11)

Step 5. Find the new element v_{n+1} by using the following minimizing sequence;

$$v_{n+1} = v_n - \beta_n J_\alpha^{'}(v_n) \tag{13}$$

where β_n is the parameter of the algorithm assures that $J_\alpha[v_n - \beta_n J_\alpha^{'}(v_n)] < J_\alpha(v_n)$.

This iteration is stopped when the stopping criteria $J_{\alpha,}(v_n) - J_\alpha(v_{n+1}) < \varepsilon$ is satisfied

The stopping parameter ε is a positive constant). If $J_\alpha^{'}(v_n) = 0,$ then v_n is a stationary element for the minimizing problem and the iteration is stopped.

Example 1. Consider the following problem on the domain $\Omega : (0,1)(0,2]$:

$$\begin{aligned}
&u_{tt} + u_{xxxx} = [-\pi^2(x^4 - 2x^3 + x) + 24]\sin \pi t, \quad (x,t) \in \Omega \\
&u(x,0) = 0, \quad u_t(x,0) = v(x) \quad x \in (0,1) \\
&u(0,t) = 0, \quad u_{xx}(0,t) = 0, \quad t \in (0,2] \\
&u(1,t) = 0, \quad u_{xx}(1,t) = 0, \quad t \in (0,2]
\end{aligned} \tag{14}$$

Find $v_* \in V_{ad}$ such that:

$$J_* = J_\alpha(v_*) = \min_{v \in V_{ad}} J_\alpha(v)$$

where

$$\begin{aligned}
J_\alpha(v) = &\int_0^1 [u(x,2;v) - 0]^2 \mathrm{d}x + \int_0^1 [u_t(x,2;v) - \pi(x^4 - 2x^3 + x)]^2 \mathrm{d}x + \\
&+\alpha \int_0^1 v^2(x) \mathrm{d}x
\end{aligned} \tag{15}$$

Firstly, let us choose $\alpha = 0$ in (15) and take:

$$J_0(v) = \int_0^1 [u(x,2;v) - 0]^2 \mathrm{d}x + \int_0^1 [u_t(x,2;v) - \pi(x^4 - 2x^3 + x)]^2 \mathrm{d}x$$

In this case, the minimum value of $J_0(v)$ is $J_{0_*} = 0$ and the optimal solution is $v_* = \pi(x^4 - 2x^3 + x)$. Choosing $\beta_n = 0.05$ and the initial element $v_0 = 10x$, we get the value of the cost functional as $J_0(v_{200}) = 0.028770,$ the norm of the distance between the approximate solution v_{200} and the element v_* as $\|v_{200} - v_*\|_{L^2(0,1)} = 2.496586$ after 200 iterations.

For another initial element $v_0 = 7\cos(x),$ we get $J_0(v_{200}) = 0.025315$ and $\|v_{200} - v_*\|_{L^2(0,1)} = 2.060566$ after 200 iterations. We plot the graphs of these solutions obtained by starting the initial element $v_0 = 10x$ and $v_0 = 7\cos(x)$ in fig. 1.

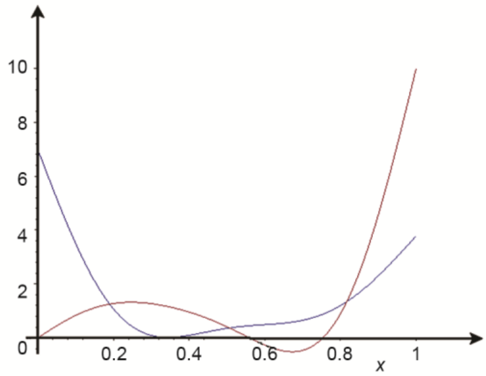

Figure 1. Graphs of the solutions for the initial element v_0 = 10x and v_0 = 7cos(x)

It can be seen from fig. 1 that the functions obtained for different initial elements are quite different. Moreover, the values of the cost functional for these elements are very close and quite small. In that case, the optimal control problem is ill-posed when $\alpha = 0$.

