GLOBAL EXISTENCE OF SOLUTIONS FOR THE SEMILINEAR KLEIN-GORDON EQUATION WITH A TIME-DEPENDENT VARIABLE COEFFICIENT

by

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The semilinear Klein-Gordon equation with initial conditions is studied in de Sitter spacetime. The L^∞ decay estimates are derived for the solutions to the linear Klein- Gordon equations with and without source term in de Sitter spacetime. It is also showed the global existence of solutions to the initial value problem with power type non-linear terms for small initial data by using these estimates.

Key words: *de Sitter spacetime, semilinear Klein-Gordon eqution, global solutions, L^∞ estimates*

Introduction

In this article, we study the semilinear Klein-Gordon equation with the following initial value problem in de Sitter spacetime:

$$\begin{gathered}\partial_t^2\xi + n\partial_t\xi - e^{-2t}\Delta\xi + m^2\xi = F(\xi), \quad (x,t) \in \mathbb{R}^n \times \mathbb{R}, \\ \xi(x,0) = \rho_0(x), \;\; \partial_t\xi(x,0) = \rho_1(x), \quad x \in \mathbb{R}^n\end{gathered} \tag{1}$$

where ρ_0, ρ_1 are in Sobolev space $W^{(n/2)+1,2}(\mathbb{R}^n)$, and $m > 0$. The model of de Sitter spacetime indicates the spatial expansion of the universe. The initial value problem for the Higgs boson equation:

$$\partial_t^2\xi + n\partial_t\xi - e^{-2t}\Delta\xi - m^2\xi = |\xi|^\alpha\,\xi, \quad (x,t) \in \mathbb{R}^n \times \mathbb{R}$$

in de Sitter spacetime is analyzed by Yagdjian [1], and some qualitative property of the solution revealed if the global solution exists. In addition, it was shown by Baskin [2] that the initial value problem for:

$$\partial_t^2 u + n\partial_t u + \frac{\partial_t\sqrt{h_t}}{\sqrt{h_t}}\partial_t u - e^{-2t}\Delta_{h_t} u + \lambda u + |u|^\alpha\, u = 0, \quad (y,t) \in Y \times R$$

admits a small amplitude global solution in the energy space $H^1 \oplus L^2$, provided $\lambda > n^2/4$ and $\alpha = 4/(n-1)$. Here h_t is a smooth family of Riemannian matrices on compact *n*-D manifold *Y*, which is characterized as an asymptotically de Sitter spacetime. When either

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$0 < m < (\sqrt{n^2 - 1}/2)$ or $m \geq n/2$, it has been shown in Yagdjian [3] that there is a constant $\epsilon_0 > 0$ such that if:

$$\|\rho_0\|_{H^s(\mathbb{R}^n)} + \|\rho_1\|_{H^s(\mathbb{R}^n)} \leq \epsilon, \quad \text{for} \quad 0 < \epsilon < \epsilon_0$$

the problem in (1) has a solution $\xi \in C([0,\infty); H^s(\mathbb{R}^n))$ satisfying:

$$e^{\gamma t} \|\xi(.,t)\|_{H^s(\mathbb{R}^n)} \leq 2\epsilon \tag{2}$$

where γ is either $\gamma = 0$ if $m \geq n/2$ or:

$$0 < \gamma < \frac{\frac{n}{2} - \sqrt{\frac{n^2}{4} - m^2}}{\alpha + 1} \quad \text{if} \quad 0 < m < \frac{\sqrt{n^2 - 1}}{2}$$

In Nakamura [4], for $m \geq n/2$, the initial value problem (1) with the power type non-linear term which satisfies the Lipschitz continuous condition was considered in de Sitter spacetime. The existence of global solutions was proved in the energy space if the power in the non-linear term satisfies the condition $4/n \leq p \leq 2/(n-2)$ for $n = 3,4$. The exponential type non-linear term was also considered for the problem (1) with $n = 3,4$ in [4]. The global existence of the energy solutions was proved under the relations between the power of the non-linear term *p*, the spatial dimension *n* and the mass *m* in [5].

Moreover, Galstian and Yagdjian [6] showed the global existence of the small data solutions to the initial value problem for the following equation:

$$\partial_t^2 \xi + n\partial_t \xi - e^{-2t} A(x, \partial_x)\xi + m^2 \xi = F(\xi), \quad t > 0, \quad x \in \mathbb{R}^n \tag{3}$$

where

$$m \in (0, \sqrt{n^2 - 1}/2) \cup [n/2, \infty) \text{ and } A(x, \partial_x) = \sum_{|\alpha| \leq 2} a_\alpha(x) \partial_x^\alpha$$

is an elliptic negative second order differential operator with the real value coefficients $a_\alpha \in \mathcal{B}^\infty$. Here, $\mathcal{B}^\infty$ denotes the space which contains all C^∞ functions with uniformly bounded all orders derivatives and *F* is Lipschitz continuous. It is also proved in [6] that if the source term $f = f(x,t)$ is added to eq. (3) with the zero initial data, the existence result is valid for $m > 0$. In [7], Nakamura proved that the non-linear Klein-Gordon equation under the quartic potential has a global solution in de Sitter spacetime. The existence of the small data global solvability of the initial value problem for the eq. (3) proved by Yagdjian [8] for:

$$m \in \left(\frac{\sqrt{n^2 - 1}}{2}, \quad \frac{n}{2} \right)$$

In this article, we study the case of $n/2 < m$. Decay estimates are crucial in proving the existence of global solutions for non-linear differential equations. Therefore, we use the L^∞ decay estimate to show the existence global solutions to the problem (1).

