A LIE SYMMETRY APPROACH FOR A NON-LINEAR ORDINARY DIFFERENTIAL EQUATION ARISING IN ENGINEERING SCIENCES

by

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A non-linear second order ODE has some applications in engineering problems. In physics, it arises in the modeling of the flux of a heated compressible fluid through a long slender tube. In this paper, we consider a non-linear second order differential equation whose analytic solution cannot be obtained directly. Therefore, we first find the canonical transformations to rewrite the equation in terms of canonical variables by using the Lie symmetry approach. We then reduce the order of the equation, which is a first type Abel equation, to one by defining a new variable.

Key words: *canonical form, non-linear ODE, Abel equation, Lie symmetry*

Introduction

The famous Norwegian mathematician Sophus Lie first developed the method of Lie groups and the symmetry method to find the solutions of non-linear differential equations, since obtaining the analytical solutions of these types of equations is not feasible by using most standard methods. The Lie symmetry method allows the systematic construction of the solutions of non-linear equations of both PDE and ODE. We also use Lie symmetry method not only to find the exact solutions of non-linear ODE, but also to reduce the order of the differential equation. In the case of PDE Lie symmetries are used to reduce a PDE to an ODE by appropriate reduction variables. The admitted symmetries of differential equations are used to find the reductions of given differential equations [1-5]. For more information on Lie groups please see Ibragimov [6, 7], Olver [8], Bluman and Kumei [9], Dresner [10], Bordag [11], Contwell [12], and Stephani [13].

Invariance of an ODE

Let:

$$y^n = f(x, y, y^{'}, y^{''}, ..., y^{n-1}) \tag{1}$$

be an n^{th} order ODE a surface in $(x, y, y^{'}, y^{''}, ..., y^{n-1}, y^n)$ -space is defined by the previosu ODE.

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Definition 1. The 1-parameter Lie group of transformations, [9]:

$$\bar{x} =: X(x, y, a) \tag{2}$$

$$\bar{y} =: Y(x, y, a) \tag{3}$$

leave ODE (1) invariant if and only if its n^{th} extension leaves the surface (1) invariant.

Theorem 1. (Infinitesimal Criteria for Invariance of an ODE, [9])

Suppose:

$$X = \xi(x, y)\frac{\partial}{\partial x} + \eta(x, y)\frac{\partial}{\partial y} \tag{4}$$

be an infinitesimal generator for (2) and (3). Let:

$$X^{(n)} = \xi(x, y)\frac{\partial}{\partial x} + \eta(x, y)\frac{\partial}{\partial y} + \zeta_1(x, y, y^{'})\frac{\partial}{\partial y^{'}} + ... + \zeta_n(x, y, y^{'}, ..., y^{n})\frac{\partial}{\partial y^{(n)}} \tag{5}$$

be the extension for infinitesimal generator of order n (4) where $\zeta_k(x, y, y^{'}, ..., y^{k})$ is inclined by:

$$\zeta_k(x, y, y^{'}, ..., y^{k}) = \frac{D\zeta_{k-1}}{Dx} - y^{k}\frac{D\xi(x, y)}{Dx} \tag{6}$$

with respect to $[\xi(x, y), \eta(x, y)]$ while $k = 1, 2, 3, ...$ where:

$$\zeta_0 = \eta(x, y) \tag{7}$$

Then, (2) and (3) is admitted by (1) if and only if:

$$X^{(n)}\{y^{(n)} - f[x, y, y^{'}, ..., y^{(n-1)}]\} = 0 \tag{8}$$

i.e.

$$\zeta_n[x, y, y^{'}, ..., y^{(n)}] - X^{(n-1)} f[x, y, y^{'}, ..., y^{(n-1)}] = 0 \tag{9}$$

where $y^{n} = f[x, y, y^{'}, ..., y^{(n-1)}]$.

The infinitesimal criterion for invariance of an ODE leads to an algorithm to determine the infinitesimals $\xi(x, y)$ and $\eta(x, y)$ admitted by a given ODE. Lie infinitesimal criterion leads to the so-called determining equations which is a linear PDE system in the unknown co-ordinates of the invariance operator. Generally the determining equations is an overdetermined system which becomes very diffucult to solve when arbitrary functions appear.

