THE α-INTERVAL VALUED FUZZY SETS DEFINED ON α-INTERVAL VALUED SET

by

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In this study, α-interval valued set is defined whose elements are closed sub-intervals including α of unit interval that is I = [0, 1]. With different order relation on this set, the properties of α-interval valued set are examined. By the help of this order relation, it is shown that α-interval valued set is complete lattice. Negation function on α-interval valued set is given in order to study the theoretical properties of this set. By means of discussions on α-interval valued set, the fundamental features of α-interval valued set are studied. By the help of α-interval valued set, α-interval valued fuzzy sets are defined. The fundamental algebraic properties of these sets are examined. The level subsets of α-interval valued fuzzy sets are defined to give the relations between α-interval valued sets and crisp sets. With the help of this definition, some propositions and examples are given.

Key words: *fuzzy sets, interval valued fuzzy sets, α-interval valued set, α-interval valued fuzzy sets*

Introduction

The concept of interval valued fuzzy set was introduced by Zadeh [1-4]. The properties of interval valued fuzzy set and fuzzy set were studied by researchers [5-12]. The properties of interval valued fuzzy sets on different structures particularly, topological properties were examined by Mondal and Samantha [13]. By means of order relation on the family of all closed sub-intervals of unit interval I = [0, 1], Biswas denoted that this family is complete lattice whose units are [0, 0] and [1, 1] [1]. For the lattice, Biswas used respectively representations rmax, rmin, rsup, and rinf for refined-max, refined-min, refined-supremum and refined-infimum [14]. The D(I) represents the family of all closed sub-intervals of unit interval $I=[0,1]$. The elements of D(I) are shown with capital letters such as M, N... In this place, M^L and M^U are called respectively lower end point and upper end point for interval $M=[M^L,M^U]$.

Definition 1. [14] $\forall M,N \in D(I)$,

i. $$M \le N :\Leftrightarrow M^L \le N^L \text{ and } M^U \le N^U$$

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It is easily seen that $M \le N \not\Leftrightarrow M \subseteq N$

Definition 2. [14] Le I be index set, $\forall i \in I, M, N, M_i \in D(I)$,

i. $\text{rmax}(M,N) = [\max(M^L, N^L), \max(M^U, N^U)]$
ii. $\text{rmin}(M,N) = [\min(M^L, N^L), \min(M^U, N^U)]$
iii. $\text{rsup}_{i\in I} M_i = [\sup_{i\in I} M_i^L, \sup_{i\in I} M_i^U]$
iv. $\text{rinf}_{i\in I} M_i = [\inf_{i\in I} M_i^L, \inf_{i\in I} M_i^U]$

Lemma 1: [14] Let be $L = [D(I), \text{ rmin}, \text{ rmax}]$, then L is complete lattice with units $[0,0]$ and $[1,1]$.

Definition 3: [7] Let X be universal set. The function $A : X \to D(I)$:

$$A = \{< x,\ A(x) > | x \in X\} = \{< x, [A^L(x), A^U(x)] > | x \in X\}$$

is called interval valued fuzzy set on X (shortly IVFS). The $A^L, A^U : X \to [0,1]$ are fuzzy sets and it is shown that $A = [A^L, A^U]$such that $A^L \le A^U$, $\forall x \in X$, $A^L(x) \le A^U(x)$. A^L and A^U are called respectively lower and upper end point for x on A. The family of all interval valued fuzzy sets on X is shown by IVFS(X).

Example 1: Let $X = \{a, b, c, d\}$:

$$A = \{< a, [0, 0.5] >, < b, [0.1, 0.3] >, < c, [0.2, 0.7] >, < d, [0.6, 1] >\}$$

is an interval valued fuzzy set.

Definition 4: [7] Let X be universal set. $\forall$ A, B $\in$ IVFS(X):

i. $A \sqsubseteq B :\Leftrightarrow \forall x \in X, A^L(x) \le B^L(x)$ and $A^U(x) \le B^U(x)$
ii. $A = B :\Leftrightarrow \forall x \in X, A^L(x) = B^L(x)$ and $A^U(x) = B^U(x)$
iii. $A \sqcap B = \left\{ < x, \begin{bmatrix} \min\{A^L(x), B^L(x)\}, \\ \min\{A^U(x),\ B^U(x)\} \end{bmatrix} > | x \in X \right\}$
iv. $A \sqcup B = \left\{ < x, \begin{bmatrix} \max\{A^L(x), B^L(x)\}, \\ \max\{A^U(x),\ B^U(x)\} \end{bmatrix} > | x \in X \right\}$
v. $A^c = \{< x, [1 - A^U(x), 1 - A^L(x)] > | x \in X\}$

Definition 5: [7] Let X be universal set. I is an index set $\forall i \in I$:

i. $\sqcap_{i\in I} A_i = \left\{ < x, \left[\min_{i\in I} \left\{ A_i{}^L(x) \right\}, \min_{i\in I} \left\{ A_i{}^U(x) \right\} \right] > | x \in X \right\}$
ii. $\sqcup_{i\in I} A_i = \left\{ < x, \left[\max_{i\in I} \left\{ A_i{}^L(x) \right\}, \max_{i\in I} \left\{ A_i{}^U(x) \right\} \right] > | x \in X \right\}$

The algebraic properties on interval valued fuzzy sets are given following proposition:

Proposition 1: [7] Let X be universal set. $\forall A, B \in$ IVFS(X) and I is an index set. $\forall i \in I, A_i, B_i \in$ IVFS(X):

i. $A \sqcap B = B \sqcap A$
ii. $A \sqcup B = B \sqcup A$

iii. $$A \sqcap \left(\sqcup_{i\in I} B_i\right) = \sqcup_{i\in I} (A \sqcap B_i)$$

iv. $$A \sqcup \left(\sqcap_{i\in I} B_i\right) = \sqcap_{i\in I} (A \sqcup B_i)$$

The level subsets take an important role on fuzzy set theory. The definition of level subset of interval valued fuzzy set is given by [15] below,

Definition 6. [15] Let X be universal set. $\forall[\lambda_1,\lambda_2]\in[0,1]$, ,

$$A_{[\lambda_1,\lambda_2]} = \left\{x \in X \middle| A^L(x) \le \lambda_1 \text{ and } A^U(x) \le \lambda_2\right\}$$

$A_{[\lambda_1,\lambda_2]}$ is called $[\lambda_1,\lambda_2]$-level subset of A.

