ON SEPARATION AXIOMS ON LEFT AND RIGHT OPERAND TOPOLOGIES

by

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In this paper, we examine separation axioms on left operand and right operand topologies generated by a raw binary operation which is weaker than the concepts of both partial and multivalued binary operation, and obtain some useful results.

Key words: *raw binary operation, left topology, right topology, separation axiom*

Introduction

Let X be a non-empty set. We denote a graph of a binary relation R by $\Gamma(R)$. A (resp. partial, multivalued) binary operation $*$ on a set X is a (resp. partial, multivalued) map from $X\times X$ to X. A group-like structure is a non-empty set X equipped with a (partial) binary operation f on X which satisfies specific axioms among such as closureness, associativity, identity, invertibility, and commutativity, for more information see [1-5]. Groupoid, semigroupoid, small category are examples of group-like structures that have a partial binary operation.

In 1969, Smithson [6] showed under which condition the topological spaces generated by binary relations satisfy the separation axioms, are compact or connected. Allam *et al.* [7] presented several methods of indirectly generating topological spaces by using operators such as closure, interior and neighborhood operators given by binary relations, and obtained useful results. Indurain and Knoblauch [8] studied categories of topological spaces induced by a binary relation, and also they prove that every metric topology is induced by a binary relation.

Now let's continue with the following concept, which is weaker than both partial and multivalued binary operations. A *raw binary operation* (or simply a *raw binop*) $*$ on X is a binary relation $*$ over sets $X\times X$ and X [9]. Also, the ordered pair $(X, *)$ is called a *raw binary structure* (or simply a *raw bistruct*). We denote by $x*y$ the set $\{z\in X \mid (x,y)*z\}$. Left operand and right operand and output set for $(a,b)\in X^2$ are defined, respectively, by $L(a,b)=\{x\in X \mid b\in x*a\}$, $R(a,b)=\{x\in X \mid b\in a*x\}$ and $a*b$. In a raw bistruct $(X, *)$, we can consider the collection of all left operand (resp. right operand, output) sets, as a subbase for a topology on X, which is called left operand (resp. right operand, output) topology and denoted by $\mathcal{L}$ (resp. $\mathcal{R}$, $\mathcal{O}$).

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Unlike concepts such as connectedness and compactness that naturally arise from the study of analysis, separation axioms that emerge from a deeper look on topology itself are extra axioms in addition to the topological space axioms that constitute more specific topological space classes. The reason they are called separation axioms is because they *separate* certain kinds of sets from each other in certain ways.

It is natural to ask the question of what condition a raw binop must satisfy in order for a particular topology generated by it to satisfy a certain separation axiom. In this study, from this point of view, we investigate the separation axioms on the left operand topology, $\mathcal{L}$, and the right operand topology, $\mathcal{R}$, generated by a raw binop (we exclude the output topology because it has a different structure than the other two topologies, which will greatly increase the size of the paper).

Preliminaries

Before the separation axioms, we will introduce some preliminary definitions [10-14]. Let X be a topological space. If two points $x, y \in X$ have exactly the same (open) neighborhoods, then they are said to be topologically indistinguishable or simply indistinguishable; otherwise they are (topologically) distinguishable. A pair of separated sets are two sets neither of which belongs to the other's closure. Given two subsets $A, B \subseteq X$. They are said to be separated by neighborhoods if they have disjoint (open) neighborhoods. If they have disjoint closed neighborhoods, then they are called sets separated by closed neighborhoods. They are said to be separated by a continuos function if there exists a continuous function $f: X \to \mathbb{R}$ such that $f(A) = 1$ and $f(B) = 0$. Again, they are called precisely separated by a continuos function if there exists a continuous function $f: X \to \mathbb{R}$ such that $f^{-1}(1) = A$ and $f^{-1}(0) = B$.

Precisely seperated by a cont. function	Precisely separated by a cont. function
$\Downarrow$	$\Downarrow$
Separated by a cont. function	Separated by a cont. function
$\Downarrow$	$\Downarrow$
Separated by closed neighborhoods	Separated by closed neighborhoods
$\Downarrow$	$\Downarrow$
Separated by neighborhoods	Separated by neighborhoods
$\Downarrow$	$\Downarrow$
Separated	Separated
$\Downarrow$	$\Downarrow$
Distinguishable	Disjoint
$\Downarrow$	
Distinct	

Figure 1. Relationships among the conditions that topologically *separate* a pair of points (left) and a pair of sets (right)

Note that the aforementioned concepts defined for a pair of sets can also be used for a pair of points using the corresponding pair of singleton sets. The relationships among the conditions given above can be illustrated with respect to a pair of points and a pair of sets as figure 0.

