APPLICATION OF THE CONFORMABLE REDUCED DIFFERENTIAL TRANSFORM METHOD TO FRACTIONAL ORDER $K(m, n)$ NON-LINEAR DIFFERENTIAL EQUATIONS

by

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In this paper, it is discussed over the method of reduced differential transform method with the help of conformable derivative of the time fractional differential equation. This method is applied to the differential equation K(m,n), which is a member of the Korteweg-de Vries equations. For these solutions, certain values have been obtained depending on the α parameter and these values are shown on the table and graph. It is shown that the method used here is effective and easy to apply.

Key words: *time-fractional differential equations, conformable derivative, reduced differential transform method*

Introduction

Russel [1] has discovered solitary waves as a result of his work in 1834. Until the end of the 19th century, the solitary wave problem could not be expressed in the form of an equation. For this purpose, Korteweg and his student have suggested a model to open this event in Gustav de Vries. This model is known as the KdV equation in [2]. Solitary waves have attracted a lot of scientists. From these, [3], one of the most important features of these waves is determined against mutual collisions and the shape has given the *soliton* because of the fact that localized waves that do not lose speed characteristics. Studies in recent years. They discovered a new soliton class. For this new solitary wave class, they recommend the name of *pure quartic soliton* [4]. With this new discovery, communications stated that frequency combs and ultrafast laser can be found in the applications.

Solitons are localized waves that are stable against mutual collisions and do not lose their properties such as shape and speed [5]. The equation of $K(m, n)$ are pioneering equations for compactons [5]. In the theory of solitary waves, compactins are defined as solitons with finite wavelengths or solitons without exponential tails [6]. Kerry Vahale *et al.* they discovered a new type of optic soliton. This new optical soliton is fed by the energy of the other wave that follows and follows the other soliton waves. The discovered new wave has been called *Stokes soliton* [7].

In recent years, fractional differential equations and fractional modeling have been observed to be effective in defining many phenomena in electrodynamics, aerodynamics, signal processing, economics, control theory, biology, fluid flow, and other science and engi-

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neering fields. As we know, it is not easy to produce solutions of partial differential equations considered as mathematical modeling. In recent years, researchers have focused on analytical approximate solutions of fractional differential equations. Numerical solutions methods have been developed for this. Some of those are adomian decomposition method (ADM) [8], variational iteration method (VIM) (He 1999) [9], homotopy perturbation method (HPM) [10, 11], differential transform method (DTM) (Zhou 1986) [12], reduced differential transform method (RDTM) [13].

Among these methods adomian introduced some definitions and theorems in the book published in 1980 and showed how to apply some differential equations to the method of adomian decomposition [8]. The VIM introduced by He [9] has been by many mathematicians and engineers to solve various functional equations. He published the HPM with his studies in 1999 and 2000 and showed how to apply it on differential equations [10, 11]. The DTM is a numerical method based on Taylor series. It can be applied to many ordinary differential and partial differential equations. Firstly, Zhou [12] used the differential transformation method to solve linear and non-linear initial value problems in electrical circuit analysis. The RDTM, which first proposed by Yildiray Keskin [13], has received much attention due to its applications to solve a widely variety of problems.

The $K(m, n)$ equations [6]:

$$u_t^l + (u^m)_x + (u^n)_{xxx} = 0 \quad m > 0, \quad 1 < n \le 3$$

present a class of solitary waves with compact supports. Here, the first term is the generalized evolution term, while the second term represent the non-linear term and third is the dispersion term. Where $U(x, t)$ is the amplitude of the wave, x is the spatial co-ordinate and t is the time. These solitary waves are solution of a two-parameter family with fully non-linear dispersive partial differential equations. Also, solitary waves are non-linear waves of finite amplitude, propagating with constant velocity and shape.

The $K(m, n)$ equations are members of the KdV [2] family. These equations are the oldest equations first modeled for the formation of shallow water waves in 1895. The role non-linear dispersion in the formation of patterns in liquid drops was studied by Rosenau and Hyman [6]. The $K(m, n)$ equations are discussed with ADM [14], HPM [15, 16], and VIM [17].

