OSCILLATION ANALYSIS OF CONFORMABLE FRACTIONAL GENERALIZED LIENARD EQUATIONS

by

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In this study, we investigate the oscillatory properties of solutions of a class of conformable fractional generalized Lienard equations. By using generalized Riccati technique, we present some new oscillation results for the equation. Illustrative examples are also given.

Key words: *oscillation, Lienard equation, fractional derivative*

Introduction

Recently fractional differential equations have been applied to the modeling of many phenomena in such diverse fields as chemistry, physics, engineering, mechanics, medical sciences, economics, and finance [1-3]. Additionally, many researchers have proposed the fractional derivative definitions. The most common definitions among them are Riemann-Liouville and Caputo fractional derivative definitions. Almost all existing fractional derivatives do not satisfy the basic properties of classical derivative. Khalil *et al.* [4] have suggested conformable fractional derivative. Unlike the other fractional derivatives, this definition satisfies almost all the requirements of standard derivative. Research on oscillation of various equations including differential equations, dynamic equations on time scales and their fractional generalizations has been a hot topic in [5-20]. In these investigations, we notice that very little attention is paid to oscillation of conformable fractional differential equations [21-24].

In this study, we investigate the following conformable fractional differential equation:

$$x^{(2\alpha)}(t)+f[x(t)][x^{(\alpha)}(t)]^2+g[x(t)]x^{(\alpha)}(t)+h[x(t)]=0 \tag{1}$$

where $0<\alpha\leq 1$, *f, g,*and *h* are continuously differentiable functions on *R*.

A solution of eq. (1) is called oscillatory if it has arbitrarily large zeros, otherwise it is called non-oscillatory. Equation (1) is called oscillatory if all of its solutions are oscillatory.

Preliminaries

In this section, we give some necessary background materials for the conformable fractional theory.

Definition 1. [19] The left conformable fractional derivative starting from t_0 of a function $f:[t_0,\infty)\to R$ of order α with $0<\alpha\leq 1$ is defined by:

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$$(\mathbf{T}_{t_0}^{\alpha} f)(t) = f^{(\alpha)}(t) = \lim_{\varepsilon \to 0} \frac{f[t + \varepsilon (t - t_0)^{1-\alpha}] - f(t)}{\varepsilon}$$

when $\alpha = 1$, this derivative of $f(t)$ coincides with $f'(t)$. If $(\mathbf{T}_{t_0}^{\alpha} f)(t)$ exists on (t_0, t_1) then $(\mathbf{T}_{t_0}^{\alpha} f)(t_0) = \lim_{t \to t_{0^+}} f^{(\alpha)}(t)$.

Definition 2. [19] Let $\alpha \in (0,1]$. Then the left conformable fractional integral of order α starting at t_0 is defined by:

$$(\mathbf{I}_{t_0}^{\alpha} f)(t) = \int_{t_0}^{t} (s - t_0)^{\alpha - 1} f(s) \mathrm{d}s := \int_{t_0}^{t} f(s) \mathrm{d}_{t_0}^{\alpha} s$$

If the conformable fractional integral of a given function f exists, we call that f is α-integrable.

Lemma 1. [23] If $\alpha \in (0,1]$ and $f \in C^1([t_0, \infty), R)$, then, for all $t > t_0$, we have:

$$\mathbf{I}_{t_0}^{\alpha} \mathbf{T}_{t_0}^{\alpha}(f)(t) = f(t) - f(t_0) \text{ and } \mathbf{T}_{t_0}^{\alpha} \mathbf{I}_{t_0}^{\alpha}(f)(t) = f(t)$$

Lemma 2. [4]:

$$\mathbf{T}_{t_0}^{\alpha}(af + bg) = a\mathbf{T}_{t_0}^{\alpha}(f) + b\mathbf{T}_{t_0}^{\alpha}(g) \text{ for real constant a, b:}$$

$$\mathbf{T}_{t_0}^{\alpha}(fg) = f\mathbf{T}_{t_0}^{\alpha}(g) + g\mathbf{T}_{t_0}^{\alpha}(f)$$

$$\mathbf{T}_{t_0}^{\alpha}(t^p) = p\mathbf{t}^{p-\alpha} \text{ for all } p$$

$$\mathbf{T}_{t_0}^{\alpha} \frac{f}{g} = \frac{g\mathbf{T}_{t_0}^{\alpha}(f) - f\mathbf{T}_{t_0}^{\alpha}(g)}{g^2}$$