Definition 1. Let $(X, *)$ be a raw bistruct and let F be a non-empty finite subset of X^2. Let $x \in X$. An element $x \in X$ is called a *left* (*right*) *common point of* F if $b \in x*a$ $(b \in a*x)$

holds for every $(a,b)\in F$. A subset $A\subseteq X$ is called a left (right) common set of *F* if every element of *A* is a left (right) common point of *F*.

Definition 2. Let $(X, *)$ be a raw bistruct, $\mathcal{F}$ be a family of non-empty finite subsets of X^2, $x\in X$ and $A\subseteq X$. The $\mathcal{F}$ is called a *left* (*right*) *common pointer of* x if there exists a set $F\in\mathcal{F}$ having x as a left (right) common point. The $\mathcal{F}$ is called a left (right) common pointer of a subset $A\subseteq X$ if it is a left (right) common pointer of each element of *A*. The $\mathcal{F}$ is called a *left* (*right*) *non-common pointer of a subset* $A\subseteq X$ if it is a left (right) common pointer of none of the elements of *A*.

Definition 3. Let $(X, *)$ be a raw bistruct. A subset $A\subseteq X$ is called a *raw-binarily* $\mathcal{L}(\mathcal{R})$ *-closed set* if there exists a family $\mathcal{F}$ of non-empty finite subsets of X^2 such that $x\in A$ if and only if $\mathcal{F}$ is a left (right) non-common pointer of *x*.

Proposition 1. If any subset of a raw bistruct $(X, *)$ is a raw-binarily $\mathcal{L}(\mathcal{R})$ -closed set, then it is a closed set in the left operand topology $\mathcal{L}$ (the right operand topology $\mathcal{R}$).

Proof. Let $(X, *)$ be a raw bistruct and let $A\subseteq X$ be a raw-binarily $\mathcal{L}$ -closed set. Then there exists a family $\mathcal{F}$ of non-empty finite subsets of X^2 such that $x\in A$ if and only if $\mathcal{F}$ is a left non-common pointer of *x*. Let $x\in X$ be chosen arbitrarily. By *Definition 2* and *Definition 1*, $x\in A$ if and only if for every $F\in\mathcal{F}$, there exists $(a,b)\in F$ such that $b\notin x*a$ and so $x\notin L(a,b)$ which is equivalent that $x\notin U$ where $U=\bigcup_{F\in\mathcal{F}}\bigcap_{(a,b)\in F}L(a,b)$ is a open set in $\mathcal{L}$. Thus this is necessary and sufficient for *A* to be a closed set in $\mathcal{L}$. We use similar arguments to those used the proof for the right operand topology $\mathcal{R}$.

Definition 4. A $\mathcal{L}(\mathcal{R})$ *-usual raw binop* $*$ on $\mathbb{R}$ is a raw binop on $\mathbb{R}$ in which every $\mathcal{L}(\mathcal{R})$ -open set is an open ray. A raw bistruct $(\mathbb{R},*)$ with equipped a $\mathcal{L}(\mathcal{R})$ -usual raw binop is called a $\mathcal{L}(\mathcal{R})$ *-usual raw bistruct*.

Proposition 2. If $(\mathbb{R},*)$ is a $\mathcal{L}(\mathcal{R})$ -usual raw bistruct, the left operand topology $\mathcal{L}$ (the right operand topology $\mathcal{R}$) induced by $(\mathbb{R},*)$ is a usual topology on $\mathbb{R}$.

Proof. Let $(\mathbb{R},*)$ be a $\mathcal{L}$ -usual raw bistruct and let $a,b\in\mathbb{R}$ be chosen arbitrarily. By *Definition 4*, $L(a,b)$ is an open ray, that is, $L(a,b)=(-\infty,p)$ or $L(a,b)=(p,\infty)$ for some $p\in\mathbb{R}$. Then a non-empty basis member for the left operand topology $\mathcal{L}$ on $\mathbb{R}$ is in the form of an open interval, since for some non-empty finite subset *F* of X^2, $\bigcap_{(a,b)\in F}L(a,b)=(p,q)$ for some $p,q\in\mathbb{R}\cup\{-\infty,\infty\}$. This implies $\mathcal{L}$ consists of arbitrarily many union of open intervals. Thus $\mathcal{L}$ is a usual topology on $\mathbb{R}$.