Theorem 1. Let $N := 2[n/2] + 3$ and $k := [n/2] + 2$ where [.] denotes the integer part. Then there is a constant $\epsilon_0 > 0$ such that if $\| \rho_0 \|_{W^{N,2}(\mathbb{R}^n)} + \| \rho_1 \|_{W^{N,2}(\mathbb{R}^n)} \leq \epsilon$ for $0 < \epsilon < \epsilon_0$, the initial value problem (1) has a solution $\xi \in C([0,\infty); W^{k,\infty}(\mathbb{R}^n))$ satisfying:

$$e^{\frac{n-1}{2}t}\|\xi(.,t)\|_{W^{k,\infty}(\mathbb{R}^n)} \le 2\epsilon \text{ where } n/2<m$$

In order to prove the theorem, we recall the following estimate for the linear Klein-Gordon equation, which is proved by Yagdjian-Galstian [9].

Lemma 1. Let $\xi=\xi(x,t)$ be the solution to the following problem:

$$\partial_t^2\xi + n\partial_t\xi - e^{-2t}\Delta\xi + m^2\xi = f, \quad \xi(x,0)=\rho_0(x), \quad \xi_t(x,0)=\rho_1(x)$$

for $(x,t)\in\mathbb{R}^n\times(0,\infty)$, where $\rho_0,\rho_1\in C_0^\infty(\mathbb{R}^n)$ and $f\in C^\infty(\mathbb{R}^{n+1})$. Let $l\in\mathbb{Z}^+\bigcup\{0\}$, $n\ge 2$ and $m\ge n/2$. Then:

$$\begin{aligned}\left\|(-\Delta)^{-s}\xi(.,t)\right\|_{W^{l,q}(\mathbb{R}^n)} &\le Ce^{-\frac{n}{2}t}(1+t)^{1-sgn\mathcal{M}}(1-e^{-t})^{\left[2s-n\left(\frac{1}{p}-\frac{1}{q}\right)\right]}\cdot \\ &\cdot\left\{e^{\frac{t}{2}}\|\rho_0\|_{W^{l,p}(\mathbb{R}^n)}+(1-e^{-t})\|\rho_1\|_{W^{l,p}(\mathbb{R}^n)}\right\}+ \\ &+Ce^{-\frac{n}{2}t}\int_0^t e^{\frac{n}{2}b}e^b(e^{-b}-e^{-t})^{1+2s-n\left(\frac{1}{p}-\frac{1}{q}\right)}\cdot \\ &\cdot(1+t-b)^{1-sgn\mathcal{M}}\|f(.,b)\|_{W^{l,p}(\mathbb{R}^n)}\,\mathrm{d}b\end{aligned} \tag{4}$$

for all $t>0$ where $C>0$ is a constant if:

$$\frac{1}{p}+\frac{1}{q}=1, \quad 1<p\le 2, \quad \frac{1}{2}(n+1)\left(\frac{1}{p}-\frac{1}{q}\right)\le 2s\le n\left(\frac{1}{p}-\frac{1}{q}\right)<2s+1 \text{and } \mathcal{M}=\sqrt{m^2-\frac{n^2}{4}}$$

Here, Sobolev space is defined:

$$W^{s,q}(\mathbb{R}^n)=\{v\in L^q(\mathbb{R}^n):D^\beta v\in L^q(\mathbb{R}^n), \quad |\beta|\le s\}$$

with the norm:

$$\|v\|_{W^{s,q}(\mathbb{R}^n)}=\left(\sum_{|\beta|\le s}\int_{\mathbb{R}^n}\left|D^\beta v\right|^q\right)^{1/q}$$

$$\|v\|_{W^{s,\infty}(\mathbb{R}^n)}=\sum_{|\beta|\le s} ess\sup_{\mathbb{R}^n}\left|D^\beta v\right|$$

The positive constants C and C_M may change and they are written by the same letters throughout the paper.

L^∞ estimates for the Klein-Gordon equation

We derive L^∞ estimates for the linear Klein-Gordon equation in de Sitter spacetime. We are going to apply the following two lemmas to show the estimates where K_0 and K_1 are denoted by Yagdjian and Galstian [9].

