OSCILLATION THEOREMS FOR CONFORMABLE FRACTIONAL DIFFERENTIAL EQUATIONS WITH DAMPING

by

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In this study, we study the oscillatory solutions of conformable fractional differential equations with damping term. Some examples have been given to illustrate the results.

Key words: *oscillation, fractional derivative, damping term*

Introduction

Fractional differential equations have recently proved to be valuable tools in the modelling of many phenomena in various fields such as mathematics, physics, biology, chemical physics, biomedical sciences, fluid flows, and finance, as well as other disciplines [1-5]. In the last few decades, many researchers have presented many fractional derivative definitions. One of them conformable derivative, Khalil *et al.* [6] have suggested this fractional derivative. Unlike the other fractional derivatives, this definition satisfies almost all the requirements of standard derivatives. As a new research field, there are many investigations with this conformable fractional derivative [6-10]. Although research on the oscillation of various equations including differential equations, difference equations, dynamic equations on time scales and their fractional generalizations has been a hot topic in the literature [11-25], we notice that very little attention is paid to oscillation of linear/non-linear conformable fractional differential equations [26-28].

In this study, we consider the following conformable fractional differential equation with damping term:

$$\{r(t)\psi[x(t)]x^{(\alpha)}(t)\}^{(\alpha)} + p(t)\psi[x(t)]x^{(\alpha)}(t) + q(t)x(t) = 0 \tag{1}$$

where $0 < \alpha \le 1$.

A solution of eq. (1) is called oscillatory if it has arbitrarily large zeros, otherwise, it is called non-oscillatory. Equation (1) is called oscillatory if all of its solutions are oscillatory.

Throughout this study, we will use the following conditions:

(H_1) $r(t) \in C^{\alpha}([t_0,\infty), R^{+})$ such that $r(t) \le k$ for some $k > 0$

(H_2) $p(t) \in C([t_0,\infty), R)$ such that $p(t) < 0$

(H_3) $\psi \in C(R,R)$, $0 < \psi(x) < m$ for some positive constant m and for all $x \ne 0$

(H_4) $q(t) \in C([t_0,\infty) \times R^{+})$ for $t \ge t_0$.

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Preliminaries

In this section, we present some background materials for the conformable fractional theory.

Definition 1. [26] The left conformable fractional derivative starting from t_0 of a function $f:[t_0,\infty)\to R$ of order α with $0<\alpha\le 1$ is defined by:

$$(\mathbf{T}_{t_0}^{\alpha} f)(t)=f^{(\alpha)}(t)=\lim_{\varepsilon\to 0}\frac{f[t+\varepsilon(t-t_0)^{1-\alpha}]-f(t)}{\varepsilon}$$

when $\alpha=1$, this derivative of $f(t)$ coincides with $f'(t)$. If $(\mathbf{T}_{t_0}^{\alpha} f)(t)$ exists on (t_0,t_1) then:

$$(\mathbf{T}_{t_0}^{\alpha} f)(t_0)=\lim_{t\to t_{0^+}} f^{(\alpha)}(t)$$

Definition 2. [26] Let $\alpha\in(0,\ 1]$. Then the left conformable fractional integral of order α starting at t_0 is defined by:

$$(\mathbf{I}_{t_0}^{\alpha} f)(t)=\int_{t_0}^{t}(s-t_0)^{\alpha-1}f(s)\mathrm{d}s:=\int_{t_0}^{t}f(s)\mathrm{d}_{t_0}^{\alpha}s$$

If the conformable fractional integral of a given function f exists, we call that f is α-integrable.