$$\mathbf{T}_{t_0}^{\alpha}(c) = 0 \text{ where } c \text{ is a constant}$$

$$\mathbf{T}_{t_0}^{\alpha}(f) = (t - t_0)^{1-\alpha} f'$$

Lemma 3. [24] Let $f, g : [t_0, t_1) \to R$ be two functions such that fg is differentiable. Then:

$$\int_{t_0}^{t_1} f(s) g^{(\alpha)}(s) \mathrm{d}_{t_0}^{\alpha} s = f(s) g(s) \Big|_{t_0}^{t_1} - \int_{t_0}^{t_1} g(s) f^{(\alpha)}(s) \mathrm{d}_{t_0}^{\alpha} s$$

Oscillation results

In this section, we present some new oscillation results for the equation.

Theorem 1. If:

$$\lim_{t \to \infty} \left(-\int_{T_1}^{t} \left\{ f[x(s)]^2 + \frac{\mathrm{d}f[x(s)]}{\mathrm{d}x} \right\} \mathrm{d}_{t_0}^{\alpha} s \right) = \infty \tag{2}$$

and

$$\lim_{t\to\infty}\int_T^t\left(\frac{\{g[x(s)]\}^2}{4\left(f[x(s)]^2+\frac{\mathrm{d}f[x(s)]}{\mathrm{d}x}\right)}-\frac{h[x(s)]}{f[x(s)]}\right)\mathrm{d}_{t_0}^{\alpha}s=\infty \tag{3}$$

then every solution of eq. (1) is oscillatory.

Proof. Let $x(t)$ be a non-oscillatory solution of eq. (1). Without loss of generality, we may assume that x is an eventually positive solution of (1). We define:

$$\varsigma(t)=\frac{x^{(\alpha)}(t)}{f[x(t)]}$$

Then we have:

$$\varsigma^{(\alpha)}(t)=\frac{x^{(2\alpha)}(t)\,f[x(t)]-x^{(\alpha)}(t)\frac{\mathrm{d}f[x(t)]}{\mathrm{d}x}x'(t)(t-t_0)^{1-\alpha}}{f^2[x(t)]}=$$

$$=-\{f[x(t)]\}^2\varsigma^2(t)-g[x(t)]\varsigma(t)-\frac{\mathrm{d}f[x(t)]}{\mathrm{d}x}\varsigma^2(t)-\frac{h[x(t)]}{f[x(t)]}=$$

$$=-\left(\left\{f[x(t)]^2+\frac{\mathrm{d}f[x(t)]}{\mathrm{d}x}\right\}^{1/2}\varsigma(t)+\frac{g[x(t)]}{2\{f[x(t)]^2+\frac{\mathrm{d}f[x(t)]}{\mathrm{d}x}\}^{1/2}}\right)^2+$$

$$+\frac{\{g[x(t)]\}^2}{4\{f[x(t)]^2+\frac{\mathrm{d}f[x(t)]}{\mathrm{d}x}\}}-\frac{h[x(t)]}{f[x(t)]}$$

Thus, for every t, T with $t\ge T\ge t_0$, we have:

$$\varsigma(t)=\varsigma(T)-\int_T^t\left(\left\{f[x(s)]^2+\frac{\mathrm{d}f[x(s)]}{\mathrm{d}x}\right\}^{1/2}\varsigma(s)+\frac{g[x(s)]}{2\{f[x(s)]^2+\frac{\mathrm{d}f[x(s)]}{\mathrm{d}x}\}^{1/2}}\right)^2\mathrm{d}_{t_0}^{\alpha}s+$$

$$+\int_T^t\left(\frac{\left(g[x(s)]\right)^2}{4\{f[x(s)]^2+\frac{\mathrm{d}f[x(s)]}{\mathrm{d}x}\}}-\frac{h[x(s)]}{f[x(s)]}\right)\mathrm{d}_{t_0}^{\alpha}s$$

By using the (3) implies there exists $T_1\ge T\ge t_0$; such that:

$$\varsigma(t)\ge-\int_T^t\left(\left\{f[x(s)]^2+\frac{\mathrm{d}f[x(s)]}{\mathrm{d}x}\right\}^{1/2}\varsigma(s)+\frac{g[x(s)]}{2\{f[x(s)]^2+\frac{\mathrm{d}f[x(s)]}{\mathrm{d}x}\}^{1/2}}\right)^2\mathrm{d}_{t_0}^{\alpha}s$$