Here, they are given that:

i. $$A^L_{\lambda_1} = \left\{x \in X \middle| A^L(x) \le \lambda_1\right\}$$

ii. $$A^U_{\lambda_2} = \left\{x \in X \middle| A^U(x) \le \lambda_2\right\}$$

iii. $$B^L_{\lambda_1} = \left\{x \in X \middle| B^L(x) \le \lambda_1\right\}$$

iv. $$B^U_{\lambda_2} = \left\{x \in X \middle| B^U(x) \le \lambda_2\right\}$$

The fundamental properties of $[\lambda_1,\lambda_2]$-level subset of A are given following,

Proposition 2. [15] Let X be universal set. The $\forall A, B \in IVFS(X)$ and $\forall[\lambda_1,\lambda_2]\in[0,1]$,

i. $$x \in A_{[\lambda_1,\lambda_2]} \Leftrightarrow A(x) \le [\lambda_1,\lambda_2]$$

ii. $$A \sqcup B_{[\lambda_1,\lambda_2]} = A_{[\lambda_1,\lambda_2]} \cap B_{[\lambda_1,\lambda_2]}$$

iii. $$A \sqcap B_{[\lambda_1,\lambda_2]} = A_{[\lambda_1,\lambda_2]} \cup B_{[\lambda_1,\lambda_2]} \cup (A^L_{\lambda_1} \cap B^U_{\lambda_2}) \cup (A^U_{\lambda_2} \cap B^L_{\lambda_1})$$

The α-interval valued sets and α-interval valued fuzzy sets

In this section, the properties of the sub-family of $D(I)$ whose elements are some special elements of $D(I)$ are studied. There are no another properties that $D(I)$ is a family of closed sub-interval of unit interval $I=[0,1]$. This situation generates that the classification of the elements of $D(I)$ makes difficult.

First and foremost, below definition is given,

Definition 8. $D(I_\alpha) = \left\{\left[M^L,\ M^U;\alpha\right] \middle| \alpha \in I\right\}$ is called α-interval valued set.

In order to make easy, it is shown that

$$\left\{\left[M^L,\ M^U;\alpha\right] \middle| \alpha \in I\right\} = \left\{[M;\alpha] \middle| M \in D(I) \text{ and } \alpha \in M\right\}$$

Order relation on $D(I_\alpha)$ is defined.

Definition 9. $\forall [M;\alpha],[N;\alpha] \in D(I_\alpha)$,

$$[M;\alpha] \le [N;\alpha] :\Leftrightarrow M^L \le N^L \text{ and } M^U \ge N^U$$

It is easily seen from definition,

$$[M;\alpha] < [N;\alpha]$$

$$\Leftrightarrow M^L < N^L,\ M^U \ge N^U \text{ or } M^L \le N^L,\ M^U > N^U \text{ or } M^L < N^L,\ M^U > N^U$$

Proposition 3. $(D(I_\alpha),\le)$ is partial ordered set.

Proof. $[M;\alpha],[N;\alpha],[P;\alpha] \in D(I_\alpha)$ are given arbitrary,

i. $M^L \le M^L$ and $M^U \ge M^U \Rightarrow [M;\alpha] \le [M;\alpha]$

ii. $$[M;\alpha] \le [N;\alpha] \text{ and } [N;\alpha] \le [M;\alpha]$$

$$\Rightarrow M^L \le N^L, M^U \ge N^U \text{ and } N^L \le M^L, N^U \ge M^U$$

$$\Rightarrow M^L = N^L \text{ and } M^U = N^U \Rightarrow [M;\alpha] = [N;\alpha]$$

iii. $$[M;\alpha] \le [N;\alpha] \text{ and } [N;\alpha] \le [P;\alpha]$$

$$\Rightarrow M^L \le N^L, M^U \ge N^U \text{ and } N^L \le P^L, N^U \ge P^U$$

$$\Rightarrow M^L \le P^L \text{ and } M^U \ge P^U \Rightarrow [M;\alpha] \le [P;\alpha]$$

By the help of order relation on $D(I_\alpha)$, the definitions of supremum and infimum on this set are given.

Definition 10. $\forall [M;\alpha],[N;\alpha] \in D(I_\alpha)$,

i. $$\inf\{[M;\alpha],[N;\alpha]\} = \left[\inf\{M^L,N^L\},\sup\{M^U,N^U\};\alpha\right]$$

ii. $$\sup\{[M;\alpha],[N;\alpha]\} = \left[\sup\{M^L,N^L\},\inf\{M^U,N^U\};\alpha\right]$$

Lemma 2. $(D(I_\alpha),\wedge,\vee)$ is complete lattice with units $[0,1;\alpha]$ and $[\alpha,\alpha;\alpha]$.

Proof. It is clear from known order relation on $\mathbb{R}$.

Proposition 4. $\forall \alpha \in I$,

$$\bigcup_{\alpha\in I} D(I_\alpha) = D(I)$$

Proof. $M \in \bigcup_{\alpha\in I} D(I_\alpha)$ is given arbitrary.

$$\forall \alpha \in I, D(I_\alpha) \subseteq D(I) \Rightarrow \bigcup_{\alpha\in I} D(I_\alpha) \subseteq D(I)$$

$M \in D(I)$ is given arbitrary.

$$\alpha = \frac{M^L + M^U}{2} \text{ is chosen,}$$

$$M^L \le \alpha \le M^U \Rightarrow D(I) \subseteq D(I_\alpha) \Rightarrow D(I) \subseteq \bigcup_{\alpha\in I} D(I_\alpha)$$

Remark 1. The intersection and union of the family of α-interval valued sets are again α-interval valued sets. If any function satisfies following the conditions then it is called negation function.

Definition 11. L is complete lattice with units 0 and 1. $\mathcal{N}: L \to L$ and $\forall a, b \in L$,

i. $$\mathcal{N}(0) = 1 \text{ and } \mathcal{N}(1) = 0$$

ii. $$\mathcal{N}(a) \le \mathcal{N}(b) :\Leftrightarrow a \ge b$$

iii. $$\mathcal{N}(\mathcal{N}(a)) = a$$

We try to define a negation function on $D(I_\alpha)$ with the following relation, $\forall [M;\alpha] \in D(I_\alpha)$

$$\mathcal{N}([M;\alpha]) = \left[\alpha - M^L, 1 + \alpha - M^U; \alpha\right]$$

this relation on $D(I_\alpha)$ is a function. Indeed,

i. $$M^L \le \alpha \Rightarrow 0 \le \alpha - M^L \le \alpha$$

ii. Assume that $1 + \alpha - M^U < \alpha$, hence
$1 - M^U < 0 \Rightarrow 1 < M^U$ is contradiction. Then, $1 + \alpha - M^U \ge \alpha$

iii. Assume that $1 + \alpha - M^U > 1$, hence
$\alpha - M^U > 0 \Rightarrow \alpha > M^U$ is contradiction. Then, $1 + \alpha - M^U \le 1$ consequences from above, we get that;

$$0 \le \alpha - M^L \le \alpha \le 1 + \alpha - M^U \le 1$$

$$\Rightarrow \left[\alpha - M^L, 1 + \alpha - M^U; \alpha\right] = \mathcal{N}([M;\alpha]) \in D(I_\alpha)$$

From previous discussions, we can claim that $\mathcal{N}$ is a negation function on $D(I_\alpha)$.

