ON IRRESOLUTE FUNCTIONS IN IDEAL TOPOLOGICAL SPACES

by

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In the ideal topological spaces (X, τ, I) and (Y, J, σ), we define (I, J)-irresolute function, weakly-(I, J)-irresolute, strongly-(I, J)-irresolute functions, semi-I-open sets and quasi-H-closed spaces modulo ideal I. We discuss properties of (I, J)-irresolute function mapping connected, closed, compact ideal spaces modulo I and semi-I-isolated points. (I, J)-Homeomorphisim on presemi-I-open sets and closure properties of quasi-H-closed spaces and semi-I-open subsets are explored in this paper.

Key words: *ideal space, (I, J)-irresolute function, quasi-H-closed space*

Introduction

Ideal topological spaces, first introduced by Kuratowski [1] in 1966 and treated by Vaidyanathaswamy [2], have a significant role in mathematics and quantum physics. An ideal I on a topological space (X, τ, I) (or simply X) is a non-empty gathering of subsets of X satisfying the conditions:

(i) If A is a subset of I and B is a subset of A , then B belongs I (heridity).

(ii) If A and B are two subsets of I, then their union belongs to I (finite additivity).

The ideal is also said to be a dual filter [3] if $X \notin I$ (*i.e.* I is a proper ideal) and $\{A : X - A \in I\}$ is filter. The simplest examples of ideals are ϕ and the power set of X. The triplet (X, τ, I) is an ideal topological space. The topologies and compatibility of ideal topological spaces were explored by Jankovic and Hamlett [4, 5], their contribution led to the topological generalization of several crucial characteristics in general topology. They were followed by Abd El-Monsef *et al.* [6], who demonstrated that a topology is not required for the gathering of all I open sets. Further research on the breakdown of continuity in ideal spaces was conducted by Keskin and Noiri [7] examined the subsequent characteristics of separation axioms in traditional topological spaces. Some examples of ideals are:

i) I_f : ideal of finite subsets of X.

ii) I_n: ideal of nowhere dense subsets of X.

iii) I_n: ideal of countable subsets of X.

iv) Is: ideal of scattered subsets of X.

For an ideal J on the ideal topological space (X, τ, I), if A is a subset of X, then the ideal $\{J \cap A : J \in I\}$, denoted by J_A, is a relative ideal on X. Now, if J is an ideal on (X, τ, I), then $\beta = \{U - J : U \in \tau,\ J \in I)$ is a base of τ^* on which an operator $(.)^*$ is defined. If $\wp(X)$

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is the collection of all subsets of *X*, then the set operator (or Kuratowski operator) $(.)^*$: $\wp(X) \to \wp(X)$ is a local function [1].

A local function of a subset *A* w.r.t. to τ and *I* is:

$$A^*(\tau, I) = \{x \in X : W \cap A \notin I \, \forall W \in \tau(x)\}$$

where $\tau(x)$ is the set of neighborhoods of *x*. A closure operator of Kuratowski [1] $cl^*(.)$ generates the * – topology $\tau^*(\tau, I)$ which is finer than (X, τ) defined as $cl^*(A) = A \bigcup A^*(\tau, I)$ for each subset *A* of *X*. A subset *A* of an ideal is τ^* closed if $A^* \subset A$ and * – closed [3] if $A = A^*$. If *A* is a subset of an ideal (X, τ, I), then *A* is:

i) *t-I*-set [8] if $int(A)= int[cl^*(A)]$
ii) σ-*I*-open [9] if $int[cl^*(A)] \subseteq int[cl^*(A)]$
iii) α-*I*-open [8] if $A \subseteq int\{cl^*[int(A)]\}$
iv) *pre-I*-open [10] if $A \subseteq int[cl^*(A)]$
v) *Semi-I*-open [8] if $A \subseteq cl^*[int(A)]$
vi) *Strong-β-I*-open [11] if $A \subseteq cl^* [intcl^*(A)]$
vii) *Semi**-*I*-open [12] if $A \subseteq cl[int^*(A)]$
viii) *Semipre** -*I*-closed [12] if $int\{cl^*[int(A)]\} \subseteq A$
ix) * -perfect [3] if $A = A^*$
x) * -dense [6] if $A \subset A^*$

