ON SOME FUNDAMENTAL PROPERTIES OF *α*-INTERVAL VALUED FUZZY SUBGROUPS

by

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In this paper, the definition of α-interval valued fuzzy subgroup is introduced by the help of α-interval valued fuzzy set which is constructed on α-interval valued set whose elements are closed sub-intervals including α of unit interval I = [0, 1]. The fundamental and structural properties of these groups are examined. Some definitions, propositions and examples about these groups are given.

Key words: *fuzzy sets, interval valued fuzzy sets, interval valued fuzzy subgroups, α-interval valued fuzzy set, α-interval valued fuzzy subgroups*

Introduction

The concept of interval valued fuzzy set was introduced by Zadeh [1-4]. Gorzalczany, Gratten-Guiness, Jahn, Sambuc, and Turksen [5-10] examined the properties of interval valued fuzzy sets. It is important to search properties of interval valued fuzzy sets on different structures. Specially, Mondal and Samantha [11] studied the topological properties of interval valued fuzzy sets.

The *α*-interval valued fuzzy set was defined by Cuvalcioglu, Bal and Citil. Algebraic properties of these sets were studied by same authors. The definition of level subset of *α*-interval valued fuzzy set was given and some properties of these level subsets were examined by same authors. Fuzzy subgroups were defined by Rosenfeld [12]. The definition of interval valued fuzzy subgroups was given by Biswas [13]. Many authors studied interval valued fuzzy subgroups and fuzzy sets [14-18]. In this paper, $D(I)$ represents all closed sub-intervals of unit interval $I=[0,1]$. The definition of $D(I_\alpha)$ is given below. The $D(I_\alpha)$ is called *α*-interval valued set.

Definition 1. $D(I_\alpha)=\{[M^L,\ M^U;\alpha]|\alpha\in I\}$ is called *α*-interval valued set. In order to make easy, it is shown that:

$$\left\{\left[M^L,\ M^U;\alpha\right]\middle|\alpha\in I\right\}=\left\{[M;\alpha]\middle|M\in D(I)\text{ and }\alpha\in M\right\}$$

Order relation on $D(I_\alpha)$ is defined.

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Definition 2. $\forall[M;\alpha],[N;\alpha]\in D(I_\alpha)$,

$$[M;\alpha]\le[N;\alpha]:\Leftrightarrow M^L\le N^L \text{ and } M^U\ge N^U$$

It is easily seen from definition:

$$[M;\alpha]<[N;\alpha]$$

$$\Leftrightarrow M^L<N^L,\ M^U\ge N^U \text{ or } M^L\le N^L,\ M^U>N^U \text{ or } M^L<N^L,\ M^U>N^U$$

Proposition 1. $[D(I_\alpha),\le]$ is partial ordered set.

By the help of order relation on $D(I_\alpha)$, the definitions of supremum and infimum on this set are given.

Definition 3. $\forall[M;\alpha],[N;\alpha]\in D(I_\alpha)$,

i. $$\inf\{[M;\alpha],[N;\alpha]\}=\left[\inf\{M^L,N^L\},\sup\{M^U,N^U\};\alpha\right]$$

ii. $$\sup\{[M;\alpha],[N;\alpha]\}=\left[\sup\{M^L,N^L\},\inf\{M^U,N^U\};\alpha\right]$$

Lemma 1. $[D(I_\alpha),\wedge,\vee]$ is complete lattice with units $[0,1;\alpha]$ and $[\alpha,\alpha;\alpha]$.

Proposition 2. $\forall\alpha\in I$,

$$\bigcup_{\alpha\in I} D(I_\alpha)=D(I)$$

The function which satisfies following conditions is called negation function.

Definition 4. L is complete lattice with units 0 and 1. $\mathcal{N}:L\to L$ and $\forall a,b\in L$,

i. $$\mathcal{N}(0)=1 \text{ and } \mathcal{N}(1)=0$$

ii. $$\mathcal{N}(a)\le\mathcal{N}(b):\Leftrightarrow a\ge b$$

iii. $$\mathcal{N}(\mathcal{N}(a))=a$$

The following function is a negation function on $D(I_\alpha)$.

Proposition 3. $\forall[M;\alpha]\in D(I_\alpha)$ and $\mathcal{N}:D(I_\alpha)\to D(I_\alpha)$,

$$\mathcal{N}([M;\alpha])=\left[\alpha-M^L,1+\alpha-M^U;\alpha\right]$$

$\mathcal{N}$ satisfies conditions of *Definition 4.*

Definition 5. Let X be universal set and $[A;\alpha]:X\to D(I_\alpha)$ be function:

$$[A;\alpha]=\left\{\left[<x,\left[A^L(x),A^U(x)\right]>;\alpha\right]\middle|x\in X\right\}$$

where; $A^L:X\to[0,1]$ and $A^U:X\to[0,1]$ are fuzzy sets.

In order to make easy, it is shown that:

$$\left\{\left[<x,\left[A^L(x),A^U(x)\right]>;\alpha\right]\middle|x\in X\right\}=\left\{\left[<x,A(x)>;\alpha\right]\middle|x\in X\right\}$$

$[A;\alpha]$ is called α-interval valued fuzzy set on X. The family of α-interval valued fuzzy sets on X is shown by α-IVFS(X).