Definition 5. Let $(X, *)$ and $(Y,\circ)$ be two raw bistructs. A function $f:X\to Y$ is called a *raw-binarily* $\mathcal{L}(\mathcal{R})$ *-continuous function* if for every $c,d\in Y$, there exists a family $\mathcal{F}$ of non-empty finite subsets of X^2 which satisfies the following condition: for every $x\in X$, $d\in f(x)*c$ if and only if $\mathcal{F}$ is a left common pointer of *x*.

Proposition 3. If a function *f* from a raw bistruct $(X, *)$ to another raw bistruct $(Y,\circ)$ is a raw-binarily $\mathcal{L}(\mathcal{R})$ -continuous function, then it is also continuous with respect to the left operand topology $\mathcal{L}$ (the right operand topology $\mathcal{R}$).

Proof. Let $(X, *)$ and $(Y,\circ)$ be two raw bistructs and let $f:X\to Y$ be a raw-binarily $\mathcal{L}(\mathcal{R})$ -continuous function. Chosen arbitrarily $c,d\in Y$. By *Definition 5*, there exists a family $\mathcal{F}$ of non-empty finite subsets of X^2 such that for every $x\in X$,

$$\mathcal{F}\text{ is a left common pointer of } x \text{ iff } d\in f(x)*c$$

Then by using *Definition 2* and *Definition 1*, for every *x* in *X*, $x\in\bigcup_{F\in\mathcal{F}}\bigcap_{(a,b)\in F}L(a.b)$ if and only if $f(x)\in L(c,d)$ which implies that $\bigcup_{F\in\mathcal{F}}\bigcap_{(a,b)\in F}L(a,b)=f^{-1}[L(c,d)]$. Since $L(c,d)$ is an arbitrarily chosen member of a subbase for the topology $\mathcal{L}_Y$ and $U = \bigcup_{F\in\mathcal{F}} \bigcap_{(a,b)\in F} L(a,b)$ is an open set in $\mathcal{L}_X$, the inverse image of an arbitrary subbase member $L(c,d)$ under f is an open set in $\mathcal{L}_X$ which means that f is also continuous with respect to the left operand topology $\mathcal{L}$. We use similar arguments to those used the proof for the right operand topology $\mathcal{R}$.

Definition 6. Let $(X, *)$ be a raw bistruct. Any two points x, y in X is called *raw-binarily $\mathcal{L}(\mathcal{R})$-distinguishable* if there exists a family of non-empty finite subsets of X^2 which is a left (right) common pointer of u but not of v for $u = x$, $v = y$ or $u = y$, $v = x$.

Definition 7. Let $(X, *)$ be a raw bistruct. Any two subsets $A, B \subseteq X$ is called *raw-binarily $\mathcal{L}(\mathcal{R})$-separated* if there exists two families of non-empty finite subsets of X^2, one of which is a left (right) common pointer of A but not of B, while the other of which is a left (right) common pointer of B but not of A.

Definition 8. Let $(X, *)$ be a raw bistruct and let F be a non-empty finite subset of X^2. The *left operand subbase with respect to F*, denoted by $\mathcal{S}_F^L$, is a family of the left operand sets of members of F, that is, $\mathcal{S}_F^L := \{L(a,b) | (a,b) \in F\}$. Similarly, The *right operand subbase with respect to F*, denoted by $\mathcal{S}_F^R$, is a family of the right operand sets of members of F, that is, $\mathcal{S}_F^R := \{R | (a,b) \in F\}$.

Before giving the definition, recall that we say a non-empty family $\mathcal{A}$ of subsets of a set X has the finite intersection property (FIP) if the intersection over any finite subfamily of $\mathcal{A}$ is non-empty.

Definition 9. Let $(X, *)$ be a raw bistruct and let F be a non-empty subset of X^2. F is called a *left* (*right*) *non-FIP index set* if $\mathcal{S}_F^L$ ($\mathcal{S}_F^R$) has not FIP.

Definition 10. Two families $\mathcal{F}$, $\mathcal{G}$ of non-empty subsets of X^2 are said to be *left* (*right*) *non-FIP indexable* if $\mathcal{F} \sqcup \mathcal{G}$ consists of left (right) non-FIP index sets where $\mathcal{F} \sqcup \mathcal{G} := \{F \cup G | F \in \mathcal{F}, G \in \mathcal{G}\}$.

Definition 11. Let $(X, *)$ be a raw bistruct. Any two subsets $A, B \subseteq X$ is called *raw-binarily $\mathcal{L}(\mathcal{R})$-separated by neighborhoods* if there exists two left (right) non-FIP indexable families of non-empty finite subsets of X^2, one of which is a left (right) common pointer of A, while the other of which is a left (right) common pointer of B.