Now, we consider the optimal control problem (14)-(15) when $\alpha > 0$. If we take $\beta_n = 0.05$ and the stopping criteria $\varepsilon = 0.1\times 10^{-8}$ in the numerical algorithm, then we can get optimal control functions for different values of the regularization parameter *α*. In tab. 1, we give the values of the functional $J_0(v)$ and the norm $\|v\|^2_{L^2(0,1)}$ obtained by different initial elements for different values of the regularization parameter *α*.

It can be seen from tab. 1 that the numerical solutions obtained from three different initial elements are close to each other.

In tab. 2, we obtain some optimal solutions of the problem (14)-(15) by using the iteration process.

Table 1. The $J_0(v)$ and $\|v\|^2_{L^2(0,1)}$ values for some different initial elements and some *α*'s

	The initial element $v_0 = 1$		The initial element $v_0 = x$		The initial element $v_0 = x^4$	
α	$J_0(v)$	$v^2_{L^2(0,1)}$	$J_0(v)$	$v^2_{L^2(0,1)}$	$J_0(v)$	$v^2_{L^2(0,1)}$
0.1	0.00773909	0.31164341	0.00773909	0.31164339	0.00773909	0.31164339
0.5	0.05999964	0.09664608	0.05999968	0.09664601	0.05999976	0.09664584
1.0	0.09939388	0.04002670	0.09993943	0.04002626	0.09939508	0.04002550
1.5	0.12154265	0.02175446	0.12154304	0.02175420	0.12154438	0.02175331
2.0	0.13550151	0.01364301	0.13550225	0.01364268	0.13550420	0.01364170

Example 2. If we take the domain $\Omega := (0,1)\times(0,1]$ in the problem (1)-(2), we write the following problem:

$$J_\alpha(v) = \int_0^1 [u(x,1;v) - 3\sin\pi x]^2 \mathrm{d}x + \int_0^1 [u_t(x,2;v) - 3\sin\pi x]^2 \mathrm{d}x + \alpha\int_0^1 v^2(x)\mathrm{d}x \tag{16}$$

subject to:

$$\begin{aligned} & u_{tt} + u_{xxxx} = \sin\pi x[\pi^4(t^2+t+1)+2], \quad (x,t)\in\Omega \\ & u(x,0) = \sin\pi x, \quad u_t(x,0) = v(x), \quad x\in(0,1) \\ & u(0,t) = 0, \quad u_{xx}(0,t) = 0, \quad t\in(0,1] \\ & u(1,t) = 0, \quad u_{xx}(1,t) = 0, \quad t\in(0,1] \end{aligned} \tag{17}$$

Rewrite the cost functional:

$$J_\alpha(v) = J_0(v) + \alpha\|v\|^2_{L^2(0,1)}$$

where

$$J_0(v) = \int_0^1 [u(x,1;v) - 3\sin\pi x]^2 \mathrm{d}x + \int_0^1 [u_t(x,1;v) - 3\sin\pi x]^2 \mathrm{d}x$$

In this example we choose the related parameter $\beta_n = 0.05$ and the stopping parameter $= 0.1\times 10^{-8}$. We give the values $J_0(v)$ and $\|v\|^2_{L^2(0,1)}$ obtained by different initial elements for different *α* in tab. 3 and the optimal solutions for these initial elements in tab. 4.

Table 2. Some optimal controls for some different initial elements and some *α*'s