Complement, inclusion, equation, intersection and union of α-interval valued fuzzy sets are given below.

Definition 6. Let X be universal set and $[A;\alpha],[B;\alpha]\in\alpha-\text{IVFS}(X)$. The Λ is index set $\forall\lambda\in\Lambda$

i. $$\left[A^c;\alpha\right]=\left\{\left[<x,\left[\alpha-A^L(x),1+\alpha-A^U(x)\right]>;\alpha\right]\middle| x\in X\right\}$$

ii. $$\left[A;\alpha\right]\sqsubseteq\left[B;\alpha\right]:\Leftrightarrow \forall x\in X, A^L(x)\le B^L(x) \text{ and } A^U(x)\ge B^U(x)$$

iii. $$\left[A;\alpha\right]=\left[B;\alpha\right]:\Leftrightarrow \forall x\in X, A^L(x)=B^L(x) \text{ and } A^U(x)=B^U(x)$$

iv. $$\left[A\sqcap B;\alpha\right]=\left\{\left[<x,\left[\inf\left\{A^L(x),B^L(x)\right\},\sup\left\{A^U(x),B^U(x)\right\}\right]>;\alpha\right]\middle| x\in X\right\}$$

v. $$\left[A\sqcup B;\alpha\right]=\left\{\left[<x,\left[\sup\left\{A^L(x),B^L(x)\right\},\inf\left\{A^U(x),B^U(x)\right\}\right]>;\alpha\right]\middle| x\in X\right\}$$

vi. $$\left[\sqcap_{\lambda\in\Lambda} A_\lambda;\alpha\right]=\left\{\left[<x,\left[\Lambda_{\lambda\in\Lambda}{A_\lambda}^L(x),V_{\lambda\in\Lambda}{A_\lambda}^U(x)\right]>;\alpha\right]\middle| x\in X\right\}$$

vii. $$\left[\sqcup_{\lambda\in\Lambda} A_\lambda;\alpha\right]=\left\{\left[<x,\left[V_{\lambda\in\Lambda}{A_\lambda}^L(x),\Lambda_{\lambda\in\Lambda}{A_\lambda}^U(x)\right]>;\alpha\right]\middle| x\in X\right\}$$

The algebraic properties of α-interval valued fuzzy sets are expressed.

Proposition 4. Let X be universal set. The $\forall[A;\alpha],[B;\alpha],[C;\alpha]\in\alpha-\text{IVFS}(X)$ and Λ is index set $\forall\lambda\in\Lambda$:

i. $$\left[A\sqcap B;\alpha\right]=\left[B\sqcap A;\alpha\right]$$

ii. $$\left[A\sqcup B;\alpha\right]=\left[B\sqcup A;\alpha\right]$$

iii. $$\left[A;\alpha\right]\sqcap\left(\left[B\sqcup C;\alpha\right]\right)=\left(\left[A\sqcap B;\alpha\right]\right)\sqcup\left(\left[A\sqcap C;\alpha\right]\right)$$

iv. $$\left[A;\alpha\right]\sqcup\left(\left[B\sqcap C;\alpha\right]\right)=\left(\left[A\sqcup B;\alpha\right]\right)\sqcap\left(\left[A\sqcup C;\alpha\right]\right)$$

v. $$\left[A;\alpha\right]\sqcap\left(\left[\sqcup_\lambda B_\lambda;\alpha\right]\right)=\left[\sqcup_\lambda\left(A\sqcap B_\lambda\right);\alpha\right]$$

vi. $$\left[A;\alpha\right]\sqcup\left(\left[\sqcap_\lambda B_\lambda;\alpha\right]\right)=\left[\sqcap_\lambda\left(A\sqcup B_\lambda\right);\alpha\right]$$

Features about complement of α-interval valued fuzzy sets are stated following proposition.

Proposition 5. Let X be universal set. The $\forall[A;\alpha],[B;\alpha]\in\alpha-\text{IVFS}(X)$ and Λ is index set $\forall\lambda\in\Lambda$:

i. $$\left[\left(\left[A^c;\alpha\right]\right)^c;\alpha\right]=\left[A;\alpha\right]$$

ii. $$\left([A \sqcap B;\alpha]\right)^c = \left[A^c \sqcup B^c;\alpha\right]$$

iii. $$\left([A \sqcup B;\alpha]\right)^c = \left[A^c \sqcap B^c;\alpha\right]$$

iv. $$\left([\sqcap_{\lambda\in\Lambda} A_\lambda;\alpha]\right)^c = \left[\sqcup_{\lambda\in\Lambda} A_\lambda^c;\alpha\right]$$

v. $$\left([\sqcup_{\lambda\in\Lambda} A_\lambda;\alpha]\right)^c = \left[\sqcap_{\lambda\in\Lambda} A_\lambda^c;\alpha\right]$$