Basic definitions

Definition 1. [18] Let the function f be continuous and integrable in every finite (α, x) range. Let $m \in \mathbb{N},\ m-1 < \alpha \le m$ and $x > a,\ a \in \mathbb{R}.$ Therefore, the Riemann-Liouville fractional derivative of the function f is defined as;

$$(\mathrm{D}_a^\alpha f)(x) = \frac{\mathrm{d}^m}{\mathrm{d}x^m}\left[\frac{1}{\Gamma(m-\alpha)}\int_a^x (x-\tau)^{m-\alpha-1} f(\tau) d\tau\right] \tag{1}$$

Definition 2. [19, 20] The fractional derivative of $f(x)$ in the Caputo sense is:

$$(\mathrm{D}_a^\alpha f)(x) = \frac{1}{\Gamma(m-\alpha)}\int_a^x (x-\tau)^{m-\alpha-1} f^{(m)}(\tau) \mathrm{d}\tau \tag{2}$$

for $m-1 < \alpha \le m, m \in \mathbb{N},\ \ x > 0,\ f \in C_{-1}^m.$

Definition 3. [21] The fractional conformable derivative α for the function $f:[0,\infty)\to\mathbb{R}$ is:

$$(T_\alpha f)(x)=\lim_{\epsilon\to 0}\frac{f(x+\epsilon x^{1-\alpha})-f(x)}{\epsilon} \tag{3}$$

for all $x>0,\ \alpha\in(0,1]$.

Lemma 1. Let $\alpha\in(0,1]$ and f,g be α-differentiable at a point $t>0.$. Then the conformable derivatives provides the given properties [21]:

i. $T_\alpha(af+bg)=a(T_\alpha f)+b(T_\alpha g)$ for $a,b\in\mathbb{R}$

ii. $T_\alpha(x^q)=qx^{q-\alpha}$ for all $q\in\mathbb{R}$,

iii. $T_\alpha[f(x)]=0,$ for all constant functions $f(x)=\lambda,$

iv. $T_\alpha(fg)=f(T_\alpha g)+g(T_\alpha f),$

v. $T_\alpha\dfrac{f}{g}=\dfrac{gT_\alpha(f)-fT_\alpha(g)}{g^2}$

vi. If $f(x)$ is differentiable a function, then:

$$T_\alpha[f(x)]=x^{1-\alpha}\frac{\mathrm{d}}{\mathrm{d}x}f(x)$$

Conformable fractional 2-D reduced differential transform method

In this section, specific theorems and definitions of the conformable fractional reduced differential transform method (CFRDTM) for fractional partial differential equations will be given. In this study, the transformation of the function $u(x,t)$ under CFRDTM is $U_h^\alpha(x)$.

Definition 4. [22] Assume $u(x,t)$ is analytical and differentiated continuously with respect to time t and space x in its domain. CFRDTM of $u(x,t)$ is:

$$u(x,t)=\sum_{h=0}^{\infty}\frac{1}{\alpha^h h!}[{}_tT_\alpha^{(h)}u]_{t=t_0} \tag{4}$$

where some $0<\alpha\le 1$, α is parameter describing the order of conformable fractional derivative:

$${}_tT_\alpha^{(h)}u=\underbrace{{}_tT_\alpha\,{}_tT_\alpha\cdots{}_tTT_\alpha}_{h\,times}u(x,t)$$

and the *t*-D spectrum function $U_h^\alpha(x)$ is the CFRDTM function.

Definition 5. [22] Let $U_h^\alpha(x)$ be the conformable fractional differential transform of $u(x,t)$. Inverse conformable fractional differential transform of $U_h^\alpha(x)$ is:

$$u(x,t)=\sum_{h=0}^{\infty}U_h^\alpha(x)(t-t_0)^{\alpha h} \tag{5}$$

from eqs. (4) and (5), we obtain:

$$u(x,t)=\sum_{h=0}^{\infty}\frac{1}{\alpha^h h!}[{}_tT_\alpha^{(h)}u]_{t=t_0} \tag{6}$$

Conformable fractional differential transform method of initial conditions for integer order derivatives are:

$$U_h^\alpha(x)=\begin{cases}\dfrac{1}{(\alpha h)!}\left[\dfrac{\partial^{\alpha h}u(x,t)}{\partial t^{\alpha h}}\right]_{t=t_0} & \text{if } \alpha h\in\mathbb{Z}^+ \text{ for } h=0,1,2,\ldots,\left(\dfrac{n}{\alpha}-1\right)\\ 0 & \text{if } \alpha h\notin\mathbb{Z}^+\end{cases} \tag{7}$$

where n is the order of the corresponding fractional equation.