Now, *A* is *Semi** *Semi**-*I*-closed iff $int[cl^*(A)] \subseteq A$ iff $int[cl^*(A)]=int(A)$, hence *Semi** -*I*-closed subsets of an ideal are *t-I*-open [10]. The set of all δ-*I*-open subsets is denoted by $\delta IO(X)$, (resp. α-*I*-open, *pre* -*I*-open, *semi* -*I*-open and *strong* -β-*I*-open by *IO*(*X*), *PIO*(*X*), *SIO*(*X*) and *sβIO*(*X*)). *pre* -*I*-interior of *A* [13] is the largest *pre* - *I* -open set that is contained in *A*. It is denoted by $pInt(A)$ and $pInt(A)=A \cap int[cl^*(A)]$. Furthermore, *A* is *Semipre** -*I*-closed iff $int\{cl^*[int(A)]\} = int(A)$ if *A* is α^*-*I*-open [8]. If *I* is an ideal on (X, τ), then *I* is a codense ideal [1] iff $\tau \cap I = \phi\}$. $X^* \subset X$, equality holds if the ideal *I* is condense. Now, the ideal *I* is compatible with [4] (denoted by $\tau \sim I$) iff for each subset *A* of *X* containing the point *x*, $\exists U \in N(x)$ (the set of all neighborhoods of *x*) : $U \cap A \in I$, then $A \in I$.

A subset *A* of a space (X, τ, I) is *I* compact [5] if for each open cover of it $\tilde{U} = \{U_\alpha : \alpha \in \Lambda\}$, $\exists \Lambda_0 = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$: *I* contains $A - \bigcup_{i \in \mathbb{N}} \{U_{\alpha_i} : i \in \mathbb{N}\}$ An ideal space is *I*-compact if *X* it self is *I* compact [6] and (X, τ, I) is quasi *H*-closed (simply QHC) [2] iff each open cover of *X* has a finite subcollection whose closure cover *X* and a QHC Hausdorff space is *H*-closed.

A space *X* is λ-compact [6] if every open cover of *X* is reduced to an open cover whose cardinality is less than λ (the least infinite cardinal number $\aleph_0$ with such property). λ is also called the compactness degree of *X* which is denoted by $d_c(X)$. Typically, a λ compact space is $\aleph_0$ and $\aleph_1$ compact spaces [14].

(*I*, *J*) – irresolute functions

If (X, τ, I) and (Y, σ, J) are two ideal spaces, then the function *f*: $(X, \tau, I) \to (Y, \sigma, J)$ is perfect modulo (*I*, *J*) [2] if *X* is Hausdorff, closed and $f^{-1}(y)$ is a compact subset of *X* $\forall y \in Y$.

Definition 1. An injective function *f* defined on a Hausdorff ideal space (X, τ, I) is perfect modulo an ideal *I* iff *f*(*X*) is a closed subset of the ideal topological space (Y, σ, J).
If (X, τ, I) is an ideal topological space, then $A \subset X$, *A* is [10]:

i) L^* perfect provided that $A - A^* \in I$

ii) R^* perfect provided that $A^* - A \in I$
iii) C^* perfect provided that A is L^* perfect and R^* perfect

The set of all L^* perfect (resp. R^* perfect and C^* perfect) is denoted by Ł (resp. R̦ and Cˇ).

It is generally known that continuous open surjective functions retain I compactness. We demonstrate an analogous finding in a more generic context in this section. If (X, τ, I) and (Y, σ, J) are two topological spaces, then a function f: $(X, \tau, I) \to (Y, \sigma, J)$ is pointwise (I, J)-continuous [13] if f: $(X, \tau^*, I) \to (Y, \sigma, J)$ is (I, J) continuous. Now, $f(I) = \{f(K) : K \in I\}$ is an ideal on Y.

