CERTAIN RESULTS OF STARLIKE AND CONVEX FUNCTIONS IN SOME CONDITIONS

by

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The theory of geometric functions was first introduced by Bernard Riemann in 1851. In 1916, with the concept of normalized function revealed by Bieberbach, univalent function concept has found application area. Assume $f(z)=z+\Sigma_{n\geq 2}^{\infty}(a_n z^n)$ *converges for all complex numbers z with* $|z|<1$, *and f(z)is one-to-one on the set of such z. Convex and starlike functions f(z) and g(z) are discussed with the help of subordination. The f(z) and g(z) are analytic in unit disc and* $f(0)=0,\ f'(0)=1$, *and* $g(0)=0$, $g'(0)-1=0$. *A single valued function f(z) is said to be univalent (or schlict or one-to-one) in domain* $D\subset\mathbb{C}$ *never gets the same value twice; that is, if* $f(z_1)-f(z_2)\neq 0$ *for all* z_1 and z_2 *with* $z_1\neq z_2$. *Let A be the class of analytic functions in the unit disk* $U=\{z:|z|<1\}$ *that are normalized with* $f(0)=0,\ f'(0)=1$. *In this paper we give the some necessary conditions for* $f(z)\in S^*[a,a^2]$ *and* $0\leq a^2\leq a\leq 1$

$$\frac{f'(z)(2^r-1)[1-f'(z)]+zf''(z)}{2^r[f'(z)]^2}$$

This condition means that convexity and starlikeness of the function f of order 2^{-r}.

Key words: *analytic, convex, starlike, unit disk*

Introduction

Let A be the class of analytic function in the unit disk that are normalized with $f(0)=0,\ f'(0)=1$. Function $f(z)=z+\Sigma_{n\geq 2}^{\infty}(a_n z^n)$ is said to be convex of order α in the open disk $U=\{z\in\mathbb{C}:|z|<1\}$ if $f(z)$ is analytic in D and satisfies:

$$1+Re\frac{zf'(z)}{f(z)}>\alpha$$

for some real $\alpha(0\leq\alpha<1)$. This family of functions was introduced by Robertson [1]. A function $f(z)=z+\Sigma_{n=2}^{\infty}(a_n z^n)$ is called starlike of order α in U.

If $f(z)$ in analytic and satisfies:

$$Re\frac{zf'(z)}{f(z)}>\alpha$$

in U for some real $\alpha(0\leq\alpha<1)$. This class was also introduced by [1] and we denote by $S^*(\alpha)$. Here [2] and [3] show that, if it is obtained as $f(z)\in K(\alpha)$, it will be $f(z)\in S^*(\alpha)$.

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If we denote the function class A then $f(z)$ is analytics on the unit disk $U=\{z:|z|<1\}$ and normalized by $f(0)=0$ and $f'(0)=1$ [4-6].

Moreover, let $f,g\in A$. Then we say that $f(z)$ is subordinate to $g(z)$, and we write $f(z)\prec g(z)$, , if function $t(z)$ in the unit disk U is analytic, so that $t(0)=0$, $|t(z)|<1$ and $f(z)=g[t(z)]$, with all $z\in U$. In particular, if $g(z)$ is univalent in the unit disk U then $f(z)\prec g(z)$ if and only if $f(0) = g(0)$ and $f(U)\subseteq g(U)$. If $0\le a^2\le a\le 1$ then important class is defined by:

$$S^*[a,a^2]=\left[f\in A:\frac{zf'(z)}{f(z)}\prec\frac{1+az}{1+a^2z}\right]$$

A new class $S^*[a,a^2]$ is defined at regular intervals using subordination properties. Here the $0\le a^2\le a\le 1$ is arranged for the desired range.