Proposition 6. Let X be universal set. The $0_{\mathbf{X}} : X \to [0,1;\alpha]$ and $1_{\mathbf{X}} : X \to [\alpha,\alpha;\alpha]$:

i. $$(0_{\mathbf{X}})^c = 1_{\mathbf{X}}$$

ii. $$(1_{\mathbf{X}})^c = 0_{\mathbf{X}}$$

Definition 7. Let X be universal set and $[A;\alpha] \in \alpha - \text{IVFS(X)}$:

$$[A;\alpha] \text{ has sup} - \text{property}$$

$$:\Leftrightarrow \forall x \in X, \exists[\lambda_1,\lambda_2;\alpha] \in D(I_\alpha) \ni \left[A(x);\alpha\right] = [\lambda_1,\lambda_2;\alpha]$$

Definition 8. Let X be universal set and $[A;\alpha] \in \alpha - \text{IVFS(X)}$.
$\forall[\lambda_1,\lambda_2;\alpha] \in D(I_\alpha)$,

$$[A;\alpha]_{[\lambda_1,\lambda_2;\alpha]} = \left\{x \in X \middle| A^L(x) \geq \lambda_1 \text{and } A^U(x) \leq \lambda_2\right\}$$

The set $[A;\alpha]_{[\lambda_1,\lambda_2;\alpha]}$ is called $[\lambda_1,\lambda_2;\alpha]$-level subset of $[A;\alpha]$. It is easily seen from definition, $[\lambda_1,\lambda_2;\alpha]$-level subsets of $[A;\alpha]$ are crisp sets.

Definition 9. Let X be universal set and $[A;\alpha] \in \alpha - \text{IVFS(X)}$.
$\forall[\lambda_1,\lambda_2;\alpha] \in D(I_\alpha)$,
$\forall[A;\alpha]_{[\lambda_1,\lambda_2;\alpha]}$ -level subsets of $[A;\alpha]$,

i. $$A^L_{\lambda_1} = \left\{x \in X \middle| A^L(x) \geq \lambda_1\right\}$$

ii. $$A^U_{\lambda_2} = \left\{x \in X \middle| A^U(x) \leq \lambda_2\right\}$$

iii. $$B^L_{\lambda_1} = \left\{x \in X \middle| B^L(x) \geq \lambda_1\right\}$$

iv. $$B^U_{\lambda_2} = \left\{x \in X \middle| B^U(x) \leq \lambda_2\right\}$$

The relations between level subsets of α-interval valued fuzzy sets and crisp sets are given.

Proposition 7. Let X be universal set and $[A;\alpha],[B;\alpha] \in \alpha - \text{IVFS(X)}$.
$\forall[\lambda_1,\lambda_2;\alpha] \in D(I_\alpha)$ and I is index set, $\forall i,j \in I,[\lambda_i,\lambda_j;\alpha] \in D(I_\alpha)$,

i. $$x \in [A;\alpha]_{[\lambda_1,\lambda_2;\alpha]} \Leftrightarrow [A(x);\alpha] \geq [\lambda_1,\lambda_2;\alpha]$$

ii. $$[A;\alpha]_{[\lambda_1,\lambda_2;\alpha]} = A^L_{\lambda_1} \cap A^U_{\lambda_2}$$

iii. $$([A \sqcup B;\alpha])_{[\lambda_1,\lambda_2;\alpha]}$$

$$= [A;\alpha]_{[\lambda_1,\lambda_2;\alpha]} \cup [B;\alpha]_{[\lambda_1,\lambda_2;\alpha]} \cup \left(A^L_{\lambda_1} \cap B^U_{\lambda_2}\right) \cup \left(B^L_{\lambda_1} \cap A^U_{\lambda_2}\right)$$

iv. $$([A \sqcap B;\alpha])_{[\lambda_1,\lambda_2;\alpha]} = [A;\alpha]_{[\lambda_1,\lambda_2;\alpha]} \cap [B;\alpha]_{[\lambda_1,\lambda_2;\alpha]}$$

v. $$A^L_{\lambda_1} \supseteq A^L_{\lambda_2}$$

vi. $$A^U_{\lambda_1} \subseteq A^U_{\lambda_2}$$

vii. $$\bigcap_{i\in I} A^L_{\lambda_i} = A^L_{\bigwedge_{i\in I}\lambda_i}$$

viii. $$\bigcup_{j\in I} A^U_{\lambda_j} = A^U_{\bigvee_{j\in I}\lambda_j}$$

The α-interval valued fuzzy subgroups

As a result of previous discussions about α-interval valued fuzzy sets, the definition of α-interval valued fuzzy subgroup is given below. The α-interval valued fuzzy subgroup on α-IVFS(G) is shown by α-IVFS(G, $[A;\alpha]$).

Definition 10. Let G be group and $[A;\alpha] \in \alpha - \text{IVFS}(G)$. $\forall x, y \in G$,

i. $$[A(xy);\alpha] \geq \inf\{[A(x);\alpha],[A(y);\alpha]\}$$

ii. $$[A(x^{-1});\alpha] \geq [A(x);\alpha]$$

Example 1. For $(\mathbb{Z},+)$ abelian group:

$\alpha = \frac{1}{3}$, the function $A:\mathbb{Z} \to D\left(I_{\frac{1}{3}}\right)$,

$$\left[A(0);\frac{1}{3}\right] = \left[\frac{1}{3},\frac{1}{3};\frac{1}{3}\right]$$

and

$$\left[A(k);\frac{1}{3}\right] = \begin{cases} \left[\frac{1}{4},\frac{2}{3};\frac{1}{3}\right]; k \text{ is even and } k \neq 0 \\ \left[\frac{1}{5},\frac{5}{6};\frac{1}{3}\right]; k \text{ is odd} \end{cases}$$

1/3-interval valued fuzzy subgroup.