By consideration of $U_0^\alpha(x)=f(x)$ as transformation of the initial condition $u(x,t)=f(x)$ straightforward iterative calculations gives the $U_h^\alpha(x)$ values for $h=0,1,2,\ldots h$.

Then the inverse transformation of the $[U_h^\alpha(x)]_{h=0}^n$ gives the approximate solution:

$$\tilde{u}_n(x,t)=\sum_{h=0}^{\infty}U_h^\alpha(x)t^{\alpha h}$$

where n represent the order of the obtained approximation solution. Hence, the CFRDTM leads a solution:

$$u(x,t)=\lim_{n\to\infty}\tilde{u}_n(x,t)$$

Fundamental operations of CRDTM are displayed in tab. 1.

Table 1. Basic operations CFRDTM [22]

Original function	Transformed function
$u(x,t)$	$U_h^\alpha(x)=\dfrac{1}{\alpha^h h!}\left[{}_tT_\alpha^{(h)}u\right]_{t=t_0}$
$u(x,t)=av(x,t)\pm bw(x,t)$	$U_h^\alpha(x)=aV_h^\alpha(x)\pm bW_h^\alpha(x)$
$u(x,t)=v(x,t)w(x,t)$	$U_h^\alpha(x)=\Sigma_{s=0}^{\infty}V_s^\alpha(x)W_{h-s}^\alpha(x)$
$u(x,t)={}_tT_\alpha v(x,t)$	$U_h^\alpha(x)=\alpha(h+1)V_{h+1}^\alpha(x)$
$u(x,t)=v(x,t)w(x,t)\varphi(x,t)$	$U_h^\alpha(x)=\Sigma_{r=0}^{h}\Sigma_{i=0}^{r}V_i^\alpha(x)W_{r-i}^\alpha(x)\Phi_{h-r}^\alpha(x)$

Numerical example

Here, CFRDTM will be applied for solving fractional order $K(3,3)$ and $K(2,2)$.

Example 1. The $K(m,n)$ equation with initial condition [14] and analytical solution [14] is:

$$\mathrm{D}_t^\alpha u+(u^m)_x+(u^n)_{xxx}=0 \tag{8}$$

If $m=n=3$ in eq. (8), the time fractional partial differential equations turn into:

$$\mathrm{D}_t^\alpha u+(u^3)_x+(u^3)_{xxx}=0, \quad -10\le x\le 10, \quad 0\le t\le 5 \tag{9}$$

$$u(x,0)=\frac{\sqrt{6c}}{2}\sin\frac{x}{3} \tag{10}$$

$$u(x,t)=\frac{\sqrt{6c}}{2}\sin\frac{x-ct}{3} \tag{11}$$

Using the initial condition at (9), we apply the CRDTM to (11) $K(3,3)$ and obtained:

$$U_{h+1}^\alpha=-\frac{1}{\alpha(h+1)}\left[\begin{array}{l}\dfrac{\partial}{\partial x}\displaystyle\sum_{r=0}^{h}\sum_{i=0}^{r}U_i^\alpha(x)U_{r-i}^\alpha(x)U_{h-r}^\alpha(x)+\\ +\dfrac{\partial^3}{\partial x^3}\displaystyle\sum_{r=0}^{h}\sum_{i=0}^{r}U_i^\alpha(x)U_{r-i}^\alpha(x)U_{h-r}^\alpha(x)\end{array}\right] \tag{12}$$

If we iterate for $h=0,1,2,\ldots$:

$$U_0^\alpha=\frac{\sqrt{6c}}{2}\sin\frac{x}{3}, \quad U_1^\alpha=-\frac{c^{3/2}t^\alpha}{\alpha\sqrt{6}}\cos\frac{x}{3} \tag{13}$$

$$U_2^\alpha=-\frac{c^{5/2}t^{2\alpha}}{\alpha^2 6\sqrt{6}}\sin\frac{x}{3}, \quad U_3^\alpha=\frac{c^{7/2}t^{3\alpha}}{\alpha^3 54\sqrt{6}}\cos\frac{x}{3}, \quad U_4^\alpha=\frac{c^{9/2}t^{4\alpha}}{\alpha^4 648\sqrt{6}}\sin\frac{x}{3} \tag{14}$$