α	v_0	v_*
	$v_0 = 1$	$0.002131 + 0.786769\sin(\pi x) - 0.504424\times 10^{-9}\sin(2\pi x) + 0.000075\sin(3\pi x) - 0.221025\times 10^{-10}\sin(4\pi x)$
0.1	$v_0 = x$	$0.003091x + 0.787515\sin(\pi x) + 0.000984\sin(2\pi x) + 0.000279\sin(3\pi x) + 0.000208\sin(4\pi x)$
	$v_0 = x^4$	$0.003218x^4 + 0.788915\sin(\pi x) + 0.000713\sin(2\pi x) + 0.000326\sin(3\pi x) + 0.000199\sin(4\pi x)$
	$v_0 = 1$	$0.000334 + 0.439223\sin(\pi x) - 0.341831\times 10^{-9}\sin(2\pi x) + 0.000135\sin(3\pi x) - 0.499829\times 10^{-10}\sin(4\pi x)$
0.5	$v_0 = x$	$0.000504x + 0.439328\sin(\pi x) + 0.000161\sin(2\pi x) + 0.000147\sin(3\pi x) + 0.000011\sin(4\pi x)$
	$v_0 = x^4$	$0.000504x^4 + 0.282891\sin(\pi x) + 0.000153\sin(2\pi x) + 0.000152\sin(3\pi x) + 0.000010\sin(4\pi x)$
	$v_0 = 1$	$0.000159 + 0.282734\sin(\pi x) - 0.244914\times 10^{-9}\sin(2\pi x) + 0.000079\sin(3\pi x) - 0.257808\times 10^{-10}\sin(4\pi x)$
1.0	$v_0 = x$	$0.000218x + 0.282796\sin(\pi x) + 0.000069\sin(2\pi x) + 0.000083\sin(3\pi x) + 0.280945\times 10^{-5}\sin(4\pi x)$
	$v_0 = x^4$	$0.000242x^4 + 0.282891\sin(\pi x) + 0.000153\sin(2\pi x) + 0.000084\sin(3\pi x) + 0.284982\times 10^{-5}\sin(4\pi x)$
	$v_0 = 1$	$0.000081 + 0.208485\sin(\pi x) - 0.191068\times 10^{-9}\sin(2\pi x) + 0.000057\sin(3\pi x) - 0.174173\times 10^{-10}\sin(4\pi x)$
1.5	$v_0 = x$	$0.000131x + 0.208503\sin(\pi x) + 0.000041\sin(2\pi x) + 0.000058\sin(3\pi x) + 0.124247\times 10^{-5}\sin(4\pi x)$
	$v_0 = x^4$	$0.000154x^4 + 0.208555\sin(\pi x) + 0.000034\sin(2\pi x) + 0.000058\sin(3\pi x) + 0.132682\times 10^{-5}\sin(4\pi x)$
	$v_0 = 1$	$0.000068 + 0.165098\sin(\pi x) - 0.156693\times 10^{-9}\sin(2\pi x) + 0.000043\sin(3\pi x) - 0.131604\times 10^{-10}\sin(4\pi x)$
2.0	$v_0 = x$	$0.000106x + 0.165115\sin(\pi x) + 0.000033\sin(2\pi x) + 0.000044\sin(3\pi x) + 0.802496\times 10^{-6}\sin(4\pi x)$
	$v_0 = x^4$	$0.000085x^4 + 0.161561\sin(\pi x) + 0.000018\sin(2\pi x) + 0.000044\sin(3\pi x) + 0.607319\times 10^{-6}\sin(4\pi x)$

Table 3. The $J_0(v)$ and $\|v\|^2_{L^2(0,1)}$ values for some different initial elements and some α's

	The initial element $v_0 = 0$		The initial element $v_0 = x^2$		The initial element $v_0 = e^x$	
α	$J_0(v)$	$\|v\|^2_{L^2(0,1)}$	$J_0(v)$	$\|v\|^2_{L^2(0,1)}$	$J_0(v)$	$\|v\|^2_{L^2(0,1)}$
0.1	0.00486772	0.39678183	0.00485915	0.39686795	0.00485915	0.39686795
0.5	0.05890076	0.19233588	0.05888349	0.19237043	0.05888349	0.19237043
1.0	0.12374806	0.10110339	0.12372810	0.10105390	0.12372809	0.10105392
1.5	0.17120556	0.06213243	0.17119142	0.06214186	0.17119130	0.06214194
2.0	0.20589454	0.04203259	0.20588304	0.04208350	0.20588282	0.04203845

We can see from tab. 3 that the values of the cost functional and the minimizing elements in the *Example 2* are close to each other.