Lemma 1. [7] If $\alpha\in(0,\ 1]$ and $f\in C^1([t_0,\infty),R)$, then, for all $t>t_0$, we have:

$$\mathbf{I}_{t_0}^{\alpha}\mathbf{T}_{t_0}^{\alpha}(f)(t)=f(t)-f(t_0)\ \text{and}\ \mathbf{T}_{t_0}^{\alpha}\mathbf{I}_{t_0}^{\alpha}(f)(t)=f(t)$$

Lemma 2. [6]

1) $\mathbf{T}_{t_0}^{\alpha}(af+bg)=a\mathbf{T}_{t_0}^{\alpha}(f)+b\mathbf{T}_{t_0}^{\alpha}(g)$ for real constant a,b
2) $\mathbf{T}_{t_0}^{\alpha}(fg)=f\mathbf{T}_{t_0}^{\alpha}(g)+g\mathbf{T}_{t_0}^{\alpha}(f)$
3) $\mathbf{T}_{t_0}^{\alpha}(t^p)=p\mathbf{t}^{p-\alpha}$ for all p
4) $\mathbf{T}_{t_0}^{\alpha}(f/g)=[g\mathbf{T}_{t_0}^{\alpha}(f)-f\mathbf{T}_{t_0}^{\alpha}(g)]/g^2$
5) $\mathbf{T}_{t_0}^{\alpha}(c)=0$, where c is a constant.

Lemma 3. [8] Let $f,g:[t_0,\ t_1)\to R$ be two functions such that fg is differentiable. Then

$$\int_{t_0}^{t_1}f(s)g^{(\alpha)}(s)\mathrm{d}_{t_0}^{\alpha}s=f(s)g(s)\big|_{t_0}^{t_1}-\int_{t_0}^{t_1}g(s)f^{(\alpha)}(s)\mathrm{d}_{t_0}^{\alpha}s$$

Main results

In this section, we will present some new oscillation theorems for conformable fractional differential equation.

Theorem 1. Suppose that $(H_1)-(H_4)$ hold. If

$$\lim_{t\to\infty}\left\{\frac{1}{m}\int_{t_0}^{t}\frac{1}{r(s)}\mathrm{d}_{t_0}^{\alpha}s\right\}=\infty \tag{2}$$

and

$$\lim_{t\to\infty}\int_{t_0}^{t}\left[q(s)-\frac{mp^2(s)}{4r(s)}\right]\mathrm{d}_{t_0}^{\alpha}s=\infty \tag{3}$$

then every solution of eq. (1) is oscillatory.

Proof. Let $x(t)$ be a non-oscillatory solution of (1) on $[t_0,\infty)$. Without loss of generality, we assume that $x(t)$ is an eventually positive solution of eq. (1). Then we define:

$$\omega(t)=-\frac{r(t)\psi[x(t)]x^{(\alpha)}(t)}{x(t)}$$

for $t\ge t_0$. Then we have:

$$\omega^{(\alpha)}(t)=-\frac{[r(t)\psi(x(t))x^{(\alpha)}(t)]^{(\alpha)}(x(t))-[r(t)\psi(x(t))x^{(\alpha)}(t)](x^{(\alpha)}(t))}{x^2(t)}.$$

From (1), we get:

$$\omega^{(\alpha)}(t)=\frac{p(t)\psi[x(t)]x^{(\alpha)}(t)}{x(t)}+q(t)+\frac{\{r(t)\psi[x(t)]x^{(\alpha)}(t)\}[x^{(\alpha)}(t)]}{x^2(t)}=$$

$$=\frac{p(t)\psi[x(t)]x^{(\alpha)}(t)}{x(t)}+q(t)+\frac{\omega^2(t)}{r(t)\psi[x(t)]}$$

thus

$$\omega^{(\alpha)}(t)=-\frac{p(t)}{r(t)}\omega(t)+q(t)+\frac{\omega^2(t)}{r(t)\psi[x(t)]}\ge$$

$$\ge-\frac{p(t)}{r(t)}\omega(t)+q(t)+\frac{\omega^2(t)}{mr(t)}$$

Thus, for every t, T with $t\ge T\ge t_0$, we have:

$$\int_T^t\omega^{(\alpha)}(s)\mathrm{d}_{t_0}^{\alpha}s\ge\int_T^t\left[-\frac{p(s)}{r(s)}\omega(s)+\frac{1}{mr(s)}\omega^2(s)\right]\mathrm{d}_{t_0}^{\alpha}s+\int_T^t q(s)\mathrm{d}_{t_0}^{\alpha}s$$