In geometric terms, this means that the diameter U is inside open unit disk centered on real axis with endpoints $(1-a)/(1-a^2)=1/(1+a)$ and $(1+a)/(1+a^2)$ by $[zf'(z)]/[f(z)]$. Special selection of a and a^2 takes us to the following classes:

$S^*[0,1]\equiv S^*$ is in the class of starlike:

$$S^*(1-2^{1-r},-1)\equiv S^*(2^{-r}),\ 0<2^{-r}\le 1$$

is the class of starlike functions of order 2^{-r}. Additionally, $K^*(2^{-r})$, $0<2^{-r}\le 1$, is the class of convex functions of order 2^{-r}, identified by $f(z)\in K(2^{-r})$ if and only if $zf'(z)\in S^*(2^{-r})$ so:

$$\operatorname{Re}\left[1+\frac{zf''(z)}{f'(z)}\right]>2^{-r}\quad z\in U$$

In this article we will study the class:

$$O_{\mu,2^{-r}}=\left\{f\in A:\left|\frac{f'(z)(2^r-1)[1-f'(z)]+zf''(z)}{2^r[f'(z)]^2}\right|<\mu,\ z\in U\right\}$$

$0<2^{-r}\le 1$, $\mu>0$ and give sufficient conditions to embed it in classes $S^*[a,a^2]$ and $K(2^{-r})$, $0<2^{-r}<1$. The results we have obtained will be compared with the previously known results. To this end, we will use the following lemma from first-order differential subordinates theory [7-10].

Instructions

Lemma 1. Let $q(z)$ be univalent in the unit disk U, and let $\theta(w)$ and $\phi(w)$ be analytic in a domain D containing $q(U)$, with $\phi(w)\ne 0$ when $w\in q(U)$. Set $Q(z)=zq'(z)\phi[q(z)]$, $h(z)=\theta[q(z)]+Q(z)$ and if we assume:

i. $Q(z)\in S^*$; and

ii. $$\operatorname{Re}\frac{zh'(z)}{Q(z)}=\operatorname{Re}\left\{\frac{\theta'[q(z)]}{\phi[q(z)]}+\frac{zQ'(z)}{Q(z)}\right\}>0,\ z\in \mathrm{U}$$

$p(z)$ is analytic in U, with $p(0)=q(0)$, $p(U)\subseteq D$ and:

$$\theta[p(z)]+zp'(z)\phi[p(z)]\prec\theta[q(z)]+zq'(z)\phi[q(z)]=h(z)\tag{1}$$

then $p(z)\prec q(z)$ and $q(z)$ is the best dominant of (1) [11].

Lemma 2. Let the $h(z)$ analytic functions with $h(0) = 1$ be class A. Open unit disk $U = U^* \cup \{0\}$ is analytic. Hence:

$$\mathrm{Re}h(z) > 0, \quad (z \in U)$$

We can say that $f(z)$ and $g(z)$ analytic in the unit disk U ,f is subordined to g and written as $f \prec g$;

$$f(z) \prec g(z), \quad (z \in U)$$

If class A has an analytic function of $v(z)$, it is $v(z) \le |z|$ and $f(z) = g[v(z)]$. Also, if the function g is univalent in unit disk U then:

$$f(z) \prec g(z) \Leftrightarrow f(0) = g(0) \text{ and } f(U) = g(U), \quad (z \in U)$$

Let $h(0) = 1$ and $h(z)$ is analytic function then $h(z)$ is in class A. The $U = U^* \cup \{0\}$ is analytic in the open unit disk and where:

$$\mathrm{Re}h(z) > 0, \quad (z \in U^*)$$

If $w(z)$ is an analytic function, we write $w(z) \le |z|$ and $f(z) = g[w(z)], \quad (z \in U^*)$. So if function $g(z)$ is univalent in U then [12]:

$$f(z) \prec g(z) \Leftrightarrow f(0) = g(0) \text{ and } f(U) \subseteq g(U), \quad (z \in U^*)$$

Main results and consequences

We will show our proof for the following result by using *Lemma 1* and *Lemma 2* with the results we obtained initially.

Theorem. Let $f \in A$, $0 \le a < 1$ and $(1+|a|)/(3+|a|) \le 2^{-r} \le 1$.