Solution.

1. $k, m \in \mathbb{Z}$ are given arbitrary.

1. situation

k and m are even.

$$k = m \Rightarrow \left[\frac{1}{4}, \frac{2}{3}; \frac{1}{3}\right] = \left[\frac{1}{4}, \frac{2}{3}; \frac{1}{3}\right] \Rightarrow \left[A(k); \frac{1}{3}\right] = \left[A(m); \frac{1}{3}\right]$$

2. situation

k and m are odd.

$$k = m \Rightarrow \left[\frac{1}{5}, \frac{5}{6}; \frac{1}{3}\right] = \left[\frac{1}{5}, \frac{5}{6}; \frac{1}{3}\right] \Rightarrow \left[A(k); \frac{1}{3}\right] = \left[A(m); \frac{1}{3}\right]$$

2.

$$\inf\left\{\left[\frac{1}{4}, \frac{2}{3}; \frac{1}{3}\right], \left[\frac{1}{5}, \frac{5}{6}; \frac{1}{3}\right]\right\} = \left[\inf\left\{\frac{1}{4}, \frac{1}{5}\right\}, \sup\left\{\frac{2}{3}, \frac{5}{6}\right\}; \frac{1}{3}\right] = \left[\frac{1}{5}, \frac{5}{6}; \frac{1}{3}\right]$$

1. situation

Let k and m be even, then $k + m$ is even.

$$\left[A(k+m); \frac{1}{3}\right] = \left[\frac{1}{4}, \frac{2}{3}; \frac{1}{3}\right] \geq \inf\left\{\left[\frac{1}{4}, \frac{2}{3}; \frac{1}{3}\right], \left[\frac{1}{4}, \frac{2}{3}; \frac{1}{3}\right]\right\} = \left[\frac{1}{4}, \frac{2}{3}; \frac{1}{3}\right] =$$
$$= \inf\left\{\left[A(k); \frac{1}{3}\right], \left[A(m); \frac{1}{3}\right]\right\}$$

2. situation

Let k be even and m be odd, then $k + m$ is odd.

$$\left[A(k+m); \frac{1}{3}\right] = \left[\frac{1}{5}, \frac{5}{6}; \frac{1}{3}\right] \geq \inf\left\{\left[\frac{1}{4}, \frac{2}{3}; \frac{1}{3}\right], \left[\frac{1}{5}, \frac{5}{6}; \frac{1}{3}\right]\right\} = \left[\frac{1}{5}, \frac{5}{6}; \frac{1}{3}\right] =$$
$$= \inf\left\{\left[A(k); \frac{1}{3}\right], \left[A(m); \frac{1}{3}\right]\right\}$$

3. situation

Let k be odd and m be even, then $k + m$ is odd.

$$\left[A(k+m); \frac{1}{3}\right] = \left[\frac{1}{5}, \frac{5}{6}; \frac{1}{3}\right] \geq \inf\left\{\left[\frac{1}{5}, \frac{5}{6}; \frac{1}{3}\right], \left[\frac{1}{4}, \frac{2}{3}; \frac{1}{3}\right]\right\} = \left[\frac{1}{5}, \frac{5}{6}; \frac{1}{3}\right] =$$
$$= \inf\left\{\left[A(k); \frac{1}{3}\right], \left[A(m); \frac{1}{3}\right]\right\}$$

4. situation
Let k and m be odd, then $k+m$ is even.

$$\left[A(k+m);\frac{1}{3}\right]=\left[\frac{1}{4},\frac{2}{3};\frac{1}{3}\right]\geq \inf\left\{\left[\frac{1}{5},\frac{5}{6};\frac{1}{3}\right],\left[\frac{1}{5},\frac{5}{6};\frac{1}{3}\right]\right\}=\left[\frac{1}{5},\frac{5}{6};\frac{1}{3}\right]=$$

$$=\inf\left\{\left[A(k);\frac{1}{3}\right],\left[A(m);\frac{1}{3}\right]\right\}$$

3.