If we define a function:

$$A(t)=-\int_T^t\left(\left\{f[x(s)]^2+\frac{\mathrm{d}f[x(s)]}{\mathrm{d}x}\right\}^{1/2}\varsigma(s)+\frac{g[x(s)]}{2\{f[x(s)]^2+\frac{\mathrm{d}f[x(s)]}{\mathrm{d}x}\}^{1/2}}\right)^2\mathrm{d}_{t_0}^{\alpha}s$$

Then $\varsigma(t) \geq A(t)$ for $t \geq T_1$. So, we get:

$$A^{(\alpha)}(t) = -\left(\left\{f[x(t)]^2 + \frac{\mathrm{d}f[x(t)]}{\mathrm{d}x}\right\}^{1/2}\varsigma(t) + \frac{g[x(t)]}{2\{f[x(t)]^2 + \frac{\mathrm{d}f[x(t)]}{\mathrm{d}x}\}^{1/2}}\right)^2 \geq$$

$$\geq -\left\{f[x(t)]^2 + \frac{\mathrm{d}f[x(t)]}{\mathrm{d}x}\right\}\varsigma^2(t) >$$

$$> -\left\{f[x(t)]^2 + \frac{\mathrm{d}f[x(t)]}{\mathrm{d}x}\right\}A^2(t)$$

We obtain:

$$-\left\{f[x(t)]^2 + \frac{\mathrm{d}f[x(t)]}{\mathrm{d}x}\right\} \leq \frac{A^{(\alpha)}(t)}{A^2(t)} \tag{4}$$

Integrating both sides of the (3) from T_1 to t, we have:

$$-\int_T^t \left\{f[x(s)]^2 + \frac{\mathrm{d}f[x(s)]}{\mathrm{d}x}\right\}\mathrm{d}_{t_0}^{\alpha} s \leq \frac{1}{A(T_1)} \tag{5}$$

Letting $t \to \infty$ in (5):

$$\lim_{t\to\infty}\left(-\int_{T_1}^t \left\{f[x(s)]^2 + \frac{\mathrm{d}f[x(s)]}{\mathrm{d}x}\right\}\mathrm{d}_{t_0}^{\alpha} s\right) \leq \frac{1}{A(T_1)}$$

which is a contradiction to eq. (2). This completes the proof of the theorem.

Theorem 2. If:

$$\lim_{t\to\infty}\int_T^t \left(\frac{g^2[x(s)]}{4\{g[x(s)]\,f[x(s)] + \frac{\mathrm{d}g[x(s)]}{\mathrm{d}x}\}} - \frac{h[x(s)]}{g[x(s)]}\right)\mathrm{d}_{t_0}^{\alpha} s = \infty \tag{6}$$

and

$$\lim_{t\to\infty}\left(-\int_{T_1}^t \left\{g[x(s)]\,f[x(s)] + \frac{\mathrm{d}g[x(s)]}{\mathrm{d}x}\right\}\mathrm{d}_{t_0}^{\alpha} s\right) = \infty \tag{7}$$

then every solution of (1) is oscillatory.

Proof. Let $x(t)$ be a non-oscillatory solution of eq. (1). Without loss of generality, we may assume that x is an eventually positive solution of (1). We define:

$$\xi(t) = \frac{x^{(\alpha)}(t)}{g[x(t)]}$$

Then we have:

$$\xi^{(\alpha)}(t) = \frac{x^{(2\alpha)}(t)\, g[x(t)] - x^{(\alpha)}(t)\frac{\mathrm{d}g[x(t)]}{\mathrm{d}x}x'(t)(t-t_0)^{1-\alpha}}{g^2[x(t)]} =$$

$$= -g[x(t)]\, f[x(t)]\xi^2(t) - g[x(t)]\xi(t) - \frac{g[x(t)]}{\mathrm{d}x}\xi^2(t) - \frac{h[x(t)]}{g[x(t)]} =$$

$$= -\left(\left\{g[x(t)]\, f[x(t)] + \frac{\mathrm{d}g[x(t)]}{\mathrm{d}x}\right\}^{1/2}\xi(t) + \frac{g[x(t)]}{2\{g[x(t)]\, f[x(t)] + \frac{\mathrm{d}g[x(t)]}{\mathrm{d}x}\}^{1/2}}\right)^2 +$$

$$+ \frac{g^2[x(t)]}{4\{g[x(t)]\, f[x(t)] + \frac{\mathrm{d}g[x(t)]}{\mathrm{d}x}\}} - \frac{h[x(t)]}{g[x(t)]}$$