If

$$\frac{f'(z)(2^r - 1)[1 - f'(z)] + zf''(z)}{2^r [f'(z)]^2} \prec 2^{-r} + (1 + 2^{1-r}) \frac{1 + a^2 z}{1 + az} + \frac{2^{-r} z(a - a^2)}{(1 + az)^2} =$$

$$= \frac{1 + a^2 z}{1 + az} + \frac{a^2 z^2 (1 + 2a) + az(5 + a) + 3}{2^r (1 + az)^2} \equiv h(z)$$

then $f \in S^*[a, a^2]$. In this way, we get our result.

Proof: We choose $r(z) = [f(z)]/[zf''(z)]$, $s(z) = (1 + a^2 z)/(1 + az)$,

The $\vartheta(t) = (1 - 2^{-r})t + 2^{-r}$ and $\theta(t) = -2^{-r}$. Then $s(z)$ is convex, thus univalent, because $1 + [zs''(z)]/[s'(z)] = (1 - az)/(1 + az)$; $\vartheta(t)$ and $\theta(t)$ are analytic in the domain $D = \mathbb{C}$ which contains $s(U)$ and $\theta(t)$ when $t \in s(U)$. Further:

$$P(z) = zs'(z)\theta[s(z)] = \frac{2^{-r}(a - a^2)z}{(1 + az)^2}$$

is starlike because $[zP'(z)]/[P(z)] = (1 - az)/(1 + az)$. Further:

$$h(z) = \vartheta[s(z)] + P(z) = \frac{1 + a^2 z}{1 + az} + \frac{a^2 z^2 (2a + 1) + az(a + 5) + 3}{2^r (1 + az)^2}$$

and

$$\operatorname{Re}\frac{zh'(z)}{P(z)} = \operatorname{Re}\left(1-2^{r}+\frac{2}{1+az}\right) > 1-2^{r}+\frac{2}{1+|a|}, \quad z\in U$$

which is equal to or greater than zero, if and only if $2^{-r} \geq (1+|a|)/(3+|a|)$. From *Lemma*, it follows that $r(z) \prec s(z)$; *i.e.* $f\in S^{*}[a,a^{2}]$. The result is sharp as the functions ze^{az} and $z(1+a^{2}z)^{1/a}$ show in the cases $a^2 = 0$ and $a \neq 0$, respectively.

Remark 1: According to the definition of subordination, the result of *Theorem 1* means that $h(U)$ is the largest region in the complex plane:

$$\frac{f'(z)(2^{r}-1)\left[1-f'(z)\right]+zf''(z)}{2^{r}[f'(z)]^{2}} \in h(U) \text{ for all } z\in U$$

then $f(z)\in S^{*}[a,a^{2}]$.

The following corollary embeds $O_{\mu,2^{-r}}$ into $S^{*}[a,a^{2}]$.

Corollary. $O_{\mu,2^{-r}} \subseteq S^{*}[a,a^{2}]$ when $(1+|a|)/(3+|a|) \leq 2^{-r} \leq 1$ and:

$$\mu = \frac{(a-a^{2})(1-2^{1-r})|a|-(1-3.2^{-r})}{(1+|a|)^{2}}$$

This result is certain, that is, given μ is the largest, so inclusion is valid.

Proof. To prove this result, due to *Theorem 1*, it is sufficient to show that: $\mu = \min\{|h(z)-(1-2^{-r})| : |z=1|\} \equiv \hat{\mu}$, , where $h(z)$ is defined as in the expression of the theorem and:

$$h(z)-(1-2^{-r}) = -az(1-a)\frac{a(1-2^{1-r})z-3.2^{-r}+1}{(1+az)^{2}}$$

Moreover, let:

$$\varphi(v) \equiv \left|h(e^{i\beta\pi/2})-(1-2^{-r})\right|^{2} =$$

$$= -(a-a^{2})^{2}\,\frac{[(1-2^{1-r})a^{2}+(2-3.2^{1-r})(1-2^{1-r})av+(1+3.2^{-r})^{2}]}{(1+2av+a^{2})^{2}}$$

$v=\cos(\beta\pi/2)\in[-1,1]$. Thus $\hat{\mu}=\min\{\sqrt{\varphi(v)} : -1\leq v\leq 1\}$.