Definition 2. For the ideal spaces (X, τ, I) and (Y, σ, J), a function f: $(X, \tau^*, I) \to (Y, \sigma, J)$ is (I, J)-irresolute if the inverse image of each *semi-J*-open subset of Y is *semi-I*-open subset of X. The f is weakly-(I, J)-irresolute function if $\forall x \in X$ and each *semi-I*-open subset of Y containing $f(x)$, $\exists U$ a *semi-I*-open subset of X containing x: $f(U) \subseteq SIO(Y)$. f is *strongly*-(I, J)-irresolute function if $\forall x \in X$ and each *pre-I*-open subset of Y containing $f(x)$, $\exists V$ a *pre-I*-open subset of X containing x: $f(U) \subseteq PIO(Y)$.

Definition 3. In an ideal space (X, τ, I), a function f: $(X,\tau,I) \to (Y,\sigma,J)$ is *presemi*-(I, J)-open if the image of each *semi-J*-open subset of Y is *semi-I*-open subset of X.

Remark 1. Every (I, J) irresolute function is (I, J) continuous, but the converse needs not to be true.

Theorem 1. In ideal spaces (X, τ, I) and (Y, σ, J), if a function $f:(X,\tau^*,I) \to (Y,\sigma,J)$ is (I, J)-continuous and open, then f is (I, J)-irresolute and *pesemi*-open.

Proof. By definition of (I, J)-continuity open function.

Corollary 1. Every I irresolute function is (I, J) continuous in the ideal topological space.

The inverse of *Corollary 1* is not true, for exmple: if a function f is given by $f(x) = y$, $f(y) = z$ and $f(z) = x$, then f is (I, J)-continuous but not an (I, J)-irresolute function.

Corollary 2. In ideal spaces (X, τ, I) and (Y,σ, J), if $f:(X,\tau^*,I) \to (Y,\sigma,J)$ is an (I, J) irresolute injective function and (Y, σ, J) is Hausdorff, then (X, τ, I) is Hausdorff.

Corollary 3. In ideal spaces (X, τ, I) and (Y, σ, J), if $f:(X,\tau^*,I) \to (Y,\sigma,J)$ is an (I, J) irresolute injective function and A is a compact subset of X modulo I, then the subset $f(K)$ of Y is a compact modulo J.

Theorem 2. In ideal spaces (X, τ, I) and (Y, σ, J), if $f:(X,\tau^*,I) \to (Y,\sigma,J)$ is an (I, J) irresolute injective function and X is connected, then so Y.

We mean by C_I the intersection of all *semi-I*-closed subsets and i_I the union of all *semi-I*-open subsets of X.

Theorem 3. For the ideal spaces (X, τ, I) and (Y, σ, J) and the function $f:(X,\tau^*,I) \to (Y,\sigma,J)$, the following are equivalent:

i) f is (I, J)-irresolute.
ii) $f^{-1}(F)$ is *semi-I*-closed subset of X $\forall$ F a *semi-I*-closed subset of Y.
iii) $f[C_I(A)] \subseteq C_I[f(A)]$ $\forall A$ an I-open subset of X.
iv) $C_I[f^{-1}(V)] \subseteq f^{-1}[C_I(V)]$ $\forall V$ an I-open subset of Y.
v) $f[C_I(A)] \subseteq C_I[f(A)]$ $\forall A$ an I-open subset of X.
vi) f is an (I, J)-irresolute function $\forall x \in X$.