Thus, for every t, T with $t \geq T \geq t_0$, we get:

$$\xi(t) = \xi(T) - \int_T^t \left(\left\{g[x(s)]\, f[x(s)] + \frac{\mathrm{d}g[x(s)]}{\mathrm{d}x}\right\}^{1/2}\xi(s) + \frac{g[x(s)]}{2\{g[x(s)]\, f[x(s)] + \frac{\mathrm{d}g[x(s)]}{\mathrm{d}x}\}^{1/2}}\right)^2 \mathrm{d}_{t_0}^{\alpha} s +$$

$$+ \int_T^t \left(\frac{g^2[x(s)]}{4\{g[x(s)]\, f[x(s)] + \frac{\mathrm{d}g[x(s)]}{\mathrm{d}x}\}} - \frac{h[x(s)]}{g[x(s)]}\right)\mathrm{d}_{t_0}^{\alpha} s$$

From (6) and there exists $T_1 \geq T \geq t_0$; such that:

$$\xi(t) \geq -\int_{T_1}^t \left(\left\{g[x(s)]\, f[x(s)] + \frac{\mathrm{d}g[x(s)]}{\mathrm{d}x}\right\}^{1/2}\xi(s) + \frac{g[x(s)]}{2\{g[x(s)]\, f[x(s)] + \frac{\mathrm{d}g[x(s)]}{\mathrm{d}x}\}^{1/2}}\right)\mathrm{d}_{t_0}^{\alpha} s$$

If we define new a function:

$$B(T) = -\int_{T_1}^t \left(\left\{g[x(s)]\, f[x(s)] + \frac{\mathrm{d}g[x(s)]}{\mathrm{d}x}\right\}^{1/2}\xi(s) + \frac{g[x(s)]}{2\{g[x(s)]\, f[x(s)] + \frac{\mathrm{d}g[x(s)]}{\mathrm{d}x}\}^{1/2}}\right)\mathrm{d}_{t_0}^{\alpha} s$$

Then $\xi(t) \geq B(t)$ for $t \geq T_1$. So, we have:

$$B^{(\alpha)}(t) = -\left(\left\{g[x(t)]\, f[x(t)] + \frac{\mathrm{d}g[x(t)]}{\mathrm{d}x}\right\}^{1/2}\xi(t) + \frac{g[x(t)]}{2\{g[x(t)]\, f[x(t)] + \frac{\mathrm{d}g[x(t)]}{\mathrm{d}x}\}^{1/2}}\right)^2 \geq$$

$$\geq -\left\{g[x(t)]\, f[x(t)] + \frac{\mathrm{d}g[x(t)]}{\mathrm{d}x}\right\}\xi^2(t) >$$

$$> -\left\{g[x(t)]\, f[x(t)] + \frac{\mathrm{d}g[x(t)]}{\mathrm{d}x}\right\}B^2(t)$$

We have:

$$-\left\{g[x(t)] f[x(t)] + \frac{\mathrm{d}g[x(t)]}{\mathrm{d}x}\right\} \leq \frac{B^{(\alpha)}(t)}{B^2(t)} \tag{8}$$

Integrating both sides of the (8) from T_1 to t, we have:

$$-\int_{T_1}^{t}\left\{g[x(s)]\,f[x(s)]+\frac{\mathrm{d}g[x(s)]}{\mathrm{d}x}\right\}\mathrm{d}_{t_0}^{\alpha}s\le\frac{1}{B(T_1)} \tag{9}$$

And letting $t\to\infty$ in (9):

$$\lim_{t\to\infty}\left(-\int_{T_1}^{t}\left\{g[x(s)]\,f[x(s)]+\frac{\mathrm{d}g[x(s)]}{\mathrm{d}x}\right\}\mathrm{d}_{t_0}^{\alpha}s\right)\le\frac{1}{B(T_1)}$$

which is a contradiction to eq. (7). This completes the proof.