then

$$\omega(t)\ge\omega(T)+\int_T^t\left[\frac{1}{mr(s)}\omega^2(s)-\frac{p(s)}{r(s)}\omega(s)\right]\mathrm{d}_{t_0}^{\alpha}s+\int_T^t q(s)\mathrm{d}_{t_0}^{\alpha}s=$$

$$=\omega(T)+\int_T^t\left[\frac{1}{\sqrt{mr(s)}}\omega(s)-\frac{\sqrt{m}}{2\sqrt{r(s)}}p(s)\right]^2\mathrm{d}_{t_0}^{\alpha}s+\int_T^t\left[q(s)-\frac{mp^2(s)}{4r(s)}\right]\mathrm{d}_{t_0}^{\alpha}s$$

By using the (3) implies there exists $T_1\ge T\ge t_0$; such that:

$$\omega(t)>\int_{T_1}^t\left[\frac{\omega(s)}{\sqrt{mr(s)}}-\frac{\sqrt{m}p(s)}{2\sqrt{r(s)}}\right]^2\mathrm{d}_{t_0}^{\alpha}s$$

If we define a function:

$$N(t)=\int_{T_1}^{t}\left[\frac{w(s)}{\sqrt{mr(s)}}-\frac{\sqrt{m}p(s)}{2\sqrt{r(s)}}\right]^2 \mathrm{d}_{t_0}^{\alpha} s \tag{4}$$

then $\omega(t)>N(t)>0$ for $t\geq T_1$. From (H_2) and eq. (4), we have:

$$N^{(\alpha)}(t)=\left[\frac{w(s)}{\sqrt{mr(s)}}-\frac{\sqrt{m}p(s)}{2\sqrt{r}}\right]^2 >$$

$$>\frac{w^2(t)}{mr(t)}>\frac{N^2(t)}{mr(t)}$$

or

$$\frac{1}{mr(t)}<\frac{N^{(\alpha)}(t)}{N^2(t)} \tag{5}$$

Integrating both sides of the (5) from T_1 to t, we have:

$$\frac{1}{m}\int_{T_1}^{t}\frac{1}{r(s)}\mathrm{d}_s^{\alpha}<\frac{1}{N(T_1)}-\frac{1}{N(t)}<\frac{1}{N(T_1)} \tag{6}$$

And letting $t\to\infty$ in (6):

$$\lim_{t\to\infty}\left[\frac{1}{m}\int_{T_1}^{t}\frac{1}{r(s)}\mathrm{d}_s^{\alpha}s\right]<\frac{1}{N(T_1)}$$

which is a contradiction to eq. (2). This completes the proof of the theorem.

Theorem 2. Let conditions $(H_1)-(H_4)$ hold. Assume that there exist a positive function $g\in C^{\alpha}[t_0,\infty)$ such that:

$$\lim_{t\to\infty}\left[\frac{1}{\sqrt{mk}}\int_{T_1}^{t}\frac{1}{\phi(s)}\mathrm{d}_{t_0}^{\alpha}s\right]=\infty \tag{7}$$

and

$$\begin{gathered}\lim_{t\to\infty}\left(\frac{mk}{4}\int_{t_0}^{t}\left\{p^2(s)\phi(s)+\frac{[\phi^{(\alpha)}(s)]^2}{\phi(s)}-\frac{2}{k}p(s)\phi^{(\alpha)}(s)-\frac{4}{mk}\phi(s)q(s)\right\}\mathrm{d}_{t_0}^{\alpha}s+\right.\\ \left.+\frac{1}{2}mr(t)\phi^{(\alpha)}(t)\right)=\infty\end{gathered} \tag{8}$$

then every solution of eq. (1) is oscillatory.