Definition 12. Let $(X, *)$ be a raw bistruct. Any two subsets $A, B \subseteq X$ is called *raw-binarily $\mathcal{L}(\mathcal{R})$-separated by closed neighborhoods* if there exists two families of non-empty finite subsets of X^2, one of which is left (right) non-FIP indexable together with a left (right) common pointer of A, and the other of which is left (right) non-FIP indexable together with a left (right) common pointer of B, and whose union is a left (right) common pointer of X.

Definition 13. Let $(X, *)$ be a raw bistruct. Any two subsets $A, B \subseteq X$ is called *raw-binarily $\mathcal{L}(\mathcal{R})$-separated by a continuous function* if there exists a raw-binarily $\mathcal{L}(\mathcal{R})$-continuous function f from X to a $\mathcal{L}(\mathcal{R})$-usual raw bistruct $(\mathbb{R}, \circ)$ such that $f(A) = \{0\}$ and $f(B) = \{1\}$.

Definition 14. Let $(X, *)$ be a raw bistruct. Any two subsets $A, B \subseteq X$ is called *raw-binarily precisely $\mathcal{L}(\mathcal{R})$-separated by a continuous function* if there exists a raw-binarily $\mathcal{L}(\mathcal{R})$-continuous function f from X to a $\mathcal{L}(\mathcal{R})$-usual raw bistruct $(\mathbb{R}, \circ)$ such that $A = f^{-1}\{(0)\}$ and $B = f^{-1}\{(1)\}$.

Main results

Theorem 1. Let $(X, *)$ be a raw bistruct. If any two points in X are raw-binarily $\mathcal{L}(\mathcal{R})$-distinguishable, then they are also topologically distinguishable with respect to the left operand topology $\mathcal{L}$ (the right operand topology $\mathcal{R}$).

Proof. Let $(X, *)$ be a raw bistruct and x, y be arbitrary two points in X. From the hypothesis, there exists a family of non-empty finite subsets of X^2 satisying the condition in *Definition 6*. Let's call this family $\mathcal{F}$. We prove it by cases. Consider the case where $u = x$, $v = y$. By the hypothesis, $\mathcal{F}$ is a left common pointer of x but not of y. By *Definition 2*, we can find a member of $\mathcal{F}$, say F, having x but not y as a left common point. It follows by *Definition 1* that there exists $(a,b) \in F$ such that $b \in x*a$ while $b \notin y*a$ which means that$x \in L(a,b) \not\ni y$. Therefore the points x, y are topologically distinguishable with regard to the left operand topology $\mathcal{L}$ induced by $(X, *)$ since $L(a, b)$ is an open set in $\mathcal{L}$. Using arguments similar to the above case for the case where $u = y$, $v = x$ it can easily be shown that the points x, y are topologically distinguishable. By similar arguments, it can be proven that the implication holds for the right operand topology $\mathcal{R}$. endpefalse A Kolmogorov (or T_0) space is one in which every distinct pair of points is topologically distinguishable. Let us give the property that a raw bistruct must have in order to generate a Kolmogorov space with regard to the left operand topology $\mathcal{L}$ (the right operand topology $\mathcal{R}$).

Corollary 1. If any distinct pair of points in a raw bistruct $(X\ *)$ is raw-binarily $\mathcal{L}(\mathcal{R})$ -distinguishable, then X is a $\mathcal{L}(\mathcal{R})$ -Kolmogorov space.

Theorem 2. If any two subsets in a raw bistruct $(X, *)$ are raw-binarily $\mathcal{L}(\mathcal{R})$ -separated, then they are also topologically separated with respect to the left operand topology $\mathcal{L}$ (the right operand topology $\mathcal{R}$).