Theorem 3. If:

$$\lim_{t\to\infty}\int_{T}^{t}\left(\frac{g^2[x(s)]}{4\{h[x(s)]\,f[x(s)]+\frac{\mathrm{d}h[x(s)]}{\mathrm{d}x}\}}-1\right)\mathrm{d}_{t_0}^{\alpha}s=\infty \tag{10}$$

and

$$\lim_{t\to\infty}\left(-\int_{T_1}^{t}\left\{h[x(s)]\,f[x(s)]+\frac{\mathrm{d}h[x(s)]}{\mathrm{d}x}\right\}\mathrm{d}_{t_0}^{\alpha}s\right)=\infty \tag{11}$$

then every solution of (1) is oscillatory.

Proof. Let $x(t)$ be a non-oscillatory solution of eq. (1). Without loss of generality, we may assume that x is an eventually positive solution of (1). We define:

$$\zeta(t)=\frac{x^{(\alpha)}(t)}{h[x(t)]}$$

So we get:

$$\zeta^{(\alpha)}(t)=\frac{x^{(2\alpha)}(t)\,h[x(t)]-x^{(\alpha)}(t)\,\frac{\mathrm{d}h[x(t)]}{\mathrm{d}x}\,x'(t)\,(t-t_0)^{1-\alpha}}{h^2[x(t)]}=$$

$$=-h[x(t)]\,f[x(t)]\,\zeta^2(t)-g[x(t)]\,\zeta(t)-\frac{h[x(t)]}{dx}\zeta^2(t)-1=$$

$$=-\left(\left\{h[x(t)]\,f[x(t)]+\frac{\mathrm{d}h[x(t)]}{\mathrm{d}x}\right\}^{1/2}\zeta(t)+\frac{g[x(t)]}{2\{h[x(t)]\,f[x(t)]+\frac{\mathrm{d}h[x(t)]}{\mathrm{d}x}\}^{1/2}}\right)^2+$$

$$+\frac{g^2[x(t)]}{4\{h[x(t)]\,f[x(t)]+\frac{\mathrm{d}h[x(t)]}{\mathrm{d}x}\}}-1$$

Thus, for every t, T with $t \ge T \ge t_0$, we have:

$$\zeta(t)=\zeta(T)-\int_T^t \left(\begin{aligned} &\left\{h[x(s)]\,f[x(s)]+\frac{\mathrm{d}h[x(s)]}{\mathrm{d}x}\right\}^{1/2}\xi(s)+ \\ &+\frac{g[x(s)]}{2\{h[x(s)]\,f[x(s)]+\frac{\mathrm{d}h[x(s)]}{\mathrm{d}x}\}^{1/2}} \end{aligned} \right)^2 \mathrm{d}_{t_0}^{\alpha}s+$$
$$+\int_T^t \left(\frac{g^2[x(s)]}{4\{h[x(s)]\,f[x(s)]+\frac{\mathrm{d}h[x(s)]}{\mathrm{d}x}\}}-1 \right) \mathrm{d}_{t_0}^{\alpha}s$$

By using the (10) implies there exists $T_1 \ge T \ge t_0$; such that:

$$\zeta(t)\ge-\int_{T_1}^t \left(\left\{h[x(s)]\,f[x(s)]+\frac{\mathrm{d}h[x(s)]}{\mathrm{d}x}\right\}^{1/2}\xi(s)+\frac{g[x(s)]}{2\{h[x(s)]\,f[x(s)]+\frac{\mathrm{d}h[x(s)]}{\mathrm{d}x}\}^{1/2}} \right) \mathrm{d}_{t_0}^{\alpha}s$$

If we consider the following function:

$$C(t)=-\int_{T_1}^t \left(\left\{h[x(s)]\,f[x(s)]+\frac{\mathrm{d}h[x(s)]}{\mathrm{d}x}\right\}^{1/2}\zeta(s)+\frac{g[x(s)]}{2\{h[x(s)]\,f[x(s)]+\frac{\mathrm{d}h[x(s)]}{\mathrm{d}x}\}^{1/2}} \right) \mathrm{d}_{t_0}^{\alpha}s$$

Then $\zeta(t) \ge C(t)$ for $t \ge T_1$. So:

$$C^{(\alpha)}(t)=-\left(\left\{h[x(t)]\,f[x(t)]+\frac{\mathrm{d}h[x(t)]}{\mathrm{d}x}\right\}^{1/2}\zeta(t)+\frac{g[x(t)]}{2\{h[x(t)]\,f[x(t)]+\frac{\mathrm{d}h[x(t)]}{\mathrm{d}x}\}^{1/2}} \right)^2 \ge$$
$$\ge-\left\{h[x(t)]\,f[x(t)]+\frac{\mathrm{d}h[x(t)]}{\mathrm{d}x}\right\}\zeta^2(t)>$$
$$>-\left\{h[x(t)]\,f[x(t)]+\frac{\mathrm{d}h[x(t)]}{\mathrm{d}x}\right\}B^2(t)$$