If $2^{-r} \leq 1/2$ then $1-2^{1-r} \geq 0$ and having in mind that $1-3.2^{-r} \leq -[(2|a|)/(3+|a|)] \leq 0$ we receive that $\varphi(v)$ is a function with a certain univalent and:

$$\hat{\mu} = \min\left\{\sqrt{\varphi(-1)},\sqrt{\varphi(-1)}\right\} = \min\left\{\left|h(-1)-(1-2^{-r})\right|,\left|h(1)-(1-2^{-r})\right|\right\} = \mu$$

The last equality is valid because $1-3.2^{-r}+a(1-2^{1-r})z \geq 0$ is equivalent to:

$$2^{-r} \geq \frac{1+|a|}{3+|a|} \geq \frac{1-|a|}{3-2|a|}$$

If $2^{-r} > 1/2$ we have the following analysis. Equation $\varphi'_v(v) = 0$ has the uniqueness of the solution:

$$v_* = -\frac{a^2(1-2^{-r})(1-2^{1-r}) + (1-3.2^{-r})(1-2^{2-r})}{2a(1-2^{1-r})(1-3.2^{-r})}$$

It can be verified that $|v_*| > 1$ is equivalent to:

$$\xi(a,2^{-r}) \equiv a^2(1-2^{-r})(1-2^{1-r}) - 2|a|(1-2^{1-r})(1-3.2^{-r}) + (1+3.2^{-r})(1-2^{2-r}) > 0$$

Now, $\xi(a,2^{-r})$ is decreasing function of $|a| \in [0,1]$ that means $\xi(a,2^{-r}) \ge \xi(1,2^{-r}) = 2^{1-2r} > 0$. Thus $|v_*| > 1$ that means $\varphi(v)$ is a univalent function on $[-1, 1]$ leading to:

$$\hat{\mu} = \min\{\sqrt{\varphi(v)} : -1 \le v \le 1\} =$$

$$= \min\left\{\sqrt{\varphi(-1)}, \sqrt{\varphi(1)}\right\} = \min\left\{\left|h(-1) - (1-2^{-r})\right|, \left|h(1) - (1-2^{-r})\right|\right\}$$

At the end, the function:

$$\gamma(a,2^{-r}) \equiv \left|h(1) - (1-2^{-r})\right| - \left|h(-1) - (1-2^{-r})\right| = 2^{1-r}\frac{1-a^2-2^{1-r}(2-a^2)}{(1+a)^2(1-a)^2}$$

has the inverse sign of the sign of coefficient A. Therefore:

$$\hat{\mu} = \begin{Bmatrix} \left|h(1) - (1-2^{-r})\right|, & a \ge 0 \\ \left|h(-1) - (1-2^{-r})\right|, & a < 0 \end{Bmatrix} = \mu$$

The sharpness of the result is due to the clarity of *Theorem 1* (in *Remark 1*) and the fact that the resulting μ is the largest that embeds the disk $\left|t - (1-2^{-r})\right| < \mu$ in $h(U)$. The example below shows some concrete conclusions that can be drawn from the results of the previous section by specifying the $2^{-r}, a, a^2$ values.

Example 1. Let $0 \le a^2 \le a \le 1$:

i) $$O_{\mu,\frac{1}{2}} \subseteq S^*[a,a^2] \text{ when } \mu = \frac{a-a^2}{2(1+|a|)^2}$$

ii) $$O_{\mu,1} \subseteq S^*[a,a^2] \text{ when } \mu = (a-a^2)\frac{2-|a|}{2(1+|a|)^2}$$

iii) $$O_{\mu,\frac{1}{2-\beta}} \subseteq S^*[a,a^2] \text{ when } \beta \ge -\frac{1-|a|}{1+|a|} \text{ and } \mu = (a-a^2)\frac{1+\beta-\beta|a|}{2(1+|a|)^2}$$

iv) $$O_{\mu,2^{-r}} \subseteq S^*[a,a^2] \text{ when } \frac{1}{2} \le 2^{-r} \le 1 \text{ and } \mu = \frac{1}{2^{1+r}}$$

v) $O_{\mu,2^{-r}} \subseteq S^*[a,a^2] \subset S^*(1/1-a^2)$ when $\frac{1}{3} \le 2^{-r} \le 1$, $0<a^2 \le 1$ and $\mu = a^2(1-3.2^{-r})$.