From the approximate solution is found from inverse transformation of the values of the set $[U_h^\alpha(x)]_{h=0}^{10}$. In order to obtained the approximate solution of this equation, if the below terms are written on the total series:

$$\tilde{u}_n(x,t)=\sum_{h=0}^{n}U_h^\alpha(x)t^{\alpha h}$$

We then arrive at the following solution:

$$\begin{aligned}\tilde{u}_{10}(x,t)&=\sum_{h=0}^{10}U_h^\alpha(x)t^{\alpha h}\cong\\ &\cong\frac{\sqrt{6c}}{2}\sin\frac{x}{3}-\frac{c^{3/2}t^\alpha}{\alpha\sqrt{6}}\cos\frac{x}{3}-\frac{c^{5/2}t^{2\alpha}}{\alpha^2 6\sqrt{6}}\sin\frac{x}{3}+\frac{c^{7/2}t^{3\alpha}}{\alpha^3 54\sqrt{6}}\cos\frac{x}{3}+\\ &+\frac{c^{9/2}t^{4\alpha}}{\alpha^4 648\sqrt{6}}\sin\frac{x}{3}-\frac{c^{11/2}t^{5\alpha}}{\alpha^5 9720\sqrt{6}}\cos\frac{x}{3}-\frac{c^{13/2}t^{6\alpha}}{\alpha^6 174960\sqrt{6}}\sin\frac{x}{3}+\\ &+\frac{c^{15/2}t^{7\alpha}}{\alpha^7 3674160\sqrt{6}}\cos\frac{x}{3}+\frac{c^{17/2}t^{8\alpha}}{\alpha^8 88179840\sqrt{6}}\sin\frac{x}{3}-\frac{c^{19/2}t^{9\alpha}}{\alpha^9 2380855680\sqrt{6}}\cos\frac{x}{3}-\\ &-\frac{c^{21/2}t^{10\alpha}}{\alpha^{10} 71425670400\sqrt{6}}\sin\frac{x}{3}+\ldots\end{aligned}$$

Here a_h is sequence. $a_0 = 1,\ a_h = a_{h-1}3(h+1)$:

$$\tilde{u}(x,t)\frac{\sqrt{6c}}{2}\sin\frac{x}{3}+\sum_{h=1}^{n}\frac{(-3)^h c^{\left(\frac{2h+1}{2}\right)}}{\alpha^h\sqrt{6}a_{h-1}}\frac{\mathrm{d}^h}{\mathrm{d}x^h}\sin\frac{x}{3}t^{h\alpha}$$

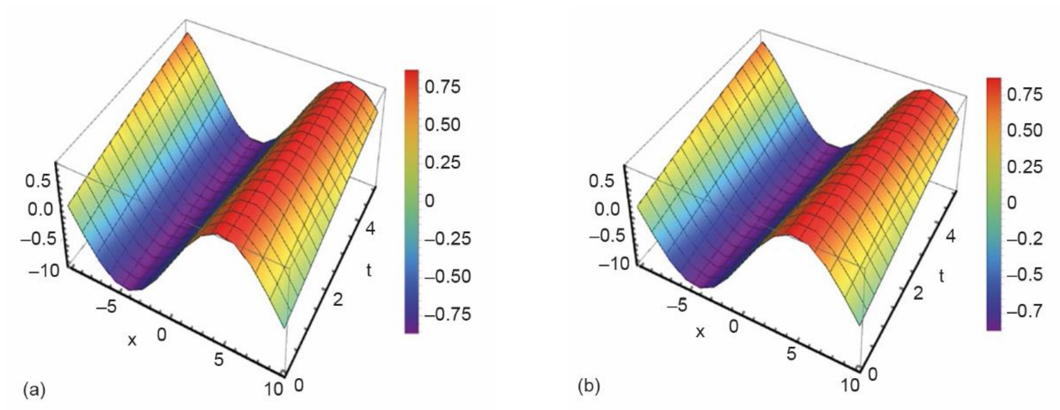

Figure 1. (a) The *K*(3, 3) equation graph for *α* = 1, *c* = 0.5 (numerical solution) and (b) the *K*(3, 3) equation graph for *α* = 1, *c* = 0.5 (exact solution)