Proof.

i) ⇒ ii) Direct from definition of (I, J)-irresolute function f.

ii)⇒iii) If A is an I-open subset of X, then $C_I[f(A)]$ is a *semi*-I-open subset of Y. Since f^{-1}(F) is *semi*-I-closed subset of X for each *semi*-I-closed subset F of Y, we get the result.

iii)⇒ iv) If V is a J-open subset of Y, then $f\{C_I[f^{-1}(V)]\} \subseteq C_I[ff^{-1}(V)] \subseteq C_I(V)$, hence $C_I[f^{-1}(V)] \subseteq f^{-1}\{fC_I[f^{-1}(V)]\} \subseteq f^{-1}[C_I(V)]$.

iv) ⇒ v) If A is an I-open subset of X and W is an I-open subset of Y, then $C_I[X - f^{-1}(W)] = C_I[f^{-1}(X-W)] \subseteq f^{-1}[C_I(X-W)] = X - C_I(X-A) = i_I(A)$ and $f^{-1}[i_I(W)] = f^{-1}[Y - C_I(Y-W)] = X - f^{-1}[C_I(Y-W)] \subseteq X - C_I[X - f^{-1}(W)] = i_I[f^{-1}(W)]$

v) ⇒ vi) Suppose that f is an (I, J)-irresolute function, if V is a *semi*-J-open subset of Y such that $f(x) \in Y \, \forall x \in X$, then $x \in f^{-1}(V) = i_I[f^{-1}(V)]$ where $f^{-1}(V)$ is a *semi*- I-open subset of X and $f[f^{-1}(V)] \subseteq V$. Thus, f is an (I, J)-irresolute function.

vi) ⇒ i) Let V be a *semi*-J-open subset of Y such that $x \in f^{-1}(V)$ for some $x \in X$, so $f(x) \in V$. Since f is an (I, J)-irresolute function $\forall x \in X$, ∃ U a *semi*-I-open subset of X containing $x: f(U) \subseteq V$. As a consequence, $x \in U \subseteq f^{-1}[f(U)] \subseteq f^{-1}(V)$ where $x \in U = i_I(U) \subseteq i_I[f^{-1}(V)]$, hence $f^{-1}(V)$ is *semi*-I-open subset of X. Thus f is an (I, J)-irresolute function.

Theorem 4. For ideal topological spaces (X, τ, I) and (Y, σ, J), if a function f: $(X, \tau^*, I) \to (Y, \sigma, j)$ is injective, then f is an (I, J)-irresolute iff $i_I[f(U)] \subseteq f[i_I(U)] \forall U$ an I-open subset of X.

Proof. Suppose that U is a *semi*-I-open subset of X, since f is injective, $f^{-1}i_I[f(U)] \subseteq i_I\{f^{-1}[f(U)]\} = i_I(U)$, hence $f(f\{f^{-1}[i_I(U)]\} \subseteq f[i_I(U)]$, thus $i_I[f(U)]f[i_I(U)]$. On the other hand, if V is a semi J open subset of Y, then $V = i_I(V)$ and $V = i_I(V) \subset i_I f[f^{-1}(V)] \subset f\{i_I[f^{-1}(V)]\}$, so $f^{-1}(V) \subset f^{-1}\{i_I[f^{-1}(V)]\}$, but f is injective, so $f^{-1}(V) \subset f^{-1}(f\{i_I[f^{-1}(V)]\}) = i_I[f^{-1}(V)]$ which implies $i_I[f^{-1}(V)] = f^{-1}f\{i_I[f^{-1}(V)]\}$ contains $f^{-1}(V)$, so $f^{-1}(V) = i_I[f^{-1}(V)]$, *i.e* $f^{-1}(V)$ is *semi*-I-open subset of X and f is (I, J)-irresolute.

Corollary 4. If (X, τ, I) and (Y, σ, J) are two ideal topological spaces and a function $f:(X, \tau^*, I) \to (Y, \sigma, j)$ is an (I, J)-irresolute, then $C_I[f^{-1}(V)] \subseteq f^{-1}[C_I(V)]$ for each *semi*-J-open subset V of Y.

The partition topology [5] is induced on the ideal space (X, τ, I) by partitioning it into disjoint subsets that form the basis of the topology.