Proof. Let $x(t)$ be a non-oscillatory solution of (1) on $[t_0,\infty)$. Without loss of generality, we assume that $x(t)$ is an eventually positive solution of eq. (1). Then we define:

$$\omega(t)=-\phi(t)\frac{r(t)\psi[x(t)]x^{(\alpha)}(t)}{x(t)}$$

for $t \geq t_0$. Then from the conditions, we have:

$$\omega^{(\alpha)}(t) = -\phi^{(\alpha)}(t)\left\{\frac{r(t)\psi[x(t)]x^{(\alpha)}(t)}{x(t)}\right\} - \phi(t)\left\{\frac{r(t)\psi[x(t)]x^{(\alpha)}(t)}{x(t)}\right\}^{(\alpha)} =$$

$$= \frac{\phi^{(\alpha)}(t)}{\phi(t)}\omega(t) - \phi(t)\left\{-\frac{p(t)\psi[x(t)]x^{(\alpha)}(t)}{x(t)} - q(t) - \frac{1}{r(t)\psi[x(t)]}\frac{\omega^2(t)}{\phi^2(t)}\right\} =$$

$$= \frac{\phi^{(\alpha)}(t)}{\phi(t)}\omega(t) - \phi(t)\left\{\frac{p(t)}{r(t)}\frac{\omega(t)}{\phi(t)} - q(t) - \frac{1}{r(t)\psi[x(t)]}[x(t)]\frac{\omega^2(t)}{\phi^2(t)}\right\} =$$

$$= \frac{\phi^{(\alpha)}(t)}{\phi(t)}\omega(t) - \frac{p(t)}{r(t)}\omega(t) + \phi(t)q(t) + \frac{1}{r(t)\psi[x(t)]}\frac{\omega^2(t)}{\phi(t)} =$$

$$= \frac{1}{\phi(t)}\left\{\phi^{(\alpha)}(t)\omega(t) - \frac{p(t)}{r(t)}\phi(t)\omega(t) + \frac{1}{r(t)\psi[x(t)]}\omega^2(t)\right\} + \phi(t)q(t) \geq$$

$$\geq \frac{1}{\phi(t)}\left[\phi^{(\alpha)}(t)\omega(t) - \frac{p(t)}{r(t)}\phi(t)\omega(t) + \frac{1}{r(t)m}\omega^2(t)\right] + \phi(t)q(t)$$

Now for $t \in [t_0, \infty)$, defining:

$$O(t) = \left[\frac{1}{\sqrt{mr(t)}}\right]\omega(t) + \frac{1}{2}\sqrt{mr(t)}\phi^{(\alpha)}(t) \tag{9}$$

we will have

$$\omega^{\alpha}(t) \geq \frac{1}{\phi(t)}\left\{\left[O(t) - \frac{1}{2}\sqrt{\frac{m}{r(t)}}p(t)\phi(t)\right]^2 - \left[\frac{1}{2}\sqrt{\frac{m}{r(t)}}p(t)\phi(t)\right]^2 - \right.$$
$$\left. - \frac{mr(t)}{4}[\phi^{(\alpha)}(t)]^2 + \frac{m}{2}p(t)\phi(t)\phi^{(\alpha)}(t)\right\} + \phi(t)q(t)$$

That is:

$$\begin{gathered}\omega^{\alpha}(t) \geq \frac{1}{\phi(t)}\left[O(t) - \frac{1}{2}\sqrt{\frac{m}{r(t)}}p(t)\phi(t)\right]^2 - \\ - \frac{mr(t)}{4}\left\{p^2(t)\phi(t) + \frac{[\phi^{(\alpha)}(t)]^2}{\phi(t)} - 2\frac{p(t)}{r(t)}\phi^{(\alpha)}(t) - \frac{4}{mr(t)}\phi(t)q(t)\right\}\end{gathered} \tag{10}$$

Thus, for every t, T with $t \geq T \geq t_0$, we get:

$$\omega(t) \geq \omega(T) + \int_T^t \frac{1}{\phi(s)}\left[O(s) - \frac{1}{2}\sqrt{\frac{m}{r(s)}}p(s)\phi(s)\right]^2 \mathrm{d}_{t_0}^{\alpha} s -$$
$$- \frac{m}{4}\int_T^t r(s)\left\{p^2(s)\phi(s) + \frac{[\phi^{(\alpha)}(s)]^2}{\phi(s)} - 2\frac{p(s)}{r(s)}\phi^{(\alpha)}(s) - \frac{4}{mr(s)}\phi(s)q(s)\right\}\mathrm{d}_{t_0}^{\alpha} s$$