Proposition 5: $\forall [M;\alpha] \in D(I_\alpha)$ and $\mathcal{N}: D(I_\alpha) \to D(I_\alpha)$,

$$\mathcal{N}([M;\alpha]) = \left[\alpha - M^L, 1 + \alpha - M^U; \alpha\right]$$

$\mathcal{N}$ satisfies conditions of *Definition 11*.

Proof. $[M;\alpha], [N;\alpha] \in D(I_\alpha)$ are given arbitrary.

1. $$[M;\alpha] = [N;\alpha] \Rightarrow M^L = N^L \text{ and } M^U = N^U$$

$$\Rightarrow \alpha - M^L = \alpha - N^L \text{ and } 1 + \alpha - M^U = 1 + \alpha - N^U$$

$$\Rightarrow \left[\alpha - M^L, 1 + \alpha - M^U; \alpha\right] = \left[\alpha - N^L, 1 + \alpha - N^U; \alpha\right]$$

$$\Rightarrow \mathcal{N}([M;\alpha]) = \mathcal{N}([N;\alpha])$$

2. Now, it is shown that $\mathcal{N}$ satisfies conditions of negation function.

i. $$\mathcal{N}([0,1;\alpha]) = [\alpha - 0, 1 + \alpha - 1; \alpha] = [\alpha, \alpha; \alpha]$$

$$\mathcal{N}([\alpha, \alpha; \alpha]) = [\alpha - \alpha, 1 + \alpha - \alpha; \alpha] = [0,1;\alpha]$$

ii.
$$\mathcal{N}([M;\alpha]) \le \mathcal{N}([N;\alpha])$$

$$\Leftrightarrow \left[\alpha - M^L, 1+\alpha - M^U;\alpha\right] \le \left[\alpha - N^L, 1+\alpha - N^U;\alpha\right]$$

$$\Leftrightarrow \alpha - M^L \le \alpha - N^L \text{ and } \quad 1+\alpha - M^U \ge 1+\alpha - N^U$$

$$\Leftrightarrow M^L \ge N^L \text{ and } M^U \le N^U \Leftrightarrow \left[M^L, M^U;\alpha\right] \ge \left[N^L, N^U;\alpha\right] \Leftrightarrow [M;\alpha] \ge [N;\alpha]$$

iii.
$$\mathcal{N}\left(\mathcal{N}([M;\alpha])\right) = \mathcal{N}\left(\left[\alpha - M^L, 1+\alpha - M^U;\alpha\right]\right)$$

$$= \left[\alpha - \left(\alpha - M^L\right), 1+\alpha - \left(1+\alpha - M^U\right);\alpha\right] = \left[M^L, M^U;\alpha\right] = [M;\alpha]$$

it is gotten that $\mathcal{N}: D(I_\alpha) \to D(I_\alpha)$ is negation function.

Definition 12. Let X be universal set and $[A;\alpha]: X \to D(I_\alpha)$ be function.

$$[A;\alpha] = \left\{\left[x, \left[A^L(x), A^U(x)\right];\alpha\right] \middle| x \in X\right\}$$

where; $A^L: X \to [0,1]$ and $A^U: X \to [0,1]$ are fuzzy sets.

In order to make easy, it is shown that;

$$\left\{\left[x, \left[A^L(x), A^U(x)\right];\alpha\right] \middle| x \in X\right\} = \left\{\left[x, A(x);\alpha\right] \middle| x \in X\right\}$$

$[A;\alpha]$ is called $\alpha-$interval valued fuzzy set on X. The family of $\alpha-$interval valued fuzzy sets on X is shown by $\alpha-$IVFS(X).

We can give below operations from previous definitions.

Definition 13. Let X be universal set and $[A;\alpha],[B;\alpha] \in \alpha-$IVFS(X).

Λ is index set $\forall\lambda \in \Lambda$,

i.
$$\left[A^c;\alpha\right] = \left\{\left[< x, \left[\alpha - A^L(x), 1+\alpha - A^U(x)\right] >;\alpha\right] \middle| x \in X\right\}$$

ii.
$$[A;\alpha] \sqsubseteq [B;\alpha] :\Leftrightarrow \forall x \in X, A^L(x) \le B^L(x) \text{ and } A^U(x) \ge B^U(x)$$

iii.
$$[A;\alpha] = [B;\alpha] :\Leftrightarrow \forall x \in X, A^L(x) = B^L(x) \text{ and } A^U(x) = B^U(x)$$

iv.
$$[A \sqcap B;\alpha] = \left\{\left[< x, \left[\inf\left\{A^L(x), B^L(x)\right\}, \sup\left\{A^U(x), B^U(x)\right\}\right] >;\alpha\right] \middle| x \in X\right\}$$

v
$$[A \sqcup B;\alpha] = \left\{\left[< x, \left[\sup\left\{A^L(x), B^L(x)\right\}, \inf\left\{A^U(x), B^U(x)\right\}\right] >;\alpha\right] \middle| x \in X\right\}$$

vi.
$$\left[\sqcap_{\lambda\in\Lambda} A_\lambda;\alpha\right] = \left\{\left[< x, \left[\Lambda_{\lambda\in\Lambda} {A_\lambda}^L(x), V_{\lambda\in\Lambda} {A_\lambda}^U(x)\right] >;\alpha\right] \middle| x \in X\right\}$$

vii.
$$\left[\sqcup_{\lambda\in\Lambda} A_\lambda;\alpha\right] = \left\{\left[< x, \left[V_{\lambda\in\Lambda} {A_\lambda}^L(x), \Lambda_{\lambda\in\Lambda} {A_\lambda}^U(x)\right] >;\alpha\right] \middle| x \in X\right\}$$