We obtain the following inequality:

$$-\left\{h[x(t)]\,f[x(t)]+\frac{\mathrm{d}h[x(t)]}{\mathrm{d}x}\right\}\le\frac{C^{(\alpha)}(t)}{C^2(t)} \tag{12}$$

Integrating both sides of the (12) from T_1 to t, we have:

$$-\int_{T_1}^t \left\{h[x(s)]\,f[x(s)]+\frac{\mathrm{d}h[x(s)]}{\mathrm{d}x}\right\}\mathrm{d}_{t_0}^{\alpha}s\le\frac{1}{C(T_1)}$$

Letting $t \to \infty$ in (13):

$$\lim_{t\to\infty}\left(-\int_{T_1}^t \left\{h[x(s)]\,f[x(s)]+\frac{\mathrm{d}h[x(s)]}{\mathrm{d}x}\right\}\mathrm{d}_{t_0}^{\alpha}s \right)\le\frac{1}{C(T_1)} \tag{13}$$

which is a contradiction to eq. (11). This completes the proof of the theorem.

Applications

In this section we present some examples.

Example 1. Consider the following conformable fractional differential equation:

$$x^{(1/2)}(t)+\cot[x(t)][x^{(1/4)}(t)]^2+x^{(1/4)}(t)-\cot[x(t)]=0 \tag{14}$$

for $t \geq 0$. This corresponds to eq. (1) with $t_0=0$, $\alpha=1/4$, $f(x)=\cot(x)$, $g(x)=1$ and $h(x)=-\cot(x)$. Then we have:

$$\lim_{t\to\infty}\left(-\int_T^t\{\cot^2[x(s)]-\csc^2[x(s)]\}\mathrm{d}_{t_0}^{\alpha}s\right)=\lim_{t\to\infty}\left(-\int_T^t-\mathrm{d}_{t_0}^{\alpha}s\right)=\infty$$

where $T>0$ and

$$\lim_{t\to\infty}\int_T^t\left(\frac{\{g[x(s)]\}^2}{4\{f[x(s)]^2+\frac{\mathrm{d}f[x(s)]}{\mathrm{d}x}\}}-\frac{h[x(s)]}{f[x(s)]}\right)\mathrm{d}_{t_0}^{\alpha}s=\lim_{t\to\infty}\int_T^t\left(\frac{1}{-4}+1\right)\mathrm{d}_{t_0}^{\alpha}s=\infty$$

(2) and (3) holds. Thus, the eq. (14) is oscillatory from *Theorem 1*.

Example 2. Consider the following conformable fractional differential equation:

$$x^{(2/3)}(t)+\left[\frac{1-3x^2(t)}{1+x^3(t)}\right][x^{(1/6)}(t)]^2-[1+x^3(t)]x^{(1/6)}(t)-[1+x^3(t)]^2=0 \tag{15}$$

for $t \geq 0$. This corresponds to eq. (1) with $t_0=0$, $\alpha=1/4$, $f(x)=\cot(x)$, $g(x)=1$ and $h(x)=-\cot(x)$. Then we have:

$$\lim_{t\to\infty}\left(-\int_T^t\{\cot^2[x(s)]-\csc^2[x(s)]\}d_{t_0}^{\alpha}s\right)=\lim_{t\to\infty}\left(-\int_T^t-\mathrm{d}_{t_0}^{\alpha}s\right)=\infty$$

where $T>0$ and

$$\lim_{t\to\infty}\int_T^t\left(\frac{\{g[x(s)]\}^2}{4\{f[x(s)]^2+\frac{\mathrm{d}f[x(s)]}{\mathrm{d}x}\}}-\frac{h[x(s)]}{f[x(s)]}\right)\mathrm{d}_{t_0}^{\alpha}s=\lim_{t\to\infty}\int_T^t\left(\frac{1}{-4}+1\right)\mathrm{d}_{t_0}^{\alpha}s=\infty$$

(2) and (3) holds. Thus, eq. (14) is oscillatory from *Theorem 2*.

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