1. situation
Let k be even, then $-k$ is even:

$$\left[A(-k);\frac{1}{3}\right]=\left[\frac{1}{4},\frac{2}{3};\frac{1}{3}\right]\geq\left[\frac{1}{4},\frac{2}{3};\frac{1}{3}\right]=\left[A(k);\frac{1}{3}\right]$$

2. situation
Let k be odd, then $-k$ is odd:

$$\left[A(-k);\frac{1}{3}\right]=\left[\frac{1}{5},\frac{5}{6};\frac{1}{3}\right]\geq\left[\frac{1}{5},\frac{5}{6};\frac{1}{3}\right]=\left[A(k);\frac{1}{3}\right]$$

A satisfies the conditions of $\alpha-\text{IVFS}(G,[A;\alpha])$.
Proposition 9. Let G be group and $\alpha-\text{IVFS}(G,[A;\alpha])$. The $\forall x\in G$,

i. $$\left[A(x^{-1});\alpha\right]=\left[A(x);\alpha\right]$$

ii. $$\left[A(e);\alpha\right]\geq\left[A(x);\alpha\right]$$

Proof. $x\in G$ is given arbitrary:

i. $$\alpha-\text{IVFS}\big(G,[A;\alpha]\big)\Rightarrow\left[A(x^{-1});\alpha\right]\geq\left[A(x);\alpha\right]\ldots \qquad (1)$$

$$\left[A\left((x^{-1})^{-1}\right);\alpha\right]\geq\left[A(x^{-1});\alpha\right]\Rightarrow\left[A(x);\alpha\right]\geq\left[A(x^{-1});\alpha\right]\ \ldots \qquad (2)$$

From inequalities (1) and (2):

$$\left[A(x^{-1});\alpha\right]=\left[A(x);\alpha\right]$$

ii. $$\left[A(e);\alpha\right]=\left[A(xx^{-1});\alpha\right]\geq\inf\left\{\left[A(x);\alpha\right],\left[A(x^{-1});\alpha\right]\right\}$$

$$=\inf\left\{\left[A(x);\alpha\right],\left[A(x);\alpha\right]\right\}=\left[A(x);\alpha\right]$$

Proposition 10. Let G be group and $[A;\alpha]\in\alpha-\text{IVFS}(G)$. $\forall x,y\in G$,

$$\alpha-\text{IVFS}\big(G,[A;\alpha]\big)\Leftrightarrow\left[A(xy^{-1});\alpha\right]\geq\inf\left[A(x);\alpha\right]\left\{,\left[A(y);\alpha\right]\right\}$$

Proof: $x, y \in G$ are given arbitrary.

"$\Rightarrow$"

$$\left[A(xy^{-1});\alpha\right] \geq \inf\left\{\left[A(x);\alpha\right],\left[A(y^{-1});\alpha\right]\right\} = \inf\left\{\left[A(x);\alpha\right],\left[A(y);\alpha\right]\right\}$$

"$\Leftarrow$"

i.
$$\left[A(e);\alpha\right] = \left[A(xx^{-1});\alpha\right] \geq \inf\left\{\left[A(x);\alpha\right],\left[A(x);\alpha\right]\right\} = \left[A(x);\alpha\right]$$

$$\Rightarrow \left[A(x^{-1});\alpha\right] = \left[A(ex^{-1});\alpha\right] \geq \inf\left\{\left[A(e);\alpha\right],\left[A(x);\alpha\right]\right\} = \left[A(x);\alpha\right]$$

ii.
$$\left[A(xy);\alpha\right] = \left[A\left(x(y^{-1})^{-1}\right);\alpha\right] \geq \inf\left\{\left[A(x);\alpha\right],\left[A(y^{-1});\alpha\right]\right\}$$

$$\geq \inf\left\{\left[A(x);\alpha\right],\left[A(y);\alpha\right]\right\}$$

Proposition 11. Let G be group and $[A;\alpha],[B;\alpha] \in \alpha - \text{IVFS}(G)$.

$$\alpha - \text{IVFS}\big(G,[A;\alpha]\big) \text{ and } \alpha - \text{IVFS}\big(G,[B;\alpha]\big) \Rightarrow \alpha - \text{IVFS}\big(G,[A \sqcap B;\alpha]\big)$$

Proof. $x, y \in G$ are given arbitrary.

$$\left[(A \sqcap B)(xy^{-1});\alpha\right] = \left[\inf\left\{A^L(xy^{-1}),B^L(xy^{-1})\right\},\sup\left\{A^U(xy^{-1}),B^U(xy^{-1})\right\};\alpha\right]$$

$$\geq \left[\inf\left\{\inf\left\{A^L(x),A^L(y)\right\},\inf\left\{B^L(x),B^L(y)\right\}\right\},\sup\left\{\sup\left\{A^U(x),A^U(y)\right\},\sup\left\{B^U(x),B^U(y)\right\}\right\};\alpha\right]$$

$$= \left[\inf\left\{\inf\left\{A^L(x),B^L(x)\right\},\inf\left\{A^L(y),B^L(y)\right\}\right\},\sup\left\{\sup\left\{A^U(x),B^U(x)\right\},\sup\left\{A^U(y),B^U(y)\right\}\right\};\alpha\right]$$

$$= \inf\left\{\left[\inf\left\{A^L(x),B^L(x)\right\},\sup\left\{A^U(x),B^U(x)\right\};\alpha\right],\left[\inf\left\{A^L(y),B^L(y)\right\},\sup\left\{A^U(y),B^U(y)\right\};\alpha\right]\right\}$$

$$= \inf\left\{\left[(A \sqcap B)(x);\alpha\right],\left[(A \sqcap B)(y);\alpha\right]\right\}$$

Proposition 12. Let G be group and I be index set, $\forall i \in I, [A_{i\in I};\alpha] \in \alpha - \text{IVFS}(G)$.