Remark 2. If an ideal space (X, τ, I) is a partition topology, then any subset of X is I-open and any function defined on X is (I, J)-irresolute.

Definition 4. In an ideal space (X, τ, I) a point x in X is semi I isolated point if U a *semi*-I open subset of X containing x and does not contain any other points of X. It is clear that each I isolated point is *semi*-I isolated point, but the converse is not true.

Example 1. Consider the standard topology with an ideal $I = \{[a,b] : a,b \in \mathbb{R}\}$, if U = (1, 2], then {2} is a *semi*-I-isolated point of U since U ∩ [2, 3) = {2} where [2, 3) is *semi*-I-open subset of X, but {2}is not an I-isolated point of U.

Definition 5. In an ideal topological space (X, τ, I), a subset A of X is *semi*-I-perfect if it is *semi*-I-closed and no *semi*-I-isolated points in it.

Theorem 5. In an ideal space (X, τ, I), if a function $f:(X, \tau^*, I) \to (Y, \sigma, J)$ is a surjective (I, J) continuous and open and a subset V of Y is *semi*-J perfect, then $f^{-1}(V)$ is *semi*-I perfect subset of X.

Proof. Let $f:(X, \tau^*, I) \to (Y, \sigma, J)$ be (I, J) continuous and open function, then it is (I, J) irresolute and *presemi*-I open by *Definition 3* and $f^{-1}(Y-V) = X - f^{-1}(V)$ is *semi*-I open, thus $f^{-1}(V)$ is *semi*-I closed. By contradiction, if X is a *semi*-I isolated point in

$f^{-1}(V)$, then *U* a *semi-I* open subset of $X : X = f^{-1}(V) \subset U$. Now, $V \subset f(U) = F(x)$ which is a *semi-I* isolated point of *V* since *f*(*U*) is a *semi-J* open subset of *Y* which is a contradiction.

Theorem 6. In an ideal space (*X, τ, I*), if *A* and *B* are *I*-open subsets of *X*, a point *x* in *X* is a *semi-I*-isolated point of *A* and *B*, then *x* is a *semi-I*-isolated point in $A \cap B$.

Corollary 5. If *A* and *B* are two subsets of *X* such that none of them have *semi-I* isolated points, then $A \cap B$ contains *semi-I* isolated points.

Theorem 7. If the point *x* in an ideal space (*X, τ, I*) is *semi-I* isolated and *U* an *I* open subset of *X* containing *x*, then *x* is a *semi-I* isolated point of *U*.

Definition 6. In an ideal space (*X, τ, I*), a bijective function $f:(X,\tau^*,I)\to(Y,\sigma,J)$ is *semi*-(*I, J*)-homeomorphism if *f* is (*I, J*)-irresolute and *presemi-I*-open.

Lemma 1. In an ideal topological space (*X, τ, I*), if $f:(X,\tau^*,I)\to(Y,\sigma,J)$ is an (*I, J*)-homeomorphism, then *f* is *semi*-(*I, J*)-homeomorphism.

Theorem 8. In an ideal space (*X, τ, I*), if $f:(X,\tau^*,I)\to(Y,\sigma,J)$ is a surjective (*I, J*) continuous function and *I* open and a subset *A* of *X* is *semi-I* perfect, then $f^{-1}(A)$ is a *semi-I* perfect subset of *X*.

Proof. Suppose that $f:(X,\tau^*,I)\to(Y,\sigma,J)$ is a surjective (*I, J*) continuous function and *I* open and a subset *A* of *X* is *semi-I* perfect, hence *f* is (*I, J*) irresolute *presemi-I* open function by *Definition 3*. Thus $f^{-1}(Y-A) = Y - f^{-1}(A)$ is *semi-I* open subset of *X* and *f*(*A*) is *semi-J* closed. By contradiction, if $f^{-1}(A)$ has a *semi-I* isolated point *x*, so $\exists V$ a *semi-I* open subset of *X* such that $\{x\} = f^{-1}(A)\cap V$, so *f*(*x*) = *A* *f*(*V*) and *f*(*V*) is semi *I* open subset of *Y*, hence *f*(*x*) is semi *I* isolated point in *A* which is a contradiction.