Proof. Let $(X, *)$ be a raw bistruct and $A, B \subseteq X$. By the hypothesis, there exists two families of non-empty finite subsets of X^2, one, say $\mathcal{F}$, of which is a left common pointer of A but not of B, while the other, say $\mathcal{G}$, of which is a left common pointer of B but not of A. By *Definition 2*, we can find a member of $\mathcal{F}$, say F, having x but not y as a left common point for every $x \in A$ and for at least one $y \in B$, and we can find a member of $\mathcal{G}$, say G, having y but not x as a left common point for at least one $x \in A$ and for every $y \in B$. Then from *Definition 1*, there exists $(a,b) \in F$ such that $b \in x*a$ while $b \notin y*a$ which means that $x \in L(a,b) \not\ni y$ for every $x \in A$ and for at least one $y \in B$, and there exists $(c,d) \in G$ such that $d \in y*c$ while $d \notin x*c$ which means that $y \in L(c,d) \not\ni x$ for at least one $x \in A$ and for every $y \in B$. Therefore it follows immediately that $A \subseteq L(a,b) \not\supseteq B$ and $A \not\subseteq L(c,d) \supseteq B$. Thus A, B are topologically separated with regard to the left operand topology $\mathcal{L}$ induced by $(X, *)$ since $L(a, b)$ and $L(c, d)$ are open in $\mathcal{L}$. We use similar arguments to those used the proof for the right operand topology $\mathcal{R}$. endpefalse Remember that a symmetric (or R_0) space is one where every topologically distinguishable pair of points are separated. Now let's give a corollary for symmetric spaces.

Corollary 2. If any raw-binarily $\mathcal{L}(\mathcal{R})$ -distinguishable pair of points in a raw bistruct $(X, *)$ is raw-binarily $\mathcal{L}(\mathcal{R})$ -separated, then X is a $\mathcal{L}(\mathcal{R})$ -symmetric space.

We write the following result using the fact that a symmetric Kolmogorov space is an accessible space (or T_1).

Corollary 3. If any distinct pair of points in a raw bistruct $(X, *)$ is raw-binarily $\mathcal{L}(\mathcal{R})$ -separated, then X is a $\mathcal{L}(\mathcal{R})$ -accessible space.

Theorem 3. If any two subsets in a raw bistruct $(X, *)$ are raw-binarily $\mathcal{L}(\mathcal{R})$ -separated by neighborhoods, then they are also topologically separated by neighborhoods with respect to the left operand topology $\mathcal{L}$ (the right operand topology $\mathcal{R}$).

Proof. Let $(X, *)$ be a raw bistruct and $A, B \subseteq X$. By the hypothesis, there exists two left non-FIP indexable families of non-empty finite subsets of X^2, one, say $\mathcal{F}$, of which is a left common pointer of A, while the other, say $\mathcal{G}$, of which is a left common pointer of B. By *Definition 2*, for every $x \in A$, we can find a member of $\mathcal{F}$, say F, having x as a left common point, and for every $y \in B$, we can find a member of $\mathcal{G}$, say G, having y as a left common point. Then from Definition 1, for every $x \in A$ and for every $(a,b) \in F$, $b \in x*a$ and so $x \in L(a,b)$, and for every $y \in B$ and for every $(c,d) \in G$, $d \in y*c$ and so $y \in L(c,d)$. Therefore it follows immediately that $A \subseteq \bigcap_{(a,b)\in F} L(a,b)$ and $B \subseteq \bigcap_{(c,d)\in G} L(c,d)$ and so $A \subseteq U$ and $B \subseteq V$ where $U = \bigcup_{F\in\mathcal{F}} \bigcap_{(a,b)\in F} L(a,b)$ and $V = \bigcup_{G\in\mathcal{G}} \bigcap_{(c,d)\in G} L(c\ d)$ are open in $\mathcal{L}$. On the other hand, by *Definition 10*, $\mathcal{F} \sqcup \mathcal{G}$ consists of left non-FIP index sets. It follows by Definition 12 that $\mathcal{S}^L_{F\cup G}$ has not FIP for every $F \in \mathcal{F}$ and for every $G \in \mathcal{G}$, that is,

$$\varnothing = \bigcap \mathrm{c}S^L_{(F\cup G)} = \bigcap_{(u,v)\in F\cup G} L(u,v) = \bigcap_{(p,q)\in F} L(p,q) \cap \bigcap_{(r,s)\in G} L(r,s)$$

for every $F \in \mathcal{F}$ and for every $G \in \mathcal{G}$ which implies that $U \cap V = \varnothing$. Thus A, B are topologically separated by neigborhoods with regard to the left operand topology $\mathcal{L}$ induced by $(X, *)$. We use similar arguments to those used the proof for the right operand topology $\mathcal{R}$ endpefalse Recall that a preregular (or R_1) space is one where every topologically distinguishable pair of points are separated by neighborhoods. So we can get a result as follows.

Corollary 4. If any raw-binarily $\mathcal{L}(\mathcal{R})$-distinguishable pair of points in a raw bistruct $(X, *)$ is raw-binarily $\mathcal{L}(\mathcal{R})$-separated by neighborhoods, then X is a $\mathcal{L}(\mathcal{R})$-preregular space.