Example 2. Let $X=\{a,b,c,d\}$.

$$[A;0.4]=\left\{\begin{array}{c}[<a,[0.1,0.5]>;0.4],[<b,[0.3,1]>;0.4],\\ [<c,[0.4,0.7]>;0.4],[<d,[0.2,0.9]>;0.4]\end{array}\right\}$$

$$[B;0.4]=\left\{\begin{array}{c}[<a,[0.2,0.8]>;0.4],[<b,[0.1,0.6]>;0.4],\\ [<c,[0.3,0.4]>;0.4],[<d,[0.4,0.5]>;0.4]\end{array}\right\}$$

$[A;0.4]$ and $[B;0.4]$ are 0.4-interval valued fuzzy sets;

i. $$[A^c;\alpha]=\left\{\begin{array}{c}[<a,[0.3,0.9]>;0.4],[<b,[0.1,0.4]>;0.4],\\ [<c,[0,0.7]>;0.4],[<d,[0.2,0.5]>;0.4]\end{array}\right\}$$

ii. $$[A\sqcap B;\alpha]=\left\{\begin{array}{c}[<a,[0.1,0.8]>;0.4],[<b,[0.1,1]>;0.4],\\ [<c,[0.3,0.7]>;0.4],[<d,[0.2,0.9]>;0.4]\end{array}\right\}$$

iii. $$[A\sqcup B;\alpha]=\left\{\begin{array}{c}[<a,[0.2,0.5]>;0.4],[<b,[0.3,0.6]>;0.4],\\ [<c,[0.4,0.4]>;0.4],[<d,[0.4,0.5]>;0.4]\end{array}\right\}$$

Proposition 6. Let X be universal set. $\forall[A;\alpha],[B;\alpha],[C;\alpha]\in\alpha-\text{IVFS}(X)$ and Λ is index set $\forall\lambda\in\Lambda$,

i. $$[A\sqcap B;\alpha]=[B\sqcap A;\alpha]$$

ii. $$[A\sqcup B;\alpha]=[B\sqcup A;\alpha]$$

iii. $$[A;\alpha]\sqcap([B\sqcup C;\alpha])=([A\sqcap B;\alpha])\sqcup([A\sqcap C;\alpha])$$

iv. $$[A;\alpha]\sqcup([B\sqcap C;\alpha])=([A\sqcup B;\alpha])\sqcap([A\sqcup C;\alpha])$$

v. $$[A;\alpha]\sqcap([\sqcup_\lambda B_\lambda;\alpha])=[\sqcup_\lambda(A\sqcap B_\lambda);\alpha]$$

vi. $$[A;\alpha]\sqcup([\sqcap_\lambda B_\lambda;\alpha])=[\sqcap_\lambda(A\sqcup B_\lambda);\alpha]$$

Proof. $[A;\alpha],[B;\alpha],[C;\alpha]\in\alpha-\text{IVFS}(X)$ are given arbitrary.

i. $$[A\sqcap B;\alpha]=\left\{\left[<x,\left[\inf\left\{A^L(x),B^L(x)\right\},\sup\left\{A^U(x),B^U(x)\right\}\right]>;\alpha\right]\middle|x\in X\right\}$$

$$=\left\{\left[<x,\left[\inf\left\{B^L(x),A^L(x)\right\},\sup\left\{B^U(x),A^U(x)\right\}\right]>;\alpha\right]\middle|x\in X\right\}=[B\sqcap A;\alpha]$$

ii. $$[A\sqcup B;\alpha]=\left\{\left[<x,\left[\sup\left\{A^L(x),B^L(x)\right\},\inf\left\{A^U(x),B^U(x)\right\}\right]>;\alpha\right]\middle|x\in X\right\}$$

$$=\left\{\left[<x,\left[\sup\left\{B^L(x),A^L(x)\right\},\inf\left\{B^U(x),A^U(x)\right\}\right]>;\alpha\right]\middle|x\in X\right\}=[B\sqcup A;\alpha]$$

iii.
$$[A;\alpha]\sqcap\left([B\sqcup C;\alpha]\right)=[A;\alpha]\sqcap\left\{\left[<x,\begin{bmatrix}\sup\left\{B^{L}(x),C^{L}(x)\right\}\\ \inf\left\{B^{U}(x),C^{U}(x)\right\}\end{bmatrix}>;\alpha\right]\middle| x\in X\right\}$$

$$=\left\{\left[<x,\begin{bmatrix}\inf\left\{A^{L}(x),\sup\left\{B^{L}(x),C^{L}(x)\right\}\right\},\\ \sup\left\{A^{U}(x),\inf\left\{B^{U}(x),C^{U}(x)\right\}\right\}\end{bmatrix}>;\alpha\right]\middle| x\in X\right\}$$

$$=\left\{\left[<x,\begin{bmatrix}\sup\left\{\inf\left\{A^{L}(x),B^{L}(x)\right\},\inf\left\{A^{L}(x),C^{L}(x)\right\}\right\},\\ \inf\left\{\sup\left\{A^{U}(x),B^{U}(x)\right\},\sup\left\{A^{U}(x),C^{U}(x)\right\}\right\}\end{bmatrix}>;\alpha\right]\middle| x\in X\right\}$$

$$=\left\{\left[<x,\begin{bmatrix}\inf\left\{A^{L}(x),B^{L}(x)\right\},\\ \sup\left\{A^{U}(x),B^{U}(x)\right\}\end{bmatrix}>;\alpha\right]\middle| x\in X\right\}\sqcup\left\{\left[<x,\begin{bmatrix}\inf\left\{A^{L}(x),C^{L}(x)\right\},\\ \sup\left\{A^{U}(x),C^{U}(x)\right\}\end{bmatrix}>;\alpha\right]\middle| x\in X\right\}$$

$$=\left([A\sqcap B;\alpha]\right)\sqcup\left([A\sqcap C;\alpha]\right)$$

iv.
$$[A;\alpha]\sqcup\left([B\sqcap C;\alpha]\right)=[A;\alpha]\sqcup\left\{\left[<x,\begin{bmatrix}\inf\left\{B^{L}(x),C^{L}(x)\right\},\\ \sup\left\{B^{U}(x),C^{U}(x)\right\}\end{bmatrix}>;\alpha\right]\middle| x\in X\right\}$$

$$=\left\{\left[<x,\begin{bmatrix}\sup\left\{A^{L}(x),\inf\left\{B^{L}(x),C^{L}(x)\right\}\right\},\\ \inf\left\{A^{U}(x),\sup\left\{B^{U}(x),C^{U}(x)\right\}\right\}\end{bmatrix}>;\alpha\right]\middle| x\in X\right\}$$