Remark 3. The composition of two (*I, J*) irresolute functions needs not to be (*I, J*) irresolute. For example, if $X=\{x,y\}$, $\tau=\{\varnothing,X,\{x\}\}$, and $\sigma=\{\varnothing,X,\{x\}\}$, consider the (*I, J*) irresolute functions $f:(X,\tau^*,I)\to(Y,\sigma,J)$ and $g:(X,\tau^*,I)\to(Y,\sigma,J)$. Now, $g\circ f:(X,\tau^*,I)\to(X,\tau^*,I)$ is not an (*I, J*) irresolute function.

Theorem 9. In an ideal space (*X, τ, I*), if $f:(X,\tau^*,I)\to(Y,\sigma,J)$ is an (*I, J*)-irresolute surjective and *presemi-I*-open function and a function $g:(Y,\sigma,J)\to(Z,\eta,K)$, then $g\circ f:(X,\tau^*,I)\to(Z,\eta,K)$ is (*I, K*)-irresolute if and only if *g* is (*J, K*)-irresolute.

Proof. For the suffecient part, suppose that $g\circ f:(X,\tau^*,I)\to(Z,\eta,K)$ is (*I, K*)-irresolute, if *F* is a *semi-K*-closed subset of *Z*, then $g\circ f^{-1}(F)$ is *semi-I*-closed subset of *X*, but *f* is *presemi-I*-open, $f\{f^{-1}[g^{-1}(F)]\}$ is *semi-J*-open subset of *Y*, hence $g^{-1}(F)$ is *semi-J*-open subset of *Y*, thus *g* is an (*J, K*)-irresolute function.

Definition 7. The ideal space (*X, τ, I*) is *I*-closed compact if every *I*-closed cover of *X* has finite subcover.

Theorem 10. In an ideal space (*X, τ, I*), if a function $f:(X,\tau^*,I)\to(Y,\sigma,J)$ is (*I, J*)-irrosolute and a subset *A* of *X* is *I*-closed compact, then *f*(*A*) is *J*-closed compact subset of *Y*.

Proof. Suppose that $\{U_\alpha:\alpha\in\Lambda\}$ is a family of *I*-open subsets of *Y* that covers $f(A)$. Now, $U = V\cap f(A)$ for some *I*-open subset V_α of $Y\,\forall\alpha\in\Lambda$. For every point $x\in A$, $\exists\alpha_x\in\Lambda:\ f(x)\in A_\alpha$. For some W_α containing *x*, $f(W_\alpha)\subset A_\alpha$. Now, $A\subset\{W_x:x\in A_0\}$ where A_0 is a finite subset of *A* since *A* is *I*-closed compact, hence $f(A)\subset\bigcup\{f(W_x):x\in A_0\}\subset\bigcup\{A_x:x\in A_0\}$. Consequently, $f(A)\bigcup\{A_x:x\in A_0\}$, thus *f*(*A*) is *J*-compact subset of *Y*.

Theorem 11. If (*X, τ, I*) and (*Y, σ, J*) are two ideal spaces, $f:(X,\tau^*,I)\to(Y,\sigma,J)$ is an (*I, J*)-irresolute function if the graph function $g:X\to X\,Y$ given by $g(x)=[x,f(x)]$ is (*I, IJ*)-irresolute for each $x\in X$.

Proof. If V is a *semi-J*-open subset of Y containing $f(x)$ for each x in X, then XV is a *semi-IJ-* open subset of XJ containing $h(x)$. But h is IJ-irresolute, $\exists$ U a *semi-I*-open subset of X containing x: $g(U) \subseteq XV$, so $f(U) \subseteq V$. Thus, f is an (I, IJ)-irresolute function.