$$\alpha - \text{IVFS}\big(G,(A_{i\in I};\alpha)\big) \Rightarrow \alpha - \text{IVFS}\big(G,[\sqcap_{i\in I} A_i;\alpha]\big)$$

Proof. $x, y \in G$ are given arbitrary.

$$\alpha - \text{IVFS}\big(G,\left[A_{i\in I};\alpha\right]\big) \Rightarrow \left[A_{i\in I}(xy^{-1});\alpha\right] \geq \inf\left\{\left[A_{i\in I}(x);\alpha\right],\left[A_{i\in I}(y);\alpha\right]\right\}$$

$$\Rightarrow \left[A_{i\in I}(xy^{-1});\alpha\right] \geq \inf\left\{\left[{A^L}_{i\in I}(x),{A^U}_{i\in I}(x);\alpha\right],\left[{A^L}_{i\in I}(y),{A^U}_{i\in I}(y);\alpha\right]\right\}$$

$$\Rightarrow \left[A_{i\in I}(xy^{-1});\alpha\right] \geq \left[\inf\left\{A^{L}{}_{i\in I}(x), A^{L}{}_{i\in I}(y)\right\}, \sup\left\{A^{U}{}_{i\in I}(x), A^{U}{}_{i\in I}(y)\right\};\alpha\right]$$

$$\Rightarrow \left[\left(\sqcap_{i\in I} A_i\right)(xy^{-1});\alpha\right] = \left[\underset{i\in I}{\Lambda} A^{L}{}_{i}(xy^{-1}), \underset{i\in I}{V} A^{U}{}_{i}(xy^{-1});\alpha\right]$$

$$\geq \left[\underset{i\in I}{\Lambda}\inf\left\{A^{L}{}_{i}(x), A^{L}{}_{i}(y)\right\}, \underset{i\in I}{V}\sup\left\{A^{U}{}_{i}(x), A^{U}{}_{i}(y)\right\};\alpha\right]$$

$$= \left[\inf\left\{\underset{i\in I}{\Lambda} A^{L}{}_{i}(x), \underset{i\in I}{\Lambda} A^{L}{}_{i}(y)\right\}, \sup\left\{\underset{i\in I}{V} A^{U}{}_{i}(x), \underset{i\in I}{V} A^{U}{}_{i}(y)\right\};\alpha\right]$$

$$= \inf\left\{\left[\underset{i\in I}{\Lambda} A^{L}{}_{i}(x), \underset{i\in I}{V} A^{U}{}_{i}(x);\alpha\right], \left[\underset{i\in I}{\Lambda} A^{L}{}_{i}(y), \underset{i\in I}{V} A^{U}{}_{i}(y);\alpha\right]\right\}$$

$$= \inf\left\{\left[(\sqcap_{i\in I} A_i)(x);\alpha\right], \left[(\sqcap_{i\in I} A_i)(y);\alpha\right]\right\}$$

$$\Rightarrow \alpha - \text{IVFS}\left(G, \left[\sqcap_{i\in I} A_i;\alpha\right]\right)$$

Proposition 13. Let G be group.

$$\alpha - \text{IVFS}\left(G,[A;\alpha]\right) \Leftrightarrow \forall[\lambda_1,\lambda_2;\alpha] \in D(I_\alpha)$$

$$[A;\alpha]_{[\lambda_1,\lambda_2;\alpha]} \neq \varnothing, [\lambda_1,\lambda_2;\alpha] - \text{level subset of } [A;\alpha] \text{ is subgroup of G.}$$

Proof.

"$\Rightarrow$"

$$x \in G,\ \exists\left[\lambda_1,\lambda_2;\alpha\right] \in D(I_\alpha),\ \left[A(x);\alpha\right] = \left[\lambda_1,\lambda_2;\alpha\right] \Rightarrow \forall x \in G,\ \left[A(e);\alpha\right] \geq \left[A(x);\alpha\right]$$

$$\Rightarrow e \in \left[A;\alpha\right]_{[\lambda_1,\lambda_2;\alpha]} \Rightarrow \left[A;\alpha\right]_{[\lambda_1,\lambda_2;\alpha]} \neq \varnothing$$

$x, y \in [A;\alpha]_{[\lambda_1,\lambda_2;\alpha]}$ are given arbitrary.

$$\left[A(x);\alpha\right] \geq \left[\lambda_1,\lambda_2;\alpha\right] \text{ and} \left[A(y);\alpha\right] \geq \left[\lambda_1,\lambda_2;\alpha\right]$$

$$\Rightarrow \left[A(xy^{-1});\alpha\right] \geq \inf\left\{\left[A(x);\alpha\right],\left[A(y);\alpha\right]\right\} \geq \left[\lambda_1,\lambda_2;\alpha\right] \Rightarrow xy^{-1} \in \left[A;\alpha\right]_{[\lambda_1,\lambda_2;\alpha]}$$

"$\Leftarrow$"