$$=\left\{\left[<x,\begin{bmatrix}\inf\left\{\sup\left\{A^{L}(x),B^{L}(x)\right\},\sup\left\{A^{L}(x),C^{L}(x)\right\}\right\},\\ \sup\left\{\inf\left\{A^{U}(x),B^{U}(x)\right\},\inf\left\{A^{U}(x),C^{U}(x)\right\}\right\}\end{bmatrix}>;\alpha\right]\middle| x\in X\right\}$$

$$=\left\{\left[<x,\begin{bmatrix}\sup\left\{A^{L}(x),B^{L}(x)\right\},\\ \inf\left\{A^{U}(x),B^{U}(x)\right\}\end{bmatrix}>;\alpha\right]\middle| x\in X\right\}\sqcap\left\{\left[<x,\begin{bmatrix}\sup\left\{A^{L}(x),C^{L}(x)\right\},\\ \inf\left\{A^{U}(x),C^{U}(x)\right\}\end{bmatrix}>;\alpha\right]\middle| x\in X\right\}$$

$$=\left([A\sqcup B;\alpha]\right)\sqcap\left([A\sqcup C;\alpha]\right)$$

v. $$[A;\alpha]\sqcap\left([\sqcup_\lambda B_\lambda;\alpha]\right)$$

$$=[A;\alpha]\sqcap\left(\left\{\left[<x,\left[\bigvee_{\lambda\in\Lambda}B_\lambda{}^L(x),\bigwedge_{\lambda\in\Lambda}B_\lambda{}^U(x)\right]>;\alpha\right]\middle| x\in X\right\}\right)$$

$$=\left\{\left[<x,\begin{bmatrix}A^L(x),\\A^U(x)\end{bmatrix}>;\alpha\right]\middle| x\in X\right\}\sqcap\left(\left\{\left[<x,\left[\bigvee_{\lambda\in\Lambda}B_\lambda{}^L(x),\bigwedge_{\lambda\in\Lambda}B_\lambda{}^U(x)\right]>;\alpha\right]\middle| x\in X\right\}\right)$$

$$=\left\{\left[<x,\left[A^L(x)\wedge\bigvee_{\lambda\in\Lambda}B_\lambda{}^L(x),A^U(x)\vee\bigwedge_{\lambda\in\Lambda}B_\lambda{}^U(x)\right]>;\alpha\right]\middle| x\in X\right\}$$

$$=\left\{\left[<x,\left[\bigvee_{\lambda\in\Lambda}A^L(x)\wedge B_\lambda{}^L(x),\bigwedge_{\lambda\in\Lambda}A^U(x)\vee B_\lambda{}^U(x)\right]>;\alpha\right]\middle| x\in X\right\}$$

$$=\sqcup_\lambda\left\{\left[<x,\left[A^L(x)\wedge B_\lambda{}^L(x),A^U(x)\vee B_\lambda{}^U(x)\right]>;\alpha\right]\middle| x\in X\right\}=\left[\sqcup_\lambda(A\sqcap B_\lambda);\alpha\right]$$

vi. $$[A;\alpha]\sqcup\left([\sqcap_\lambda B_\lambda;\alpha]\right)$$

$$=[A;\alpha]\sqcup\left(\left\{\left[<x,\left[\bigwedge_{\lambda\in\Lambda}B_\lambda{}^L(x),\bigvee_{\lambda\in\Lambda}B_\lambda{}^U(x)\right]>;\alpha\right]\middle| x\in X\right\}\right)$$

$$=\left\{\left[<x,\begin{bmatrix}A^L(x),\\A^U(x)\end{bmatrix}>;\alpha\right]\middle| x\in X\right\}\sqcup\left(\left\{\left[<x,\left[\bigwedge_{\lambda\in\Lambda}B_\lambda{}^L(x),\bigvee_{\lambda\in\Lambda}B_\lambda{}^U(x)\right]>;\alpha\right]\middle| x\in X\right\}\right)$$

$$=\left\{\left[<x,\left[A^L(x)\vee\bigwedge_{\lambda\in\Lambda}B_\lambda{}^L(x),A^U(x)\wedge\bigvee_{\lambda\in\Lambda}B_\lambda{}^U(x)\right]>;\alpha\right]\middle| x\in X\right\}$$

$$=\left\{\left[<x,\left[\bigwedge_{\lambda\in\Lambda}A^L(x)\vee B_\lambda{}^L(x),\bigvee_{\lambda\in\Lambda}A^U(x)\wedge B_\lambda{}^U(x)\right]>;\alpha\right]\middle| x\in X\right\}$$

$$=\sqcap_{\lambda\in\Lambda}\left\{\left[<x,\left[A^L(x)\vee B_\lambda{}^L(x),A^U(x)\wedge B_\lambda{}^U(x)\right]>;\alpha\right]\middle| x\in X\right\}=\left[\sqcap_{\lambda\in\Lambda}(A\sqcup B_\lambda);\alpha\right]$$

Proposition 7. Let X be universal set. $\forall[A;\alpha],[B;\alpha]\in\alpha-\text{IVFS}(X)$ and Λ is index set $\forall\lambda\in\Lambda$,

i. $$\left[\left([A^c;\alpha]\right)^c;\alpha\right]=[A;\alpha]$$

ii. $$\left([A\sqcap B;\alpha]\right)^c=\left[A^c\sqcup B^c;\alpha\right]$$

iii. $$\left([A\sqcup B;\alpha]\right)^c=\left[A^c\sqcap B^c;\alpha\right]$$

iv. $$\left(\left[\sqcap_{\lambda\in\Lambda} A_\lambda;\alpha\right]\right)^c = \left[\sqcup_{\lambda\in\Lambda} A_\lambda^c;\alpha\right]$$

v. $$\left(\left[\sqcup_{\lambda\in\Lambda} A_\lambda;\alpha\right]\right)^c = \left[\sqcap_{\lambda\in\Lambda} A_\lambda^c;\alpha\right]$$

Proof. $[A;\alpha],[B;\alpha] \in \alpha - \text{IVFS}(X)$ are given arbitrary.