The following result is immediately followed from the fact that a preregular Kolmogorov space is a Hausdorff space.

Corollary 5. If any distinct pair of points in a raw bistruct $(X, *)$ is raw-binarily $\mathcal{L}(\mathcal{R})$-separated by neighborhoods, then X is a $\mathcal{L}(\mathcal{R})$-Hausdorff space.

Corollary 6. If any point and a raw-binarily $\mathcal{L}(\mathcal{R})$-closed set not containing that point in a raw bistruct $(X, *)$ is raw-binarily $\mathcal{L}(\mathcal{R})$-separated by neighborhoods, then X is a $\mathcal{L}(\mathcal{R})$-regular space.

We write the following result using the fact that a regular Kolmogorov space is an regular Hausdorff space (or T_3).

Corollary 7. Any of topological spaces $(X,\mathcal{L})$ and $(X,\mathcal{R})$ induced by a raw bistruct $(X, *)$ is both regular and Kolmogorov, then it is also a regular Hausdorff space.

Corollary 8. If any two disjoint raw-binarily $\mathcal{L}(\mathcal{R})$-closed sets in a raw bistruct $(X, *)$ is raw-binarily $\mathcal{L}(\mathcal{R})$-separated by neighborhoods, then X is a $\mathcal{L}(\mathcal{R})$-normal space.

Remember that a completely normal space is one where every pair of separated sets are separated by neighborhoods. Now let's give a corollary for completely normal spaces.

Corollary 9. If any two raw-binarily $\mathcal{L}(\mathcal{R})$-separated sets in a raw bistruct $(X, *)$ is raw-binarily $\mathcal{L}(\mathcal{R})$-separated by neighborhoods, then X is a completely $\mathcal{L}(\mathcal{R})$-normal space.

The following result is immediately followed from the fact that an accessible and completely normal space is a completely normal Hausdorff space (or T_5).

Corollary 10. Any of topological spaces $(X,\mathcal{L})$ and $(X,\mathcal{R})$ induced by a raw bistruct $(X, *)$ is both accessible and completely normal, then it is also a completely normal Hausdorff space.

Theorem 4. If any two subsets in a raw bistruct $(X, *)$ are raw-binarily $\mathcal{L}(\mathcal{R})$-separated by closed neighborhoods, then they are also topologically separated by closed neighborhoods with respect to the left operand topology $\mathcal{L}$ (the right operand topology $\mathcal{R}$).

Proof. Let $(X, *)$ be a raw bistruct and $A, B \subseteq X$. By the hypothesis, there exists two families of non-empty finite subsets of X^2, one, say $\mathcal{F}$, of which is left non-FIP indexable together with a left common pointer, say $\mathcal{R}$, of A, and the other, say $\mathcal{G}$, of which is left non-

-FIP indexable together with a left common pointer, say $\mathcal{S}$, of *B*, and whose union is a left common pointer of *X*. By *Definition 2*, for every $x \in A$, we can find a member of $\mathcal{R}$, say *R*, having *x* as a left common point, and for every $y \in B$, we can find a member of $\mathcal{S}$, say *S*, having *y* as a left common point. Then from *Definition 1*, for every $x \in A$ and for every $(a,b) \in R$, $b \in x{*}a$ and so $x \in L(a,b)$, and for every $y \in B$ and for every $(c,d) \in S$, $d \in y{*}c$ and so $y \in L(c,d)$. Therefore it follows immediately that $A \subseteq \bigcap_{(a,b)\in R} L(a,b)$ and $B \subseteq \bigcap_{(c,d)\in S} L(c,d)$ and so $A \subseteq M$ and $B \subseteq N$ where $M = \bigcup_{R\in\mathcal{R}} \bigcap_{(a,b)\in R} L(a,b)$ and $N = \bigcup_{S\in\mathcal{S}} \bigcap_{(c,d)\in S} L(c,d)$ are open sets in $\mathcal{L}$.