Quasi *H* closed ideal spaces

Definition 8.

i) An ideal topological space (X, τ, I) is quasi H-closed modulo I if each open cover by *semi-I*-open subsets of X has a finite subcover.
ii) A subspace Y of X endowed with topology σ is quasi H-closed relative to X if every family of *semi-I*-open subsets of X covering Y has a finite subfamily whose union is $*$-dense in Y.

Remark 4. Quasi H closure modulo an ideal I in the ideal topological space (X, τ, I) is not a hereditary property.

Lemma 2. If the ideal topological space (X, τ, I) is quasi H-closed, then:

i) The finite union of quasi H-closed subsets of X modulo I is quasi H-closed modulo I.
ii) The closure of a quasi H-closed subset of X modulo I is also quasi H-closed modulo I.
iii) The closure of a *semi-I*-open subset is quasi H-closed modulo I.

Definition 9. For the ideal topological space (X, τ, I):

i) X is T_2 if no filterbase has more than one cluster point.
ii) X is T_1 if no filterbase with an element x converges to any point $y \neq x$.

Thoerem 12. For the ideal topological space (X, τ, I), if X is Hausdorff and F is an H-closed subset relative to X modulo I, $\forall y \in X - F$, $\exists$ U a *semi-I*-subset of X such that $F \subseteq U \subseteq X - (y)$. Particularly, $F = \bigcap_{y \in x - F} U$.

Remark 5. For the *semi-I*-closed subset F of a quasi H-closed ideal topological space (X, τ, I), if the boundary of F is quasi H-closed modulo I, then F is quasi H-closed modulo I.

Theorem 13. For the ideal topological space (X, τ, I), X is an H-closed modulo an ideal I iff X is T_1 and compact modulo I.

Definition 10. An ideal topological space (X, τ, I) is C-compact modulo I if each *semi-I*-closed subset of X is H-closed subset.

Theorem 14. If the ideal topological spaces (X, τ, I) and (Y, σ, J) are *quasi-H*-closed, $f:(X,\tau,I) \to (Y,\sigma,J)$ is an (I, J)-irresolute function and (I, J)-continuous, then Y is so.

Theorem 15. The non-empty product of *quasi-H*-closed ideal spaces is *quasi-H*-closed.

Proof. Suppose that $\{(X_\alpha, \tau_\alpha, I_\alpha) : \alpha \in \Lambda\}$ be a family of *quasi-H*-closed ideal topological spaces. By *Theorem 13*, each space is compact modulo I and by Tyconoff theorem, $\Pi_{\alpha \in \Lambda} X_\alpha$ is compact modulo $I_{\alpha_1} \times I_{\alpha_2} \times \ldots \times I_{\alpha_n} \forall \alpha \in \Lambda$ and $n \in \mathbb{N}$. Now, if $x \in \Pi_{\alpha \in \Lambda} X_\alpha$, then $\Pi_{\alpha \in \Lambda} \Pi cl\{U_{\alpha_i} : i \in \mathbb{N}\} \Pi_{\beta \neq \alpha_i} X_\beta$ for each *semi-I*-open subset U_α of X. Hence, $\Pi_{\alpha \in \Lambda}(X_\alpha, \tau_\alpha, I_\alpha)$ is compact modulo $I_{\alpha_1} \times I_{\alpha_2} \times \ldots \times I_{\alpha_n}$ $\forall \alpha \in \Lambda$ and $n \in \mathbb{N}$.

Conclusion

For the ideal spaces (X, τ, I) and (Y, σ, J), a function $f:(X,\tau,I) \to (Y,\sigma,J)$ is (I, J)-irresolute if the inverse image of each *semi-J*-open subset of Y is *semi-I*-open subset of X, and an (I, J) irresolute injective function and A is a compact subset of X modulo I, then the subset f(K) of Y is a compact modulo J. A surjective (I, J) continuous function and I open and a subset A of X is *semi-I* perfect, then $f^{-1}(A)$ is a *semi-I* perfect subset of X.

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