From (9) and (10), (H_1), (H_2) and (H_4), we get:

$$O(t) \geq \frac{1}{\sqrt{mk}}\omega(T) + \left[\frac{1}{\sqrt{mk}}\int_T^t \frac{1}{\phi(s)}(O(s) - \frac{1}{2}\sqrt{\frac{m}{r(s)}}p(s)\phi(s)\right]^2 \mathrm{d}_{t_0}^{\alpha} s -$$

$$-\frac{mk}{4}\int_T^t p^2(s)\phi(s) + \frac{[\phi^{(\alpha)}(s)]^2}{\phi(s)} - \frac{2}{k}p(s)\phi^{\alpha}(s) - \frac{4}{mk}\phi(s)q(s)\mathrm{d}_{t_0}^{\alpha} s + \frac{1}{2}\sqrt{mr(t)}\phi^{(\alpha)}(t)$$

Then by using (8), we get:

$$O(t) > \left\{\frac{1}{\sqrt{mk}}\int_T^t \frac{1}{\phi(s)}\left[O(s) - \frac{1}{2}\sqrt{\frac{m}{r(s)}}p(s)\phi(s)\right]^2\right\}\mathrm{d}_{t_0}^{\alpha} s$$

Define a function $P(t)$ for $t \geq T$ by:

$$P(t) = \left\{\frac{1}{\sqrt{mk}}\int_T^t \frac{1}{\phi(s)}\left[O(s) - \frac{1}{2}\sqrt{\frac{m}{r(s)}}p(s)\phi(s)\right]^2\right\}\mathrm{d}_{t_0}^{\alpha} s$$

Using the (H_2), we have $O(t) > P(t) > 0$. Then we get:

$$P^{(\alpha)}(t) \geq \left(\frac{1}{\sqrt{mk}}\frac{1}{\phi(t)}\left[O(t) - \frac{1}{2}\sqrt{\frac{m}{r(t)}}p(t)\phi(t)\right]^2 > \right.$$

$$> \frac{1}{\sqrt{mk}}\frac{1}{\phi(t)}O^2(t)$$

or

$$\frac{1}{\sqrt{mk}}\frac{1}{\phi(t)} < \frac{O^{(\alpha)}(t)}{O^2(t)}$$

Integrating both sides of the previous inequality from T to t, we obtain:

$$\frac{1}{\sqrt{mk}}\int_T^t \frac{1}{\phi(s)}\mathrm{d}_{t_0}^{\alpha} s < \frac{1}{O(T)} - \frac{1}{O(t)} < \frac{1}{O(T)}$$

Letting $t \to \infty$, we get a contradiction to (7). This completes the proof of the theorem.

Illustrative example

To confirm our obtained results in the previous section, we present herein some numerical examples.

Example 1.

$$\left[\frac{1}{t^4}e^{-x^4(t)}x^{(1/7)}(t)\right]^{(1/7)} - \frac{1}{t^2}e^{-x^4(t)}x^{(1/7)}(t) + tx(t) = 0 \tag{11}$$

for $t \geq 1$. This corresponds to eq. (1) with $t_0 = 1$, $\alpha = 1/7$, $r(t) = 1/t^4$, $\psi(x) = e^{-x^4}$, $p(t) = -(1/t^2)$, and $q(t) = t$. So we have $r(t) \leq 1 = k$ and $\psi(x) \leq 1 = m$. Then:

$$\lim_{t\to\infty}\left\{\frac{1}{m}\int_{t_0}^{t}\frac{1}{r(s)}\mathrm{d}_{t_0}^{\alpha}s\right\} = \lim_{t\to\infty}\left\{\int_{1}^{t}s^4\mathrm{d}_{t_0}^{\alpha}s\right\} = \infty$$

and

$$\lim_{t\to\infty}\int_{t_0}^{t}\left[q(s) - \frac{mp^2(s)}{4r(s)}\right]\mathrm{d}_{t_0}^{\alpha}s = \lim_{t\to\infty}\int_{1}^{t}\left[s - \frac{1}{4}\right]\mathrm{d}_{t_0}^{\alpha}s = \infty$$

Thus, eq. (11) satisfies the inequalities (2) and (3) in *Theorem* 1 and is oscillatory.

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