Given μ is the greatest value so inclusions are valid.

Remark 2. The result from *Example 1* (i) is the same as in *Corollary* in [13]. Also, for $2^{-r}=1/2$ in *Example 1* (v) we receive the same result as in *Theorem 1* from [14]. Eventually, for $2^{-r}=1$ and $a^2=1$ in *Example 1* (v) we receive the same result as in *Corollary* from [15]. Next theorem studies connection between $O_{\mu,2^{-r}}$ and the class of convex functions of some order.

Theorem 2. $O_{\mu,2^{-r}} \subseteq K(2-2^r)$ when $1/2 \le 2^{-r} \le 1$ and:

$$\mu = \frac{(1-2^{-r})(3.2^{-r}-1)}{\sqrt{2(5.2^{-2r}-2^{2-r}+1)}}$$

Proof. Let $f \in O_{\mu,2^{-r}}$ and $a^2=\mu/(1-3.2^{-r})$. Then, by *Example 1* (v) we have $f \in S^*[0,a^2]$, *i.e.*:

$$\left|\frac{f(z)}{zf'(z)}-1\right| < a^2, \quad z \in U$$

Further:

$$1+\frac{zf''(z)}{f'(z)}-(2-2^r)=\frac{zf'(z)}{2^{-r}f(z)}\frac{1-2^{-r}+\dfrac{2^{-r}zf''(z)}{f'(z)}}{\dfrac{zf'(z)}{f(z)}}$$

and for all $z \in U$ we obtain:

$$\left|\arg\left[1+\frac{zf''(z)}{f'(z)}-2+2^r\right]\right| \le \left|\arg\frac{zf'(z)}{f(z)}\right| + \left|\arg\frac{1-2^{-r}+\dfrac{2^{-r}zf''(z)}{f'(z)}}{\dfrac{zf'(z)}{f(z)}}\right|$$

$$\le \arcsin a^2 + \arcsin\frac{\mu}{1-2^{-r}} = \arcsin\left[\frac{\mu}{1-2^{-r}}\sqrt{1-a^4}+a^2\sqrt{1-\frac{\mu^2}{[(1-a)^2}}\right] = \arcsin 1 = \frac{\pi}{2}$$

so $f \in K(2-2^r)$.

Example 2. For $2^{-r}=1/2$ and $2^{-r}=1/(2-\beta)$ from the previous theorem with the results we obtained:

i) $$O_{\mu,\frac{1}{2}} \subseteq K \text{ when } \mu=\frac{\sqrt{2}}{4}$$

ii) $$O_{\mu,\frac{1}{2-\beta}} \subseteq K(\beta) \text{ when } 0 \le \beta < 1 \text{ and } \mu=\frac{1-\beta^2}{(2-\beta)\sqrt{2(1+\beta^2)}}.$$

Remark 3. By putting $2^r=2-\beta$, $0 \le \beta < 1$, we receive the result from *Theorem 2* in [16].

Conclusion

In this study, the principles of functions starlike and convex are investigated. In addition, analytical functions and their properties are discussed. Convexity of analytic functions are examined and proofs for 2^{-r} grade cases are obtained. By considering functions K and S, proofs of these functions are obtained by subordination. The relationship and transformations between the $O_{\mu,2}{}^{-r} \subseteq K(2-2^r)$ shaped $O_{\mu,2}{}^{-r}$ and $K(2-2^r)$ were obtained by subordination method. Examples of these are shown. A new subclass $O_{\mu,2}{}^{-r}$ has been created.

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