In this paper, we consider the following second order non-linear ODE:

$$3y^{''} + xyy^{'} - y^2 = 0 \tag{10}$$

Since the analytic solution of this equation cannot be obtained directly, we apply the second extension $X^{(2)}$ of the infinitesimal generator X to obtain the corresponding Lie symmetries of the equation. Then, $X^{(2)}$ is given as:

$$X^{(2)} = \xi(x, y)\frac{\partial}{\partial x} + \eta(x, y)\frac{\partial}{\partial y} + \zeta_1(x, y, y^{'})\frac{\partial}{\partial y^{'}} + \zeta_2(x, y, y^{'}, y^{''})\frac{\partial}{\partial y^{''}} \tag{11}$$

If we apply $X^{(2)}$ to the ODE, we get the following equations:

$$X^{(2)}(3y^{''}+xyy^{'}-y^2)=0$$

$$3\zeta_2+xy\zeta_1+yy^{'}\xi+\eta(xy^{'}-2y)=0 \tag{12}$$

In the last equation if we write ζ_1 and ζ_2 we get:

$$yy^{'}\xi+(xy^{'}-2y)\eta+xy[\eta_x+y^{'}(\eta_y-\xi_x)-\xi_y y^{'2}]+$$
$$+3[\eta_{xx}+(2\eta_{xy}-\xi_{xx})y^{'}+(\eta_{yy}-2\xi_{xy})y^{'2}-$$
$$-\xi_{yy}y^{'3}+y^{''}(\eta_y-2\xi_x)-3y^{'}\xi_y]=0 \tag{13}$$

In eq. (13) we use the symmetry condition where the symmetry condition is:

$$y^{''}=\frac{y^2-xyy^{'}}{3} \tag{14}$$

Putting the symmetry condition in eq. (13) leads the following equation:

$$yy^{'}\xi+(xy^{'}-2y)\eta+xy[\eta_x+y^{'}(\eta_y-\xi_x)-\xi_y y^{'2}]+3[\eta_{xx}+(2\eta_{xy}-\xi_{xx})y^{'}+$$
$$+(\eta_{yy}-2\xi_{xy})y^{'2}-\xi_{yy}y^{'3}+\frac{1}{3}(y^2-xyy^{'})+$$
$$+(\eta_y-2\xi_x-3y^{'}\xi_y)]=0 \tag{15}$$

If we equate the coefficients of the powers of *y'* to zero then we get the following equations:

$$(y^{'})^3:\ \xi_{yy}=0 \tag{16}$$

$$(y^{'})^2:3\eta_{yy}-6\xi_{xy}+2xy\xi_y=0 \tag{17}$$

$$(y^{'})^1:y\xi+x\eta+xy\xi_x+6\eta_{xy}-3\xi_{xx}-3y^2\xi_y=0 \tag{18}$$

$$(y^{'})^0:-2y\eta+xy\eta_x+3\eta_{xx}+y^2(\eta_y-2\xi_x)=0 \tag{19}$$

We start from eq. (16) and integrating it twice with respect to y gives:

$$\xi(x,y)=a(x)y+b(x) \tag{20}$$

where $a(x)$ and $b(x)$ are arbitrary functions of x. If we use the result which we get for $\xi(x, y)$ in eq. (20) by taking the necessary derivation of $\xi(x, y)$ in eq. (17) we get:

$$3\eta_{yy}-6a^{'}+2xya=0 \tag{21}$$

Now, similarly if we use the result which we get for $\xi(x, y)$ in eq. (20) by taking the necessary derivatives of $\xi(x, y)$ in eq. (18) leads to the following equation:

$$x\eta + 6\eta_{xy} - 2y^2 a + yb + xyb' + xy^2 a' - 3ya'' - 3b'' = 0 \tag{22}$$

From eq. (21) if we write:

$$3\eta_{yy} = 6a' - 2xya$$

and integrate it with respect to y two times gives:

$$\eta(x,y) = a' y^2 - \frac{1}{9} xy^3 a + c(x)y + d(x) \tag{23}$$

where $c(x)$ and $d(x)$ are arbitrary functions of x.

Now taking necessary derivatives of $\eta(x, y)$ in (23) and put in eq. (22) gives:

$$\begin{aligned} &xa' y^2 - \frac{1}{9} x^2 ay^3 + cxy + dx + 12a'' y - \\ &-2ay^2 - 2xa' y^2 + 6c' - 2ay^2 + by + \\ &+xb' y + xa' y^2 - 3a'' y - 3b'' = 0 \end{aligned} \tag{24}$$

Equation (24) is a polynomial in powers of y where the coefficients are functions of x. In eq. (24) we can equate the coefficients of powers of y to the zero:

$$y^3 : \frac{1}{9} x^2 a = 0 \tag{25}$$

$$y^2 : 4a = 0 \tag{26}$$

$$y^1 : cx + 12a'' + b + xb' - 3a'' = 0 \tag{27}$$

$$y^0 : dx + 6c' - 3b'' = 0 \tag{28}$$

From eqs. (25) or (26), $a(x)$ is easily determined as:

$$a(x) = 0 \tag{29}$$

By using this result, we can reduce (27) to the following form:

$$cx + b + xb' = 0 \tag{30}$$

Before we solve eq. (30), first of all we use (29) in eqs. (20) and (23) then $\xi(x, y)$ and $\eta(x, y)$ have the following forms:

$$\xi(x,y) = b(x) \tag{31}$$

$$\eta(x,y) = c(x)y + d(x) \tag{32}$$

Now, If we use the results in the (31) and (32) into eq. (19) then it gives:

$$-2cy^2 - 2dy + xc' y^2 + xd' y + 3c'' y + 3d'' + cy^2 - 2b' y^2 = 0 \tag{33}$$

If we write eq. (33) as a polynomial in terms of powers of y, and equate the coefficients of powers of y to zero, we have the following equations:

$$y^2 : -2c + xc' + c - 2b' = 0 \tag{34}$$

$$y^1 : -2d + xd' + 3c'' = 0 \tag{35}$$

$$y^0 : d'' = 0 \tag{36}$$

From eq. (36), $d(x)$ can be found as:

$$d(x) = k_1 x + k_2, \quad k_1, k_2 \in \mathbb{R} \tag{37}$$

Using (37) in eq. (35) gives:

$$3c'' = k_1 x + 2k_2. \tag{38}$$

If we integrate (38) with respect to x two times we get for $c(x)$ following:

$$c(x) = \frac{1}{18}k_1 x^3 + \frac{1}{3}k_2 x^2 + k_3 x + k_4, \quad k_3, k_4 \in \mathbb{R} \tag{39}$$

If we use (39) in eq. (34), and solve it for $b(x)$ we obtain:

$$b(x) = \frac{1}{72}k_1 x^4 + \frac{1}{18}k_2 x^3 - \frac{1}{2}k_4 x + k_5, \quad k_5 \in \mathbb{R} \tag{40}$$

Now, we will use the results for $b(x)$, $c(x)$, and $d(x)$, which we get respectively in (40), (39) and (37), in eqs. (28) and (30), respectively.

First putting these results in eq. (30) leads to the following equation:

$$\frac{9}{72}k_1 x^4 + \frac{10}{18}k_2 x^3 + k_3 x^2 + k_5 = 0 \tag{41}$$

From (41) we can easily determine that:

$$k_1 = k_2 = k_3 = k_5 = 0 \tag{42}$$

Using (42), $b(x)$, $c(x)$, and $d(x)$ are calculated as:

$$\begin{aligned} b(x) &= -\frac{1}{2}k_4 x \\ c(x) &= k_4 \\ d(x) &= 0 \end{aligned} \tag{43}$$

If we use the results in (43) in eq. (28) we get:

$$xd + 6c' - 3b'' = x \cdot 0 + 6 \cdot 0 - 3 \cdot 0 = 0$$

This shows that eq. (28) is also satisfied. Finally $\xi(x, y)$ and $\eta(x, y)$ are found as:

$$\begin{aligned} \xi(x,y) &= -\frac{1}{2}k_4 x \\ \eta(x,y) &= k_4 y \end{aligned} \tag{44}$$

Infinitesimal symmetry generator of the ODE is:

$$X = -\frac{1}{2}k_4 x\frac{\partial}{\partial x} + k_4 y\frac{\partial}{\partial y}$$

For instance, if we choose $k_4 = 2$ then the infinitesimal symmetry generator takes the following form:

$$X = -x\frac{\partial}{\partial x} + 2y\frac{\partial}{\partial y} \tag{45}$$

The solution of the two PDE:

$$\begin{aligned} Xr &= 0 \\ Xs &= 1 \end{aligned} \tag{46}$$

gives the canonical transformations which leave the ODE invariant. The solutions of the PDE in (46) are obtained as:

$$x = e^{-s}, \quad y = re^{2s} \tag{47}$$

Under the transformations in (47), y' and y'' are transformed as:

$$\begin{aligned} y' &= -\frac{e^{3s}}{s'} - 2re^{3s} \\ y'' &= e^{4s}\left[\frac{s}{s'} - \frac{s''}{(s')^3} + 6r\right] \end{aligned} \tag{48}$$

Writing the ODE in canonical co-ordinates by using the derivatives in (48), they together transform the ODE to the following form:

$$-3s'' + 15(s')^2 - r(s')^2 - 3r^2(s')^3 + 18r(s')^3 = 0 \tag{49}$$

Now defining $\omega = s'$ reduces the second order ODE in (49) to a first order ODE:

$$\omega' + \left(\frac{r-15}{3}\right)\omega^2 + (r^2 - 6r)\omega^3 = 0 \tag{50}$$

which is an Abel equation of first type.

Conclusion

The Abel equation is important because it is thought to be to be the simplest non-linear differential equation which cannot be transformed into a linear ODE. Moreover, Abel's differential equation arises in different areas such as cosmology [14], control theory [15], fluids [16], problems of magnetostatistic [17]. In this work, we reduced the order of the second order non-linear ODE to the Abel equation of first type. The reduced equation is a degenerate type of equation since there is a missing term, *w*, in the equation. On the other hand, there is a substantial amount of existing research about methods for solving Abel's equation [18-21]. Our future study will be on the solution of the obtained degenerate Abel equation of the first kind in this paper.

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