Furthermore, by *Definition 10*, $\mathcal{F} \sqcup \mathcal{R}$ consists of left non-FIP index sets. It follows by *Definition 9* that $\mathcal{S}^L_{F\cup R}$ has not FIP for every $F \in \mathcal{F}$ and for every $R \in \mathcal{R}$, that is,

$$\varnothing = \bigcap \mathcal{S}^L_{F\cup R} = \bigcap_{(u,v)\in F\cup R} L(u,v) = \bigcap_{(p,q)\in F} L(p,q) \cap \bigcap_{(r,s)\in R} L(r,s)$$

for every $F \in \mathcal{F}$ and for every $R \in \mathcal{R}$ which implies that $M \cap U = \varnothing$ where $U = \bigcup_{F\in\mathcal{F}} \bigcap_{(a,b)\in F} L(a,b)$ is open in $\mathcal{L}$. Thus $M \subseteq K$ holds where $K = U^c$ is a closed set in $\mathcal{L}$. Similarly, it follows from the fact that $\mathcal{G} \sqcup \mathcal{S}$ consists of left non-FIP index sets that $\varnothing = \bigcap \mathcal{S}^L_{G\cup S} = \bigcap_{(p,q)\in G} L(p,q) \cap \bigcap_{(r,s)\in S} L(r,s)$ for every $G \in \mathcal{G}$ and for every $S \in \mathcal{S}$ which implies that $N \cap V = \varnothing$ where $V = \bigcup_{G\in\mathcal{G}} \bigcap_{(a,b)\in G} L(a,b)$ is open in $\mathcal{L}$. Thus $N \subseteq L$ holds where $L = V^c$ is a closed set in $\mathcal{L}$.

On the other hand, by *Definition 2*, for every $x \in X$, there exists a member *H* of $\mathcal{F} \cup \mathcal{G}$ having *x* as a left common point, that is, $b \in x{*}a$ and so $x \in L(a,b)$ for every $(a,b) \in H$. It follows that for every $x \in X$, there exists $H \in \mathcal{F} \cup \mathcal{G}$ such that $x \in \bigcap_{(a,b)\in H} L(a,b)$ which implies that $\bigcup_{H\in\mathcal{F}\cup\mathcal{G}} \bigcap_{(a,b)\in H} L(a,b) = X$. Therefore, we get

$$X = \bigcup_{H\in\mathcal{F}\cup\mathcal{G}} \bigcap_{(a,b)\in H} L(a,b) = \bigcup_{F\in\mathcal{F}} \bigcap_{(a,b)\in F} L(a,b) \cup \bigcup_{G\in\mathcal{G}} \bigcap_{(c,d)\in G} L(c,d) = U \cup V$$

which implies that $K \cap L = \varnothing$. Thus *A*, *B* are topologically separated by closed neigborhoods with regard to the left operand topology $\mathcal{L}$ induced by $(X,*)$ since *K* and *L* are closed sets in $\mathcal{L}$. We use similar arguments to those used the proof for the right operand topology $\mathcal{R}$. endpefalse We give the following corollary by using the fact a Urysohn, or $T_{2(1/2)}$, space is one in which every distinct pair of points is separated by closed neighborhoods.

Corollary 11. If any distinct pair of points in a raw bistruct $(X, *)$ is raw-binarily $\mathcal{L}(\mathcal{R})$ -separated by closed neighborhoods, then *X* is a $\mathcal{L}(\mathcal{R})$ -Urysohn space.

Theorem 5. If any two subsets in a raw bistruct $(X, *)$ are raw-binarily $\mathcal{L}(\mathcal{R})$ -separated by a continuous function, then they are also separated by a continuous function with respect to the left operand topology $\mathcal{L}$ (the right operand topology $\mathcal{R}$).

Proof. Let $(X, *)$ be a raw bistruct, $(\mathbb{R}, \circ)$ be a $\mathcal{L}$ -usual raw bistruct and *A*, *B* be two arbitrary subsets of *X*. By the hypothesis, there exists a raw-binarily $\mathcal{L}$ -continuous function $f : X \to \mathbb{R}$ such that $f(A) = \{0\}$ and $f(B) = \{1\}$. From *Proposition 2*, $\mathcal{L}_{\mathbb{R}}$ is the usual topology on $\mathbb{R}$. Also, by *Proposition 3*, $f : (X, \mathcal{L}_X) \to (\mathbb{R}, \mathcal{L}_{\mathbb{R}})$ is also a continuous function. Thus *A*, *B* are separated by *f* which is a continuous function. We use similar arguments to those used the proof for the right operand topology $\mathcal{R}$.

Corollary 12. If any distinct pair of points in a raw bistruct $(X, *)$ is raw-binarily $\mathcal{L}(\mathcal{R})$ -separated by a continuous function, then *X* is a completely $\mathcal{L}(\mathcal{R})$ -Hausdorff space (or completely T_2).

Corollary 13. If any point and a raw-binarily $\mathcal{L}(\mathcal{R})$-closed set not containing that point in a raw bistruct (X, $*$) is raw-binarily $\mathcal{L}(\mathcal{R})$-separated by a continuous function, then X is a $\mathcal{L}$ completely ($\mathcal{R}$)-regular space.