Assume $\exists x_0, y_0 \in G, [A(x_0 y_0{}^{-1});\alpha] < [\inf\left\{A(x_0), A(y_0);\alpha\right\}]$

$$\left[A(x_0);\alpha\right] = \left[a_1,a_2;\alpha\right],\ \left[A(y_0);\alpha\right] = \left[b_1,b_2;\alpha\right] \text{ is taken,}$$

$$\left[\lambda_1,\lambda_2;\alpha\right] = \inf\left\{\left[a_1,a_2;\alpha\right],\left[b_1,b_2;\alpha\right]\right\} \text{ is chosen,}$$

$$\left[A(x_0);\alpha\right] = \left[a_1,a_2;\alpha\right] \geq \inf\left\{\left[a_1,a_2;\alpha\right],\left[b_1,b_2;\alpha\right]\right\} = \left[\lambda_1,\lambda_2;\alpha\right] \Rightarrow x_0 \in \left[A;\alpha\right]_{[\lambda_1,\lambda_2;\alpha]}$$

$$\left[A(y_0);\alpha\right] = \left[b_1,b_2;\alpha\right] \geq \inf\left\{\left[a_1,a_2;\alpha\right],\left[b_1,b_2;\alpha\right]\right\} = \left[\lambda_1,\lambda_2;\alpha\right] \Rightarrow y_0 \in \left[A;\alpha\right]_{[\lambda_1,\lambda_2;\alpha]}$$

$$\left[A(x_0 {y_0}^{-1});\alpha\right] < \inf\left\{[A(x_0);\alpha],[A(y_0);\alpha]\right\} = \inf\left\{[a_1,a_2;\alpha],[b_1,b_2;\alpha]\right\}$$

$$= [\lambda_1,\lambda_2;\alpha] \Rightarrow x_0 {y_0}^{-1} \notin [A;\alpha]_{[\lambda_1,\lambda_2;\alpha]}$$

is contradiction. Then,

$$\forall x_0, y_0 \in G, \left[A(x_0 {y_0}^{-1});\alpha\right] \geq \inf\left\{[A(x_0);\alpha],[A(y_0);\alpha]\right\}$$

Proposition 14. Let G be group and $\alpha-\mathrm{IVFS}(G,[A;\alpha])$. The $\forall[\lambda_1,\lambda_2;\alpha]\in D(I_\alpha)$, $[A;\alpha]_{[\lambda_1,\lambda_2;\alpha]}$ is subgroup of G $\Leftrightarrow A^L_{\lambda_1}$ and $\left((A^U)^c\right)_{1-\lambda_2}$ are subgroups of G.

Proof.

"$\Rightarrow$"

$$\forall[\lambda_1,\lambda_2;\alpha]\in D(I_\alpha), e\in[A;\alpha]_{[\lambda_1,\lambda_2;\alpha]} = A^L_{\lambda_1}\cap A^U_{\lambda_2} \Rightarrow e\in A^L_{\lambda_1} \text{ and } e\in A^U_{\lambda_2}$$

$$\Rightarrow A^U(e)\leq\lambda_2 \Rightarrow 1-A^U(e)\geq 1-\lambda_2 \Rightarrow e\in\left((A^U)^c\right)_{1-\lambda_2}$$

$$A^L_{\lambda_1}\neq\varnothing \text{ and } \left((A^U)^c\right)_{1-\lambda_2}\neq\varnothing$$

$x, y \in A^L_{\lambda_1}$ are given arbitrary,

$$A^L(x)\geq\lambda_1 \text{ and } A^L(y)\geq\lambda_1$$

$$\Rightarrow A^L(xy^{-1})\geq\inf\left\{A^L(x),A^L(y)\right\}\geq\lambda_1 \Rightarrow xy^{-1}\in A^L_{\lambda_1}$$

$x, y \in \left((A^U)^c\right)_{1-\lambda_2}$ are given arbitrary,

$$1-A^U(x)\geq 1-\lambda_2 \text{ and } 1-A^U(y)\geq 1-\lambda_2 \text{ and}$$

$$A^U(xy^{-1})\leq\sup\left\{A^U(x),A^U(y)\right\}$$

$$\Rightarrow 1-A^U(xy^{-1})\geq 1-\sup\left\{A^U(x),A^U(y)\right\}$$

$$\Rightarrow 1-A^U(xy^{-1})\geq\inf\left\{1-A^U(x),1-A^U(y)\right\}\geq 1-\lambda_2$$

$$\Rightarrow xy^{-1}\in\left((A^U)^c\right)_{1-\lambda_2}$$

"$\Leftarrow$"

$$A^L_{\lambda_1} \text{ and } \left((A^U)^c\right)_{1-\lambda_2} \text{ are subgroups of } G \Rightarrow e\in A^L_{\lambda_1} \text{ and } e\in\left((A^U)^c\right)_{1-\lambda_2}$$

$$\Rightarrow 1-A^U(e)\geq 1-\lambda_2 \Rightarrow A^U(e)\leq\lambda_2 \Rightarrow e\in A^U_{\lambda_2}$$

$$\Rightarrow e \in A^L_{\lambda_1} \cap A^U_{\lambda_2} = [A;\alpha]_{[\lambda_1,\lambda_2;\alpha]} \Rightarrow [A;\alpha]_{[\lambda_1,\lambda_2;\alpha]} \neq \emptyset$$