Lemma 2. Let $\mathcal{M} > 0$ and $\tau(t) = 1 - e^{-t}$. Then:

$$\int_0^1 [1+\tau(t)s]^{-\frac{n-1}{2}} |K_1(\tau(t)s,t)| \tau(t) \mathrm{d}s \le C_M e^{\frac{t}{2}} \tag{5}$$

for all $t > 0$.

Proof. Changing the variable by $1+\tau(t)s = r$ and using the definition of K_1, we get:

$$\int_0^1 [1+\tau(t)s]^{-\frac{n-1}{2}} |K_1(\tau(t)s,t)| \tau(t) \mathrm{d}s =$$

$$= \int_1^{2-e^{-t}} r^{-\frac{n-1}{2}} [(1+e^{-t})^2 - (r-1)^2]^{-\frac{1}{2}} \left| F\left(\frac{1}{2}+i\mathcal{M}, \frac{1}{2}+i\mathcal{M}; 1; \frac{(1-e^{-t})^2-(r-1)^2}{(1+e^{-t})^2-(r-1)^2}\right)\right| \mathrm{d}r \le$$

$$\le C \int_0^{e^t-1} [(e^t+1)^2 - y^2]^{-\frac{1}{2}} \left| F\left(\frac{1}{2}+i\mathcal{M}, \frac{1}{2}+i\mathcal{M}; 1; \frac{(e^t-1)^2-y^2}{(e^t+1)^2-y^2}\right)\right| \mathrm{d}y$$

where we have changed the variable by $e^t(r-1) = y$ in the last inequality. In [9], the hypergeometric function (see e.g., [10]) obeys the estimate:

$$\left| F\left(\frac{1}{2}+i\mathcal{M}, \frac{1}{2}+i\mathcal{M}; 1; \frac{(e^t-1)^2-y^2}{(e^t+1)^2-y^2}\right)\right| \le C_M$$

for $\mathcal{M} > 0$. Hence:

$$\int_0^1 [1+\tau(t)s]^{-\frac{n-1}{2}} |K_1(\tau(t)s,t)| \tau(t) \mathrm{d}s \le C_M \int_0^{e^t-1} [(e^t+1)^2 - y^2]^{-\frac{1}{2}} \mathrm{d}y \le C_M e^{\frac{t}{2}} (1-e^{-t})$$

which leads to (5).

Lemma 3. Let $\mathcal{M} > 1/2$ and $\tau(t) = 1 - e^{-t}$. Then:

$$\int_0^1 [1+\tau(t)s]^{-\frac{n-1}{2}} |K_0(\tau(t)s,t)| \tau(t) \mathrm{d}s \le C_M e^{\frac{t}{2}} \tag{6}$$

for all $t > 0$.

Proof. Similarly to the proof of *Lemma 2*, we obtain:

$$\int_0^1 [1+\tau(t)s]^{-\frac{n-1}{2}} |K_0(\tau(t)s,t)| \tau(t) \mathrm{d}s \le C_M \int_0^{e^t-1} [(e^t+1)^2 - y^2]^{-\frac{1}{2}} [(e^t-1)^2 - y^2]^{-1} \cdot$$

$$\cdot \left\| \left[[e^t - e^{2t} - i\mathcal{M}(1-e^{2t}-y^2)] F\left(\frac{1}{2}+i\mathcal{M}, \frac{1}{2}+i\mathcal{M}; 1; \frac{(e^t-1)^2-y^2}{(e^t+1)^2-y^2}\right) + \right.\right.$$

$$\left.\left. +(e^{2t}-1+y^2)\left(\frac{1}{2}-i\mathcal{M}\right) F\left(-\frac{1}{2}+i\mathcal{M}, \frac{1}{2}+i\mathcal{M}; 1; \frac{(e^t-1)^2-y^2}{(e^t+1)^2-y^2}\right)\right]\right\| \mathrm{d}y$$

From [9], we have

$$\begin{aligned}&\int_0^{z-1}[(z+1)^2-y^2]^{-\frac{1}{2}}[(z-1)^2-y^2]^{-1}\cdot\\ &\cdot\left|[z-z^2-i\mathcal{M}(1-z^2-y^2)]F\left(\frac{1}{2}+i\mathcal{M},\frac{1}{2}+i\mathcal{M};1;\frac{(z-1)^2-y^2}{(z+1)^2-y^2}\right)+\right.\\ &\left.+(z^2-1+y^2)\left(\frac{1}{2}-i\mathcal{M}\right)F\left(-\frac{1}{2}+i\mathcal{M},\frac{1}{2}+i\mathcal{M};1;\frac{(z-1)^2-y^2}{(z+1)^2-y^2}\right)\right|\mathrm{d}y\\ &\le C_M(z+1)^{-\frac{1}{2}}(z-1)\end{aligned}\tag{7}$$

for all $z:=e^t>1$. Hence (7) leads to (6). This completes the proof.