The following result is immediately followed from the fact that a completely regular Kolmogorov space is Tychonoff space, or $T_{3(1/2)}$.

Corollary 14. Each of topological spaces $(X,\mathcal{L})$ and $(X,\mathcal{R})$ induced by a raw bistruct (X, $*$) is both completely regular and Kolmogorov, then it is also a Tychonoff space.

We give the following two corollaries by using the facts that a symmetric normal space is a normal regular space, and that an accessible normal space is a normal Hausdorff, or T_4 space.

Corollary 15. Each of topological spaces $(X,\mathcal{L})$ and $(X,\mathcal{R})$ induced by a raw bistruct (X, $*$) is both symmetric and normal, then it is also a normal regular space.

Corollary 16. Each of topological spaces $(X,\mathcal{L})$ and $(X,\mathcal{R})$ induced by a raw bistruct (X, $*$) is both accessible and normal, then it is also a normal Hausdorff space.

Theorem 6. If any two subsets in a raw bistruct (X, $*$) are raw-binarily precisely $\mathcal{L}(\mathcal{R})$-separated by a continuous function, then they are also precisely separated by a continuous function with respect to the left operand topology $\mathcal{L}$ (the right operand topology $\mathcal{R}$).

Proof. Let (X, $*$) be a raw bistruct, $(\mathbb{R},\circ)$ be a $\mathcal{L}$-usual raw bistruct and A, B be two arbitrary subsets of X. By the hypothesis, there exists a raw-binarily $\mathcal{L}$-continuous function $f: X \to \mathbb{R}$ such that $A = f^{-1}(\{0\})$ and $B = f^{-1}(\{1\})$. From *Proposition 6*, $\mathcal{L}_{\mathbb{R}}$ is the usual topology on $\mathbb{R}$. Also, by *Proposition* 8, $f:(X,\mathcal{L}_X) \to (\mathbb{R},\mathcal{L}_{\mathbb{R}})$ is also a continuous function. Thus A, B are separated by f which is a continuous function. We use similar arguments to those used the proof for the right operand topology $\mathcal{R}$. endpefalse Recall that a perfectly normal space is one where every disjoint pair of closed sets are precisely separated by a continuous function. So we can get a result as follows.

Corollary 17. If any two disjoint raw-binarily $\mathcal{L}(\mathcal{R})$-closed subsets in a raw bistruct (X, $*$) is raw-binarily precisely $\mathcal{L}(\mathcal{R})$-separated by a continuous function, then X is a perfectly $\mathcal{L}(\mathcal{R})$-normal space.

The following result is immediately followed from the fact that an accessible and perfectly normal space is a perfectly normal Hausdorff, or T_6, space.

Corollary 18. Any of topological spaces $(X,\mathcal{L})$ and $(X,\mathcal{R})$ induced by a raw bistruct (X, $*$) is both accessible and perfectly normal, then it is also a perfectly normal Hausdorff space.

Further work

We plan our next study to be a detailed study of the characteristics of each topological space created by a raw binop when it satisfies a certain one of the group-like axioms such as totality (closureness), associativity, identity, invertibility, and commutativity.

Conclusions

We now give some of useful following conclusions obtained on the axioms of separation in the left operand and right operand topologies produced by a raw binary operation. A raw bistruct where every pair of points is raw-binarily $\mathcal{L}(\mathcal{R})$-distinguishable induces a $\mathcal{L}(\mathcal{R})$-Kolmogorov space. A topological space induced by a raw bistruct in which each distinct pair of points is raw-binarily $\mathcal{L}(\mathcal{R})$-separated is a $\mathcal{L}(\mathcal{R})$-accessible. In a raw bistruct where each distinct pair of points is raw-binarily $\mathcal{L}(\mathcal{R})$-separated by neighborhoods, the space induced by it is a $\mathcal{L}(\mathcal{R})$ -Hausdorff. In a raw bistruct where each pair of raw-binarily $\mathcal{L}(\mathcal{R})$ -separated sets is raw-binarily $\mathcal{L}(\mathcal{R})$ -separated by neighborhoods, the induced topological space is a completely $\mathcal{L}(\mathcal{R})$ -normal. The topological space induced by a raw bistruct in which each disjoint pair of raw-binarily $\mathcal{L}(\mathcal{R})$ -closed subsets is raw-binarily precisely $\mathcal{L}(\mathcal{R})$ -separated by a continuous function is a perfectly $\mathcal{L}(\mathcal{R})$ -normal.

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