$x, y \in G$ are given arbitrary,

$$A^L(xy^{-1}) \geq \inf\left\{A^L(x), A^L(y)\right\} \geq \lambda_1 \text{ and}$$

$$1 - A^U(xy^{-1}) \geq \inf\left\{1 - A^U(x), 1 - A^U(y)\right\} \geq 1 - \lambda_2$$

$$\Rightarrow A^L(xy^{-1}) \geq \inf\left\{A^L(x), A^L(y)\right\} \geq \lambda_1 \text{ and}$$

$$1 - A^U(xy^{-1}) \geq 1 - \sup\left\{A^U(x), A^U(y)\right\} \geq 1 - \lambda_2$$

$$\Rightarrow A^L(xy^{-1}) \geq \inf\left\{A^L(x), A^L(y)\right\} \geq \lambda_1 \text{and } A^U(xy^{-1}) \leq \sup\left\{A^U(x), A^U(y)\right\} \leq \lambda_2$$

$$\Rightarrow \left[A^L(xy^{-1}), A^U(xy^{-1}); \alpha\right] \geq [\lambda_1, \lambda_2; \alpha]$$

$$\Rightarrow xy^{-1} \in [A;\alpha]_{[\lambda_1,\lambda_2;\alpha]}$$

Example 2. For $(\mathbb{Z},+)$ abelian group:

$$\alpha = \frac{1}{3}, \text{ the function } A : \mathbb{Z} \to D\left(I_{\frac{1}{3}}\right),$$

$$\left[A(0); \frac{1}{3}\right] = \left[\frac{1}{3}, \frac{1}{3}; \frac{1}{3}\right]$$

and

$$\left[A(k); \frac{1}{3}\right] = \begin{cases} \left[\frac{1}{4}, \frac{2}{3}; \frac{1}{3}\right]; k \text{ is even and } k \neq 0 \\ \left[\frac{1}{5}, \frac{5}{6}; \frac{1}{3}\right]; k \text{ is odd} \end{cases}$$

The 1/3-interval valued fuzzy subgroup:

$$\left[\frac{1}{6}, \frac{4}{5}; \frac{1}{3}\right] \in D\left(I_{\frac{1}{3}}\right)$$

$$\left[A; \frac{1}{3}\right]_{\left[\frac{1}{6}, \frac{4}{5}; \frac{1}{3}\right]} = 2\mathbb{Z} \text{ is subgroup of } \mathbb{Z}$$

$$A^L_{\frac{1}{6}} = \mathbb{Z} \text{ and} \left((A^U)^c\right)_{1-\frac{4}{5}} = \left((A^U)^c\right)_{\frac{1}{5}} = 2\mathbb{Z} \text{ are subgroups of } \mathbb{Z}$$

Definition 11. Let X be universal set and $[A;\alpha] \in \alpha - \text{IVFS}(X)$.

$$A^* = \left\{ x \in X \middle| [A(x);\alpha] > [0,1;\alpha] \right\}$$

is called support set of A.

Definition 12. Let G be group and $\alpha - \text{IVFS}(G,[A;\alpha])$.

$$A_* = \left\{ x \in G \middle| [A(x);\alpha] = [A(e);\alpha] \right\}$$

Proposition 15. Let G be group and $\alpha - \text{IVFS}(G,[A;\alpha])$.

i. A^* is subgroup of G.
ii. A_* is subgroup of G.

Proof.

i. $x, y \in A^*$ are given arbitrary.

$$[A(x);\alpha] > [0,1;\alpha] \text{ and } [A(y);\alpha] > [0,1;\alpha] \Rightarrow \inf\{[A(x);\alpha],[A(y);\alpha]\} > [0,1;\alpha]$$

$$\Rightarrow \left[A(xy^{-1});\alpha\right] \ge \inf\{[A(x);\alpha],[A(y);\alpha]\} > [0,1;\alpha] \Rightarrow xy^{-1} \in A^*$$

ii. $x, y \in A_*$ are given arbitrary.

$$[A(x);\alpha] = [A(e);\alpha] \text{ and } [A(y);\alpha] = [A(e);\alpha] \Rightarrow \inf\{[A(x);\alpha],[A(y);\alpha]\} = [A(e);\alpha]$$

$$\Rightarrow \left[A(xy^{-1});\alpha\right] \ge \inf\{[A(x);\alpha],[A(y);\alpha]\} = [A(e);\alpha]\ldots \tag{3}$$

$$\text{and } \alpha - \text{IVFS}\left(G,[A;\alpha]\right) \Rightarrow [A(e);\alpha] \ge \left[A(xy^{-1});\alpha\right]\ldots \tag{4}$$

from (3) and (4), we get below equality:

$$\left[A(xy^{-1});\alpha\right] = [A(e);\alpha] \Rightarrow xy^{-1} \in A_*$$

Conclusion

In this paper, the definition of α-interval valued fuzzy subgroups was introduced. It was studied that an α-interval valued fuzzy set of G group under which conditions this set is α-interval valued fuzzy subgroup. The structural properties of these groups were examined. An example of α-interval valued fuzzy subgroup was given. It was searched that when the level subset of α-interval valued fuzzy subgroup is a classical subgroup of G group.

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