i. $$\left[A^c;\alpha\right] = \left\{\left[< x, \left[\alpha - A^L(x), 1+\alpha - A^U(x)\right] >;\alpha\right] \middle| x \in X\right\}$$

$$\Rightarrow \left[\left(\left[A^c;\alpha\right]\right)^c;\alpha\right]$$

$$= \left\{\left[< x, \left[\alpha - \left(\alpha - A^L(x)\right), 1+\alpha - \left(1+\alpha - A^U(x)\right)\right] >;\alpha\right] \middle| x \in X\right\}$$

$$\Rightarrow \left[\left(\left[A^c;\alpha\right]\right)^c;\alpha\right] = \left\{\left[< x, \left[A^L(x), A^U(x)\right] >;\alpha\right] \middle| x \in X\right\} = [A;\alpha]$$

ii. $$\left(\left[A \sqcap B;\alpha\right]\right)^c = \left\{\left[< x, \begin{bmatrix} \alpha - \inf\left\{A^L(x), B^L(x)\right\}, \\ 1+\alpha - \sup\left\{A^U(x), B^U(x)\right\} \end{bmatrix} >;\alpha\right] \middle| x \in X\right\}$$

$$= \left\{\left[< x, \begin{bmatrix} \sup\left\{\alpha - A^L(x), \alpha - B^L(x)\right\}, \\ \inf\left\{1+\alpha - A^U(x), 1+\alpha - B^U(x)\right\} \end{bmatrix} >;\alpha\right] \middle| x \in X\right\}$$

$$= \left\{\left[< x, \begin{bmatrix} \alpha - A^L(x), \\ 1+\alpha - A^U(x) \end{bmatrix} >;\alpha\right] \middle| x \in X\right\} \sqcup \left\{\left[< x, \begin{bmatrix} \alpha - B^L(x), \\ 1+\alpha - B^U(x) \end{bmatrix} >;\alpha\right] \middle| x \in X\right\}$$

$$= \left[A^c \sqcup B^c;\alpha\right]$$

iii. $$\left(\left[A \sqcup B;\alpha\right]\right)^c = \left\{\left[< x, \begin{bmatrix} \alpha - \sup\left\{A^L(x), B^L(x)\right\}, \\ 1+\alpha - \inf\left\{A^U(x), B^U(x)\right\} \end{bmatrix} >;\alpha\right] \middle| x \in X\right\}$$

$$= \left\{\left[< x, \begin{bmatrix} \inf\left\{\alpha - A^L(x), \alpha - B^L(x)\right\}, \\ \sup\left\{1+\alpha - A^U(x), 1+\alpha - B^U(x)\right\} \end{bmatrix} >;\alpha\right] \middle| x \in X\right\}$$

$$= \left\{\left[< x, \begin{bmatrix} \alpha - A^L(x), \\ 1+\alpha - A^U(x) \end{bmatrix} >;\alpha\right] \middle| x \in X\right\} \sqcap \left\{\left[< x, \begin{bmatrix} \alpha - B^L(x), \\ 1+\alpha - B^U(x) \end{bmatrix} >;\alpha\right] \middle| x \in X\right\}$$

$$= \left[A^c \sqcap B^c;\alpha\right]$$

iv. $$[\sqcap_{\lambda\in\Lambda} A_\lambda;\alpha]=\left\{\left[<x,\left[\bigwedge_{\lambda\in\Lambda} A_\lambda{}^L(x),\bigvee_{\lambda\in\Lambda} A_\lambda{}^U(x)\right]>;\alpha\right]\middle| x\in X\right\}$$

$$\Rightarrow\left([\sqcap_{\lambda\in\Lambda} A_\lambda;\alpha]\right)^c=\left\{\left[<x,\left[\alpha-\bigwedge_{\lambda\in\Lambda} A_\lambda{}^L(x),1+\alpha-\bigvee_{\lambda\in\Lambda} A_\lambda{}^U(x)\right]>;\alpha\right]\middle| x\in X\right\}$$

$$=\left\{\left[<x,\left[\bigvee_{\lambda\in\Lambda}\alpha-A_\lambda{}^L(x),\bigwedge_{\lambda\in\Lambda}1+\alpha-A_\lambda{}^U(x)\right]>;\alpha\right]\middle| x\in X\right\}=\left[\sqcup_{\lambda\in\Lambda} A_\lambda^c;\alpha\right]$$

v. $$[\sqcup_{\lambda\in\Lambda} A_\lambda;\alpha]=\left\{\left[<x,\left[V_\lambda A_{\lambda\in\Lambda}{}^L(x),\Lambda_{\lambda\in\Lambda}A_\lambda{}^U(x)\right]>;\alpha\right]\middle| x\in X\right\}$$

$$\Rightarrow\left([\sqcup_{\lambda\in\Lambda} A_\lambda;\alpha]\right)^c=\left\{\left[<x,\left[\alpha-\bigvee_{\lambda\in\Lambda} A_\lambda{}^L(x),1+\alpha-\bigwedge_{\lambda\in\Lambda} A_\lambda{}^U(x)\right]>;\alpha\right]\middle| x\in X\right\}$$

$$=\left\{\left[<x,\left[\bigwedge_{\lambda\in\Lambda}\alpha-A_\lambda{}^L(x),\bigvee_{\lambda\in\Lambda}1+\alpha-A_\lambda{}^U(x)\right]>;\alpha\right]\middle| x\in X\right\}=\left[\sqcap_{\lambda\in\Lambda} A_\lambda^c;\alpha\right]$$

Proposition 8. Let X be universal set. $0_X : X\to[0,1;\alpha]$ and $1_X : X\to[\alpha,\alpha;\alpha]$.

i. $$(0_X)^c=1_X$$

ii. $$(1_X)^c=0_X$$

Proof.

i. $$(0_X)^c=([0,1;\alpha])^c=[\alpha-0,1+\alpha-1;\alpha]=[\alpha,\alpha;\alpha]=1_X$$

ii. $$(1_X)^c=([\alpha,\alpha;\alpha])^c=[\alpha-\alpha,1+\alpha-\alpha;\alpha]=[0,1;\alpha]=0_X$$

Definition 14. Let X be universal set and $[A;\alpha]\in\alpha-\text{IVFS(X)}$. $[A;\alpha]$ has sup-property

$$:\Leftrightarrow \forall x\in X,\exists[\lambda_1,\lambda_2;\alpha]\in D(I_\alpha)\ni[A(x);\alpha]=[\lambda_1,\lambda_2;\alpha]$$

Definition 15. Let X be universal set and $[A;\alpha]\in\alpha-\text{IVFS(X)}$.

$$\forall[\lambda_1,\lambda_2;\alpha]\in D(I_\alpha),$$

$$[A;\alpha]_{[\lambda_1,\lambda_2;\alpha]}=\left\{x\in X\middle| A^L(x)\geq\lambda_1\text{and } A^U(x)\leq\lambda_2\right\}$$

The set $[A;\alpha]_{[\lambda_1,\lambda_2;\alpha]}$ is called $[\lambda_1,\lambda_2;\alpha]$-level subset of $[A;\alpha]$. It is easily seen from definition, $[\lambda_1,\lambda_2;\alpha]$-level subsets of $[A;\alpha]$ are crisp sets.