Table 2. When *α* = 1 the *u*(*x*, *t*) numerical solution of time-fractional differential eq. (9)

x-value	*t*-value	Numerical solutions	Exact solutions	Absolute error
1	0.5	0.21425811	0.21425811	0
2	1.0	0.41519469	0.41519469	$1.6653 \cdot 10^{-16}$
3	1.5	0.59031648	0.59031648	$2.9976 \cdot 10^{-15}$
4	2.0	0.72873524	0.72873524	$3.2196 \cdot 10^{-14}$
5	2.5	0.82184478	0.82184478	$8.7152 \cdot 10^{-14}$
6	3.0	0.86385599	0.86385599	$4.0005 \cdot 10^{-12}$
7	3.5	0.85215682	0.85215682	$3.7770 \cdot 10^{-11}$
8	4.0	0.78747467	0.78747467	$2.1607 \cdot 10^{-10}$
9	4.5	0.67383115	0.67383115	$8.9583 \cdot 10^{-10}$
19	5.0	0.51829208	0.51829208	$2.8922 \cdot 10^{-9}$

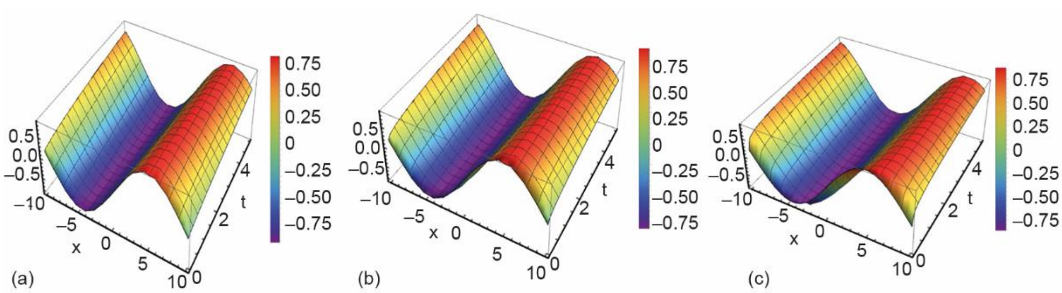

Figure 2. (a) The *K*(3, 3) equation graph for *α* = 0.75, *c* = 0.5 (numerical solution), (b) the *K*(3, 3) equation graph for *α* = 0.50, *c* = 0.5 (numerical solution), and (c) the *K*(3, 3) equation graph for *α* = 0.25, *c* = 0.5 (numerical solution)

Table 3. When $\alpha = 1$, $\alpha = 0.75$, $\alpha = 0.50$, $\alpha = 0.25$ the $u(x, t)$ numerical solution of time-fractional differential eq. (9)

x-value	t-value	$\alpha = 0.75$	$\alpha = 0.50$	$\alpha = 0.25$
1	0.5	0.17307040	0.08441673	– 0.19512677
2	1.0	0.37235313	0.28335891	$-2.05 \cdot 10^{-16}$
3	1.5	0.55711319	0.48308259	0.22448905
4	2.0	0.70924277	0.65739952	0.44564740
5	2.5	0.81471759	0.78676234	0.63811831
6	3.0	0.86343573	0.85653926	0.78049472
7	3.5	0.84979702	0.85768133	0.85638894
8	4.0	0.77316485	0.78747467	0.85590068
9	4.5	0.63800353	0.64988401	0.77668731
10	5.0	0.45362286	0.45532666	0.62435280

Example 2. The $K(2, 2)$ [23] equation with initial condition and analytical solution is:

$$\mathrm{D}_t^\alpha u + (u^2)_x + (u^2)_{xxx} = 0, \quad 0 < x < 8, \quad 0 < t < 1/2 \tag{15}$$

$$\boldsymbol{u}(\boldsymbol{x},0) = \boldsymbol{x} \tag{16}$$

$$\boldsymbol{u}(\boldsymbol{x},\boldsymbol{t}) = \frac{\boldsymbol{x}}{1+2\boldsymbol{t}}, \quad \boldsymbol{t} \in \left(\frac{1}{2}\right) \tag{17}$$