Theorem 2. Let $\xi=\xi(x,t)$ be the solution to the following problem:

$$\partial_t^2\xi+n\partial_t\xi+e^{-2t}\Delta\xi+m^2\xi=f,\quad \xi(x,0)=\rho_0(x),\quad \xi_t(x,0)=\rho_1(x)$$

for $(x,t)\in\mathbb{R}^n\times(0,\infty)$, where $\rho_0,\rho_1\in C_0^\infty(\mathbb{R}^n)$ and $f\in C^\infty(\mathbb{R}^{n+1})$. Let $n/2<m$ and $2\le n$. Then, for all $t>0$, we obtain for $C>0$:

$$\|\xi(.,t)\|_{L^\infty(\mathbb{R}^n)}\le Ce^{-\frac{n-1}{2}t}\{\|\rho_0\|_{W^{[n/2]+1,2}(\mathbb{R}^n)}+\|\rho_1\|_{W^{[n/2]+1,2}(\mathbb{R}^n)}\}+Ce^{-\frac{n-1}{2}t}\int_0^t e^{\frac{n-1}{2}b}\|f(.,b)\|_{W^{[n/2]+1,1}(\mathbb{R}^n)}\,\mathrm{d}b\tag{8}$$

Proof. First, we consider the solution of the initial value problem:

$$\partial_t^2\xi+n\partial_t\xi+e^{-2t}\Delta\xi+m^2\xi=0,\quad \xi(x,0)=\rho_0(x),\quad \partial_t\xi(x,0)=\rho_1(x)\tag{9}$$

for $(x,t)\in\mathbb{R}^n\times(0,\infty)$, with $\rho_0,\rho_1\in C_0^\infty(\mathbb{R}^n)$. If $\rho_1=0$, from the solution of (9) in [9], we get:

$$\xi(x,t)=e^{-\frac{n-1}{2}t}v_{\rho_0}(x,\tau(t))+e^{-\frac{n}{2}t}\int_0^1 v_{\rho_0}(x,\tau(t)s)(2K_0(\tau(t)s,t)+nK_1(\tau(t)s,t))\tau(t)\mathrm{d}s$$

where for $\rho\in C_0^\infty(\mathbb{R}^n), v_\rho(x,t)$ denotes the solution to the following problem:

$$\partial_t^2 v-\Delta v=0,\ v(x,0)=\rho(x),\quad v_t(x,0)=0,\quad (x,t)\in\mathbb{R}^n\times(0,\infty)\tag{10}$$

Then, we get:

$$\begin{aligned}\|\xi(.,t)\|_{L^\infty(\mathbb{R}^n)}\le e^{-\frac{n-1}{2}t}\left\|v_{\rho_0}(.,\tau(t))\right\|_{L^\infty(\mathbb{R}^n)}+e^{-\frac{n}{2}t}\int_0^1\left\|v_{\rho_0}(.,\tau(t)s)\right\|_{L^\infty(\mathbb{R}^n)}\cdot\\ \cdot\left|\{2K_0(\tau(t)s,t)+nK_1(\tau(t)s,t)\}\right|\tau(t)\mathrm{d}s\end{aligned}\tag{11}$$

As is well known, the solution $v(x,t)$ of the initial value problem (10) satisfies:

$$\|v(.,t)\|_{L^\infty(\mathbb{R}^n)}\le C(1+t)^{-\frac{n-1}{2}}\|\rho\|_{W^{[n/2]+1,1}(\mathbb{R}^n)}\tag{12}$$

for $t \geq 0$, if $n \geq 2$ (see e.g. [11]). For all $t \geq 0$, we have:

$$e^{-\frac{n-1}{2}t} \| v_{\rho_0}(.,\tau(t)) \|_{L^\infty(\mathbb{R}^n)} \leq Ce^{-\frac{n-1}{2}t}(1+\tau(t))^{-\frac{n-1}{2}} \|\rho_0\|_{W^{[n/2]+1,1}(\mathbb{R}^n)}$$
$$\leq Ce^{-\frac{n-1}{2}t} \|\rho_0\|_{W^{[n/2]+1,1}(\mathbb{R}^n)}$$

Hence, we get:

$$e^{-\frac{n-1}{2}t} \left\| v_{\rho_0}(.,\tau(t)) \right\|_{L^\infty(\mathbb{R}^n)} \leq Ce^{-\frac{n-1}{2}t} \|\rho_0\|_{W^{[n/2]+1,1}(\mathbb{R}^n)} \tag{13}$$

On the other hand, we obtain:

$$e^{-\frac{n}{2}t} \int_0^1 \left\| v_{\rho_0}(.,\tau(t)s) \right\|_{L^\infty(\mathbb{R}^n)} \left| (2K_0(\tau(t)s,t) + nK_1(\tau(t)s,t)) \right| \tau(t)\mathrm{d}s \leq$$

$$\leq C\|\rho_0\|_{W^{[n/2]+1,1}(\mathbb{R}^n)} e^{-\frac{n}{2}t} \int_0^1 (\tau(t)s,t)^{-\frac{n-1}{2}} \mid (2K_0(\tau(t)s,t) + nK_1(\tau(t)s,t)) \mid \tau(t)\mathrm{d}s$$