Table 4. Some optimal controls for some different initial elements and some α's

α	v_0	v_*
	$v_0 = 0$	$0.890821\sin(\pi x) - 0.507572\times10^{-11}\sin(2\pi x) +$ $+0.822976\times10^{-10}\sin(3\pi x) - 0.592222\times10^{-10}\sin(4\pi x)$
0.1	$v_0 = x^2$	$0.003317x^2 + 0.889662\sin(\pi x) + 0.000969\sin(2\pi x) -$ $-0.000672\sin(3\pi x) + 0.000527\sin(4\pi x)$
	$v_0 = e^x$	$0.001109e^x + 0.888533\sin(\pi x) + 0.000561\sin(2\pi x) -$ $-0.000865\sin(3\pi x) + 0.000301\sin(4\pi x)$
	$v_0 = 0$	$0.620219\sin(\pi x) + 0.144111\times10^{-10}\sin(2\pi x) +$ $+0.514902\times10^{-10}\sin(3\pi x) - 0.374614\times10^{-10}\sin(4\pi x)$
0.5	$v_0 = x^2$	$0.000531x^2 + 0.620073\sin(\pi x) + 0.000082\sin(2\pi x) -$ $-0.000107\sin(3\pi x) + 0.000084\sin(4\pi x)$
	$v_0 = e^x$	$0.000201e^x + 0.619843\sin(\pi x) + 0.000056\sin(2\pi x) -$ $-0.000156\sin(3\pi x) + 0.000054\sin(4\pi x)$
	$v_0 = 0$	$0.449519\sin(\pi x) + 0.138580\times10^{-10}\sin(2\pi x) +$ $+0.353502\times10^{-10}\sin(3\pi x) - 0.258358\times10^{-10}\sin(4\pi x)$
1.0	$v_0 = x^2$	$0.000218x^2 + 0.449481\sin(\pi x) + 0.000022\sin(2\pi x) -$ $-0.000043\sin(3\pi x) + 0.000034\sin(4\pi x)$
	$v_0 = e^x$	$0.000084e^x + 0.449382\sin(\pi x) + 0.000015\sin(2\pi x) -$ $-0.000065\sin(3\pi x) + 0.000022\sin(4\pi x)$
	$v_0 = 0$	$0.352512\sin(\pi x) + 0.118530\times10^{-10}\sin(2\pi x) +$ $+0.269790\times10^{-10}\sin(3\pi x) - 0.197570\times10^{-10}\sin(4\pi x)$

→

α	v_0	v_*
1.5	$v_0 = x^2$	$0.000131x^2 + 0.352489\sin(\pi x) + 0.000011\sin(2\pi x) - -0.000024\sin(3\pi x) + 0.000019\sin(4\pi x)$
	$v_0 = e^x$	$0.000058e^x + 0.352414\sin(\pi x) + 0.822448\times10^{-5}\sin(2\pi x) - -0.000043\sin(3\pi x) + 0.000015\sin(4\pi x)$
	$v_0 = 0$	$0.289939\sin(\pi x) + 0.101698\times10^{-10}\sin(2\pi x) + +0.218303\times10^{-10}\sin(3\pi x) - 0.160043\times10^{-10}\sin(4\pi x)$
2.0	$v_0 = x^2$	$0.000085x^2 + 0.289927\sin(\pi x) + 0.553812\times10^{-5}\sin(2\pi x) - -0.000015\sin(3\pi x) + 0.000012\sin(4\pi x)$
	$v_0 = e^x$	$0.000034e^x + 0.289885\sin(\pi x) + 0.412020\times10^{-5}\sin(2\pi x) - -0.000024\sin(3\pi x) + 0.881472\times10^{-5}\sin(4\pi x)$

Conclusion

As it is known, the vibration problems of beams can be used to describe many engineering phenomena, in particular, for building, mechanical and aircraft engineering. It is important to study optimal control of the initial condition for the beam. This paper investigates the theoretical and numerical studies regarding the controllability of the initial condition in the beam problem. The gradient of the cost functional to be minimized is derived via an adjoint problem. In order to find numerical solution of the problem (1)-(2), we propose an iteration process based on the gradient of the cost functional. In the numerical examples, we show that the regularized parameter α has an important role in minimizing process. When $\alpha = 0$, there may be two different solutions to the optimal control problem (1)-(2). The uniqueness and stability of the optimal solution for the problem (1)-(2) are achieved for $\alpha > 0$.

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