Using the initial condition at (16), we apply the CRDTM to (15) $K(2,2)$ and obtained:

$$U_{h+1}^\alpha = -\frac{1}{\alpha(h+1)}\left[\frac{\partial}{\partial x}\sum_{r=0}^{h} U_r^\alpha(x) U_{h-r}^\alpha(x) + \frac{\partial^3}{\partial x^3}\sum_{r=0}^{h} U_r^\alpha(x) U_{h-r}^\alpha(x)\right] \tag{18}$$

Thus, for $h = 0,1,2,\ldots,n,$, we have, respectively, $(U_0^\alpha), (U_1^\alpha), (U_2^\alpha),\ldots, (U_n^\alpha)$:

$$\boldsymbol{U}_0^{\boldsymbol{\alpha}}(x) = \boldsymbol{x}, \quad \boldsymbol{U}_1^{\boldsymbol{\alpha}}(x) = -\frac{2\boldsymbol{x}}{\boldsymbol{\alpha}}, \quad \boldsymbol{U}_2^{\boldsymbol{\alpha}}(x) = \frac{4\boldsymbol{x}}{\boldsymbol{\alpha}^2}, \quad \boldsymbol{U}_3^{\boldsymbol{\alpha}}(x) = -\frac{8\boldsymbol{x}}{\boldsymbol{\alpha}^3}, \quad \boldsymbol{U}_4^{\boldsymbol{\alpha}}(x) = \frac{16\boldsymbol{x}}{\boldsymbol{\alpha}^4}$$

$$\boldsymbol{U}_5^{\boldsymbol{\alpha}}(x) = -\frac{32\boldsymbol{x}}{\boldsymbol{\alpha}^5},\ldots, \boldsymbol{U}_{\boldsymbol{h}}^{\boldsymbol{\alpha}}(x) = \left(-\frac{2}{\boldsymbol{\alpha}}\right)^{\boldsymbol{n}} \boldsymbol{x} \tag{19}$$

From the approximate solution is found from inverse transformation of the values of the set $[U_h^\alpha(x)]_{h=0}^\infty$. In order to obtained the approximate solution of this equation, if the above terms are written on the total series:

$$\tilde{u}_n(x,t) = \sum_{h=0}^{n} U_h^\alpha(x) t^{\alpha h}$$

and we then arrive at the following solution:

$$u(x,t)=\sum_{h=0}^{\infty}\left(-\frac{2}{\alpha}\right)^{h}xt^{\alpha h}=x-\frac{2x}{\alpha}t^{\alpha}+\frac{4x}{\alpha^{2}}t^{2\alpha}-\frac{8x}{\alpha^{3}}t^{3\alpha}+\frac{16x}{\alpha^{4}}t^{\alpha}-\frac{32x}{\alpha^{5}}t^{5\alpha}+\ldots$$

$$=x\left[1-\frac{2t^{\alpha}}{\alpha}+\left(\frac{2t^{\alpha}}{\alpha}\right)^{2}-\left(\frac{2t^{\alpha}}{\alpha}\right)^{3}+\left(\frac{2t^{\alpha}}{\alpha}\right)^{4}-\left(\frac{2t^{\alpha}}{\alpha}\right)^{5}+\ldots\right]$$

$$u(x,t)=x\left[\sum_{h=0}^{\infty}\left(-\frac{2t^{\alpha}}{\alpha}\right)^{h}\right]=\frac{x}{1+\dfrac{2t^{\alpha}}{\alpha}}$$

The approximate solution of (15) and (16) by CRDTM is the same as the exact solution.

Conclusion

In this article, the conformable fractional reduced differential transformation method (CFRDTM) was applied to solve the time-fractional differential equation $K(3,3)$ and $K(2,2)$. The obtained results are compared on figs. 1 and 2 and tabs. 2 and 3. It has been observed that the numerical results for $\alpha=1$ are quite close to analytical values. In other words, it has been observed that this method is effective and suitable for the solution of the fractional differential equation $K(3,3)$ and $K(2,2)$.

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