From *Lemma 2* and *Lemma 3*, we have:

$$e^{-\frac{n}{2}t} \int_0^1 [1+\tau(t)s]^{-\frac{n-1}{2}} \left| 2K_0(\tau(t)s,t) \right| \tau(t)\mathrm{d}s \leq Ce^{-\frac{n-1}{2}t} \tag{14}$$

$$e^{-\frac{n}{2}t} \int_0^1 [1+\tau(t)s]^{-\frac{n-1}{2}} \left| nK_1(\tau(t)s,t) \right| \tau(t)\mathrm{d}s \leq Ce^{-\frac{n-1}{2}t} \tag{15}$$

Hence, from (13)-(15) we get:

$$\| \xi(.,t) \|_{L^\infty(\mathbb{R}^n)} \leq Ce^{-\frac{n-1}{2}t} \|\rho_0\|_{W^{[n/2]+1,1}(\mathbb{R}^n)} \tag{16}$$

when $\rho_1 = 0$. For the case $\rho_0 = 0$:

$$\| \xi(.,t) \|_{L^\infty(\mathbb{R}^n)} \leq Ce^{-\frac{n-1}{2}t} \|\rho_1\|_{W^{[n/2]+1,1}(\mathbb{R}^n)} \tag{17}$$

in a similar way. Since $\rho_0, \rho_1 \in C_0^\infty(\mathbb{R}^n)$, from Holder inequality we have:

$$\|\rho_0\|_{W^{[n/2]+1,1}(\mathbb{R}^n)} + \|\rho_1\|_{W^{[n/2]+1,1}(\mathbb{R}^n)} \leq C[\|\rho_0\|_{W^{[n/2]+1,2}(\mathbb{R}^n)} + \|\rho_1\|_{W^{[n/2]+1,2}(\mathbb{R}^n)}]$$

Then, from (16) and (17) we get:

$$\| \xi(.,t) \|_{L^\infty(\mathbb{R}^n)} \leq Ce^{-\frac{n-1}{2}t} [\|\rho_0\|_{W^{[n/2]+1,2}(\mathbb{R}^n)} + \|\rho_1\|_{W^{[n/2]+1,2}(\mathbb{R}^n)}] \tag{18}$$

for the solution of (9).

Next, we consider the solution to the initial value problem:

$$\partial_t^2 \xi + n\partial_t \xi + e^{-2t}\Delta\xi + m^2\xi = f, \quad \xi(x,0) = 0, \quad \partial_t \xi(x,0) = 0$$

for $(x,t)\in\mathbb{R}^n\times(0,\infty)$, with $f\in C^\infty(\mathbb{R}^{n+1})$. The solution of the initial value problem in [9] has the following form:

$$\xi(x,t)=2e^{-\frac{n}{2}t}\int_0^t \mathrm{d}b\int_0^{e^{-b}-e^{-t}}\mathrm{d}re^{\frac{n}{2}b}v(x,r;b)4^{i\mathcal{M}}e^{-i\mathcal{M}(b+t)}[(e^{-t}+e^{-b})^2-r^2]^{-\frac{1}{2}-i\mathcal{M}}\cdot$$
$$\cdot F\left(\frac{1}{2}+i\mathcal{M},\frac{1}{2}+i\mathcal{M};1;\frac{(e^{-b}-e^{-t})^2-r^2}{(e^{-b}+e^{-t})^2-r^2}\right)\mathrm{d}r$$

where $v(x,t;b)$ is the solution to the following problem:

$$\partial_t^2 v-\Delta v=0,\quad v(x,0;b)=f(x,b),\quad v_t(x,0;b)=0,(x,t)\in\mathbb{R}^n(0,\infty)$$

where $b > 0$. From (12), we get:

$$\|v(.,r;b)\|_{L^\infty(\mathbb{R}^n)}\le C(1+r)^{-\frac{n-1}{2}}\|f(.,b)\|_{W^{[n/2]+1,1}(\mathbb{R}^n)}$$

for all $r>0$. Hence:

$$\|\xi(.,t)\|_{L^\infty(\mathbb{R}^n)}\le C_M e^{-\frac{n}{2}t}\int_0^t e^{\frac{n}{2}b}\|f(.,b)\|_{W^{[n/2]+1,1}(\mathbb{R}^n)}\mathrm{d}b\cdot$$
$$\int_0^{e^{-b}-e^{-t}}(1+r)^{-\frac{n-1}{2}}[(e^{-t}+e^{-b})^2-r^2]^{-\frac{1}{2}}\left|F\left(\frac{1}{2}+i\mathcal{M},\frac{1}{2}+i\mathcal{M};1;\frac{(e^{-b}-e^{-t})^2-r^2}{(e^{-b}+e^{-t})^2-r^2}\right)\right|\mathrm{d}r\le$$
$$\le C_M e^{-\frac{n}{2}t}\int_0^t e^{\frac{n}{2}b}\|f(.,b)\|_{W^{[n/2]+1,1}(\mathbb{R}^n)}\mathrm{d}b\int_0^{e^{-b}-e^{-t}}[(e^{-t}+e^{-b})^2-r^2]^{-\frac{1}{2}}\cdot$$
$$\left|F\left(\frac{1}{2}+i\mathcal{M},\frac{1}{2}+i\mathcal{M};1;\frac{(e^{-b}-e^{-t})^2-r^2}{(e^{-b}+e^{-t})^2-r^2}\right)\right|\mathrm{d}r$$

If the variable $r=e^{-t}y$ is used, then we get:

$$\|\xi(.,t)\|_{L^\infty(\mathbb{R}^n)}\le C_M e^{-\frac{n}{2}t}\int_0^t e^{\frac{n}{2}b}\|f(.,b)\|_{W^{[n/2]+1,1}(\mathbb{R}^n)}\mathrm{d}b\cdot$$
$$\cdot\int_0^{e^{t-b}-1}[(e^{t-b}+1)^2-y^2]^{-\frac{1}{2}}\left|F\left(\frac{1}{2}+i\mathcal{M},\frac{1}{2}+i\mathcal{M};1;\frac{(e^{t-b}-1)^2-y^2}{(e^{t-b}+1)^2-y^2}\right)\right|\mathrm{d}y$$

Then we have the following estimate for the second integral of the last inequality:

$$\int_0^{e^{t-b}-1}[(e^{t-b}+1)^2-y^2]^{-\frac{1}{2}}\left|F\left(\frac{1}{2}+i\mathcal{M},\frac{1}{2}+i\mathcal{M};1;\frac{(e^{t-b}-1)^2-y^2}{(e^{t-b}+1)^2-y^2}\right)\right|\mathrm{d}y\le$$
$$\le C_M\int_0^{e^{t-b}-1}[(e^{t-b}+1)^2-y^2]^{-\frac{1}{2}}\mathrm{d}y\le C_M e^{\frac{t-b}{2}}[1-e^{-(t-b)}]$$

for $b < t$. Thus, we obtain:

$$\| \xi(.,t) \|_{L^\infty(\mathbb{R}^n)} \le C_M e^{-\frac{n-1}{2}t} \int_0^t e^{\frac{n-1}{2}b} \| f(.,b) \|_{W^{[n/2]+1,1}(\mathbb{R}^n)} \, \mathrm{d}b \tag{19}$$

Hence (18) and (19) lead to (8). This completes the proof.

Global existence with small data

From (1), it follows that:

$$\partial_{tt}\partial_x^\eta \xi + n\partial_t \partial_x^\eta \xi - e^{-2t}\Delta \partial_x^\eta \xi + m^2 \partial_x^\eta \xi = \partial_x^\eta F(\xi), \quad (x,t) \in \mathbb{R}^n \times (0,T),$$
$$\partial_x^\eta \xi(x,0) = \partial_x^\eta \rho_0(x), \quad (\partial_x^\eta \xi)_t(x,0) = \partial_x^\eta \rho_1(x), \quad x \in \mathbb{R}^n$$

where η is a multi-index. Therefore:

$$\partial_x^\eta \xi(x,t) = \xi_0(x,t) + L[\partial_x^\eta F(\xi)](x,t) \text{ for } (x,t) \in \mathbb{R}^n \times [0,T) \tag{20}$$

where ξ_0 is the solution of:

$$\partial_{tt}\xi + n\partial_t \xi - e^{-2t}\Delta \xi + m^2 \xi = 0, \quad (x,t) \in \mathbb{R}^n \times (0,T)$$
$$\xi(x,0) = \partial_x^\eta \rho_0(x), \quad \xi_t(x,0) = \partial_x^\eta \rho_1(x), \quad x \in \mathbb{R}^n$$

and for a smooth function *f*, set:

$$L[f](x,t) = 2e^{-\frac{n}{2}t} \int_0^t \mathrm{d}b \int_0^{e^{-b}-e^{-t}} \mathrm{d}r e^{\frac{n}{2}b} v(x,r;b) E(r,t;0,b) \text{ for } (x,t) \in \mathbb{R}^n \times (0,T)$$

where $E(r, t)$ is defined in [9]. Here $v(x,t;b)$ is the solution to the problem:

$$\partial_t^2 v - \Delta v = 0, \quad v(x,0;b) = f(x,b), \quad v_t(x,0;b) = 0, \quad (x,t) \in \mathbb{R}^n \times (0,T)$$