Definition 16. Let X be universal set and $[A;\alpha]\in\alpha-\text{IVFS(X)}$.

$$\forall[\lambda_1,\lambda_2;\alpha]\in D(I_\alpha),$$

$\forall[A;\alpha]_{[\lambda_1,\lambda_2;\alpha]}$-level subsets of $[A;\alpha]$,

i. $$A^L_{\lambda_1} = \left\{ x \in X \middle| A^L(x) \geq \lambda_1 \right\}$$

ii. $$A^U_{\lambda_2} = \left\{ x \in X \middle| A^U(x) \leq \lambda_2 \right\}$$

iii. $$B^L_{\lambda_1} = \left\{ x \in X \middle| B^L(x) \geq \lambda_1 \right\}$$

iv. $$B^U_{\lambda_2} = \left\{ x \in X \middle| B^U(x) \leq \lambda_2 \right\}$$

Proposition 9. Let X be universal set and $[A;\alpha],[B;\alpha] \in \alpha - \text{IVFS(X)}$. $\forall[\lambda_1,\lambda_2;\alpha] \in D(I_\alpha)$ and I is index set, $\forall i, j \in I, [\lambda_i,\lambda_j;\alpha] \in D(I_\alpha)$,

i. $$x \in [A;\alpha]_{[\lambda_1,\lambda_2;\alpha]} \Leftrightarrow [A(x);\alpha] \geq [\lambda_1,\lambda_2;\alpha]$$

ii. $$[A;\alpha]_{[\lambda_1,\lambda_2;\alpha]} = A^L_{\lambda_1} \cap A^U_{\lambda_2}$$

iii. $$([A \sqcup B;\alpha])_{[\lambda_1,\lambda_2;\alpha]}$$

$$= [A;\alpha]_{[\lambda_1,\lambda_2;\alpha]} \cup [B;\alpha]_{[\lambda_1,\lambda_2;\alpha]} \cup \left(A^L_{\lambda_1} \cap B^U_{\lambda_2}\right) \cup \left(B^L_{\lambda_1} \cap A^U_{\lambda_2}\right)$$

iv. $$([A \sqcap B;\alpha])_{[\lambda_1,\lambda_2;\alpha]} = [A;\alpha]_{[\lambda_1,\lambda_2;\alpha]} \cap [B;\alpha]_{[\lambda_1,\lambda_2;\alpha]}$$

v. $$A^L_{\lambda_1} \supseteq A^L_{\lambda_2}$$

vi. $$A^U_{\lambda_1} \subseteq A^U_{\lambda_2}$$

vii. $$\bigcap_{i\in I} A^L{}_{\lambda_i} = A^L{}_{\bigwedge_{i\in I}\lambda_i}$$

viii. $$\bigcup_{j\in I} A^U{}_{\lambda_j} = A^U{}_{\bigvee_{j\in I}\lambda_j}$$

Proof**.**

i. $$x \in [A;\alpha]_{[\lambda_1,\lambda_2;\alpha]} \Leftrightarrow A^L(x) \geq \lambda_1 \text{and } A^U(x) \leq \lambda_2$$

$$\Leftrightarrow \left[A^L(x), A^U(x);\alpha\right] \geq [\lambda_1,\lambda_2;\alpha] \Leftrightarrow [A(x);\alpha] \geq [\lambda_1,\lambda_2;\alpha]$$

ii. $x \in [A;\alpha]_{[\lambda_1,\lambda_2;\alpha]}$ is given arbitrary.

$$[A(x);\alpha] \geq [\lambda_1,\lambda_2;\alpha] \Leftrightarrow A^L(x) \geq \lambda_1 \text{and } A^U(x) \leq \lambda_2 \Leftrightarrow x \in A^L_{\lambda_1} \text{ and } x \in A^U_{\lambda_2}$$

$$\Leftrightarrow x \in A^{L}_{\lambda_1} \cap A^{U}_{\lambda_2}$$

iii. $x \in \left(\left[A \sqcup B;\alpha\right]\right)_{[\lambda_1,\lambda_2;\alpha]}$ is given arbitrary.

$$\left[(A \sqcup B)(x);\alpha\right] \ge [\lambda_1,\lambda_2;\alpha]$$

$$\Leftrightarrow \left[\sup\left\{A^{L}(x),B^{L}(x)\right\},\inf\left\{A^{U}(x),B^{U}(x)\right\};\alpha\right] \ge [\lambda_1,\lambda_2;\alpha]$$

$$\Leftrightarrow \sup\left\{A^{L}(x),B^{L}(x)\right\} \ge \lambda_1 \text{and} \inf\left\{A^{U}(x),B^{U}(x)\right\} \le \lambda_2$$

$$\Leftrightarrow \left\{A^{L}(x) \ge \lambda_1 \text{ or } B^{L}(x) \ge \lambda_1\right\} \text{ and} \left\{A^{U}(x) \le \lambda_2 \text{ or } B^{U}(x) \le \lambda_2\right\}$$

$$\Leftrightarrow \left\{A^{L}(x) \ge \lambda_1 \text{ and } A^{U}(x) \le \lambda_2\right\} \text{ or} \left\{B^{L}(x) \ge \lambda_1 \text{ and } B^{U}(x) \le \lambda_2\right\}$$

$$\text{or} \left\{A^{L}(x) \ge \lambda_1 \text{ and } B^{U}(x) \le \lambda_2\right\} \text{ or} \left\{B^{L}(x) \ge \lambda_1 \text{ and } A^{U}(x) \le \lambda_2\right\}$$

$$\Leftrightarrow x \in [A;\alpha]_{[\lambda_1,\lambda_2;\alpha]} \text{ or } x \in [B;\alpha]_{[\lambda_1,\lambda_2;\alpha]} \text{ or} \left\{x \in A^{L}_{\lambda_1} \cap B^{U}_{\lambda_2}\right\} \text{ or} \left\{x \in B^{L}_{\lambda_1} \cap A^{U}_{\lambda_2}\right\}$$

$$\Leftrightarrow x \in [A;\alpha]_{[\lambda_1,\lambda_2;\alpha]} \cup [B;\alpha]_{[\lambda_1,\lambda_2;\alpha]} \cup \left(A^{L}_{\lambda_1} \cap B^{U}_{\lambda_2}\right) \cup \left(B^{L}_{\lambda_1} \cap A^{U}_{\lambda_2}\right)$$

iv. $x \in \left(\left[A \sqcap B;\alpha\right]\right)_{[\lambda_1,\lambda_2;\alpha]}$ is given arbitrary.

$$\left[(A \sqcap B)(x);\alpha\right] \ge [\lambda_1,\lambda_2;\alpha]$$

$$\Leftrightarrow \left[\inf\left\{A^{L}(x),B^{L}(x)\right\},\sup\left\{A^{U}(x),B^{U}(x)\right\};\alpha\right] \ge [\lambda_1,\lambda_2;\alpha]$$

$$\Leftrightarrow \inf\left\{A^{L}(x),B^{L}(x)\right\} \ge \lambda_1 \text{ and} \sup\left\{A^{U}(x),B^{U}(x)\right\} \le \lambda_2$$

$$\Leftrightarrow \left\{A^{L}(x) \ge \lambda_1 \text{and } B^{L}(x) \ge \lambda_1\right\} \text{ and} \left\{A^{U}(x) \le \lambda_2 \text{ and } B^{U}(x) \le \lambda_2\right\}$$

$$\Leftrightarrow \left\{A^{L}(x) \ge \lambda_1 \text{and } A^{U}(x) \le \lambda_2\right\} \text{ and} \left\{B^{L}(x) \ge \lambda_1 \text{ and } B^{U}(x) \le \lambda_2\right\}$$

$$\Leftrightarrow x \in [A;\alpha]_{[\lambda_1,\lambda_2;\alpha]} \cap [B;\alpha]_{[\lambda_1,\lambda_2;\alpha]}$$

v. $x \in A^{L}_{\lambda_2}$ is given arbitrary.

$$A^{L}(x) \ge \lambda_2 \ge \lambda_1 \Rightarrow x \in A^{L}_{\lambda_1} \Rightarrow A^{L}_{\lambda_1} \supseteq A^{L}_{\lambda_2}$$

vi. $x \in A^{U}_{\lambda_1}$ is given arbitrary.

$$A^U(x) \le \lambda_1 \le \lambda_2 \Rightarrow x \in A^U_{\lambda_2} \Rightarrow A^U_{\lambda_1} \subseteq A^U_{\lambda_2}$$

vii. $x \in \bigcap_{i\in I} A^L{}_{\lambda_i}$ is given arbitrary.

$$x \in \bigcap_{i\in I} A^L_{\lambda_i} \Leftrightarrow \forall i \in I, x \in A^L_{\lambda_i} \Leftrightarrow \forall i \in I, A^L(x) \ge \lambda_i$$

$$\Leftrightarrow A^L(x) \ge \underset{i\in I}{\Lambda} \lambda_i \Leftrightarrow x \in A^L{}_{\underset{i\in I}{\Lambda}\lambda_i}$$

viii. $x \in \bigcup_{j\in I} A^U_{\lambda_j}$ is given arbitrary.

$$x \in \bigcup_{j\in I} A^U_{\lambda_j} \Leftrightarrow \exists j \in I, x \in A^U_{\lambda_j} \Leftrightarrow \exists j \in I, A^U(x) \le \lambda_j$$

$$\Leftrightarrow A^U(x) \le \underset{j\in I}{V} \lambda_j \Leftrightarrow x \in A^U{}_{\underset{j\in I}{V}\lambda_j}$$

Example 3. Let $X = \{a, b, c, d\}$.

$$[A;0.4] = \left\{ \begin{matrix} [<a,[0.1,0.5]>;0.4],[<b,[0.3,1]>;0.4], \\ [<c,[0.4,0.7]>;0.4],[<d,[0.2,0.9]>;0.4] \end{matrix} \right\}$$

$[A;0.4]$ is 0.4-interval valued fuzzy set;

i. $[A;0.4]_{[0,1;0.4]} = \{x \in X | A^L(x) \ge 0 \text{ and } A^U(x) \le 1\} = \{a,b,c,d\} = X$

ii. $[A;0.4]_{[0,0.5;0.4]} = \{x \in X | A^L(x) \ge 0 \text{ and } A^U(x) \le 0.5\} = \{a\}$

iii. $[A;0.4]_{[0,0.9;0.4]} = \{x \in X | A^L(x) \ge 0 \text{ and } A^U(x) \le 0.9\} = \{a,c,d\}$

iv. $[A;0.4]_{[0.3,0.5;0.4]} = \{x \in X | A^L(x) \ge 0.3 \text{ and } A^U(x) \le 0.5\} = \varnothing$

v. $[A;0.4]_{[0.3,0.7;0.4]} = \{x \in X | A^L(x) \ge 0.3 \text{ and } A^U(x) \le 0.7\} = \{c\}$

vi $[A;0.4]_{[0.2,0.9;0.4]} = \{x \in X | A^L(x) \ge 0.2 \text{ and } A^U(x) \le 0.9\} = \{c,d\}$

Proposition 10. Let X be universal set and $[A;\alpha] \in \alpha - \text{IVFS(X)}$.

$$M \notin D(I_\alpha) \Rightarrow [A;\alpha]_M = \varnothing$$

Proof: Let $M \notin D(I_\alpha)$. Assume that $[A;\alpha]_M \ne \varnothing$, hence,

$$\exists x \in [A;\alpha]_M \Rightarrow [A(x);\alpha] \ge M$$

$$\Rightarrow A^L(x) \ge M^L,\ A^U(x) \le M^U \Rightarrow \alpha \ge A^L(x) \ge M^L \text{ and } \alpha \le A^U(x) \le M^U$$

$$\Rightarrow M \in D(I_\alpha)$$

is contradiction. Then:

$$[A;\alpha]_M = \varnothing$$

Conclusion

In this study, the definition of α-interval valued set was given. It was shown that α-interval valued set is lattice by giving of definition of partial ordered relation on this set. In terms of definitions and discussions, α-interval valued fuzzy set was introduced. The definitions of intersection and union on this set were introduced. Afterwards, the complement of α-interval valued fuzzy sets was given by the help of negation function on α-interval valued set. Thus, the fundamental algebraic properties of this set were studied. In addition, the level subset of α-interval valued fuzzy set was given.

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