In other words, $L[f]$ is the solution of:

$$\partial_{tt}\xi + n\partial_t \xi - e^{-2t}\Delta \xi + m^2 \xi = f, \quad (x,t) \in \mathbb{R}^n \times (0,T),$$
$$\xi(x,0) = 0, \quad \xi_t(x,0) = 0, \quad x \in \mathbb{R}^n$$

Let $|\eta| \le [n/2] + 2$. Then, from (19) we get:

$$\left\| L[\partial_x^\eta F(\xi)](.,t) \right\|_{L^\infty(\mathbb{R}^n)} \le C e^{-\frac{n-1}{2}t} \int_0^t e^{\frac{n-1}{2}b} \| F(\xi)(.,b) \|_{W^{N,1}(\mathbb{R}^n)} \, \mathrm{d}b$$

for $t \in [0,T)$, where we put $N = 2[n/2] + 3$. From (18) , we also have:

$$\left\| \xi_0(.,t) \right\|_{L^\infty(\mathbb{R}^n)} \le C e^{-\frac{n-1}{2}t} [\left\| \rho_0 \right\|_{W^{N,2}(\mathbb{R}^n)} + \left\| \rho_1 \right\|_{W^{N,2}(\mathbb{R}^n)}], \quad t \in [0,T)$$

Therefore, we obtain:

$$\| \xi(.,t) \|_{W^{k,\infty}(\mathbb{R}^n)} \le C e^{-\frac{n-1}{2}t} [\left\| \rho_0 \right\|_{W^{N,2}(\mathbb{R}^n)} + \left\| \rho_1 \right\|_{W^{N,2}(\mathbb{R}^n)}] + C e^{-\frac{n-1}{2}t} \int_0^t e^{\frac{n-1}{2}b} \| F(\xi)(.,b) \|_{W^{N,1}(\mathbb{R}^n)} \, \mathrm{d}b \tag{21}$$

We use the following *Lemma* to estimate the non-linear term $F(\xi)$.

Lemma 4. Let $F(\xi)=|\xi|^{\alpha}\xi$ with an even integer $\alpha>0$, $N=2[n/2]+3$ and $k=[N/2]+1=[n/2]+2$. If ξ is a solution of (1) with compactly supported initial data, then we have:

$$\|F(\xi)\|_{W^{N,1}(\mathbb{R}^n)}\le C\|\xi\|^{\alpha}_{W^{k,\infty}(\mathbb{R}^n)}\|\xi\|_{W^{N,2}(\mathbb{R}^n)} \tag{22}$$

where C is a positive constant.

Proof. Since $F(0)=0$, from the finite speed propagation property of the solution, we get:

$$\|F(\xi)\|_{W^{N,1}(\mathbb{R}^n)}\le C\|F(\xi)\|_{W^{N,2}(\mathbb{R}^n)}$$

from Holder's inequality where C is a positive constant independent of t. It follows that:

$$\|F(\xi)\|_{W^{N,2}(\mathbb{R}^n)}=\sum_{|\sigma|\le N}\left\{\int_{\mathbb{R}^n}\left|\partial^{\sigma}((\xi\bar{\xi})^{\alpha/2}\xi)\right|^2\mathrm{d}x\right\}^{1/2}=$$

$$=\sum_{|\sigma|\le N}\left[\int_{\mathbb{R}^n}\left|\sum_{\sigma_1+\ldots+\sigma_{\alpha+1}=\sigma}C_{\sigma_\alpha}(\partial^{\sigma_1}\xi\partial^{\sigma_2}\bar{\xi}\ldots\partial^{\sigma_{\alpha-1}}\xi\partial^{\sigma_\alpha}\bar{\xi}\partial^{\sigma_{\alpha+1}}\xi)\right|^2\mathrm{d}x\right]^{1/2}$$

where C_{σ_α} is a suitable constant. Without loss of generality, we may assume $|\sigma_1|,\ldots,|\sigma_\alpha|\le[N/2]+1$. Hence we have:

$$\|F(\xi)\|_{W^{N,2}(\mathbb{R}^n)}\le C\|\xi\|_{W^{N,2}(\mathbb{R}^n)}\left[\sum_{|\nu|\le[N/2]+1}\partial^{\nu}\xi_{L^{\infty}(\mathbb{R}^n)}\right]^{\alpha}$$

This completes the proof.

Proof of Theorem 1. Since the local smooth solution of (1) exists, we need to derive is a suitable apriori estimate for proving the global solvability of (1). Let $N=2[n/2]+3$ and $k=[N/2]+1$. We assume that the solution of (1) satisfies:

$$e^{\frac{n-1}{2}t}\|\xi(.,t)\|_{W^{k,\infty}(\mathbb{R}^n)}\le 2\epsilon \text{ for } t\in[0,T) \tag{23}$$

for $n/2<m$, $T>0$, and $\epsilon>0$. We have from (20) and (21):

$$\begin{aligned}&e^{\frac{n-1}{2}t}\|\xi(.,t)\|_{W^{k,\infty}(\mathbb{R}^n)}\le C[\|\rho_0\|_{W^{N,2}(\mathbb{R}^n)}+\|\rho_1\|_{W^{N,2}(\mathbb{R}^n)}]+\\&+C\int_0^t e^{\frac{n-1}{2}b}\|\xi(.,b)\|^{\alpha}_{W^{k,\infty}(\mathbb{R}^n)}\|\xi(.,b)\|_{W^{N,2}(\mathbb{R}^n)}\,\mathrm{d}b\end{aligned} \tag{24}$$

where $m>n/2$. Therefore, we need to evaluate $\|\xi(.,b)\|_{W^{N,2}(\mathbb{R}^n)}$. Let $|\eta|\le N$ in (20). Then, from *Lemma 1* with $\rho_0\equiv\varphi_1\equiv 0, p=q=2$ and $s=l=0$ we have:

$$L\left[\partial_x^{\eta}F(\xi)\right](.,t)_{L^2(\mathbb{R}^n)}\le Ce^{-\frac{n-1}{2}t}\int_0^t e^{\frac{n-1}{2}b}\|F(\xi)(.,b)\|_{W^{N,2}(\mathbb{R}^n)}\,\mathrm{d}b$$

In view of the proof of *Lemma 4* we get:

$$\| F[\xi(.,b)] \|_{W^{N,2}(\mathbb{R}^n)} \le C \| \xi(.,b) \|^{\alpha}_{W^{k,\infty}(\mathbb{R}^n)} \| \xi(.,b) \|_{W^{N,2}(\mathbb{R}^n)}$$

On the other hand, from *Lemma 1* with $f \equiv 0, s = 0, l = N$ and $p = q = 2,$ we get:

$$\|\xi_0(.,t)\|_{W^{N,2}(\mathbb{R}^n)} \le Ce^{-\frac{n-1}{2}t}[\|\rho_0\|_{W^{N,2}(\mathbb{R}^n)} + \|\rho_1\|_{W^{N,2}(\mathbb{R}^n)}]$$

for $t \in [0,T).$ Summing up, we obtain:

$$\begin{aligned}\left\|\partial_x^{\eta}\xi(.,t)\right\|_{L^2(\mathbb{R}^n)} &\le Ce^{-\frac{n-1}{2}t}[\|\rho_0\|_{W^{N,2}(\mathbb{R}^n)} + \|\rho_1\|_{W^{N,2}(\mathbb{R}^n)}] + \\ &+Ce^{-\frac{n-1}{2}t}\int_0^t e^{\frac{n-1}{2}b} \| \xi(.,b) \|^{\alpha}_{W^{k,\infty}(\mathbb{R}^n)} \| \xi(.,b) \|_{W^{N,2}(\mathbb{R}^n)} \, \mathrm{d}b\end{aligned}$$

so that (23) yields:

$$\begin{aligned}e^{\frac{n-1}{2}t} \| \xi(.,t) \|_{W^{N,2}(\mathbb{R}^n)} &\le C[\|\rho_0\|_{W^{N,2}(\mathbb{R}^n)} + \|\rho_1\|_{W^{N,2}(\mathbb{R}^n)}] + \\ &+C\int_0^t e^{-\alpha\left(\frac{n-1}{2}\right)b}(2\epsilon)^{\alpha}\left[e^{\frac{n-1}{2}b} \| \xi(.,b) \|_{W^{N,2}(\mathbb{R}^n)}\right]\mathrm{d}b\end{aligned}$$

for $t \in [0,T).$ Since we assumed $\|\rho_0\|_{W^{N,2}(\mathbb{R}^n)} + \|\rho_1\|_{W^{N,2}(\mathbb{R}^n)} \le \epsilon,$ we see from Gronwall's inequality that for $t \in [0,T)$:

$$e^{\frac{n-1}{2}t} \| \xi(.,t) \|_{W^{N,2}(\mathbb{R}^n)} \le C\epsilon e^{\int_0^t C\epsilon^{\alpha} e^{-\alpha\left(\frac{n-1}{2}\right)b} db} \le C\epsilon$$

for $n \ge 2.$ Using these bounds in (24) , we get:

$$e^{\frac{n-1}{2}t} \| \xi(.,t) \|_{W^{k,\infty}(\mathbb{R}^n)} \le CC_0\epsilon + C\int_0^t \epsilon^{\alpha+1} e^{-\alpha\left(\frac{n-1}{2}\right)b} \mathrm{d}b \le C_0\epsilon + CC_1\epsilon^{\alpha+1}$$

for $t \in [0,T).$ If we choose ϵ and C_0 such that $CC_1\epsilon^{\alpha} \le C_0/3$, then we obtain:

$$e^{\frac{n-1}{2}t} \| \xi(.,t) \|_{W^{k,\infty}(\mathbb{R}^n)} \le \frac{4C_0\epsilon}{3}$$

for $n/2 < m.$ Combining the existence of the local solution, we find that the initial value problem (1) admits a global solution.

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