COEFFICIENTS BOUNDS FOR A SUBCLASS OF BI-UNIVALENT FUNCTIONS DEFINED BY AL-OBOUDI DIFFERENTIAL OPERATOR

by

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In this paper, we investigate a new subclass $\Sigma_\delta^n(\lambda,\gamma,\varphi)$ *of analytic and bi-univalent functions in the open unit disk* $\mathcal{U}=\{z:|z|<1\}$ *defined by Al-Oboudi differential operator. We obtain coefficient bounds* $|a_2|$ *and* $|a_3|$ *for functions belonging to subclass* $\Sigma_\delta^n(\lambda,\gamma,\varphi)$. *Relevant connections of the results presented here with various well-known results are briefly indicated.*

Key words: *analytic functions, univalent functions, Bi-univalent functions, subordination, Al-Oboudi differential operator*

Introduction

Let $\mathcal{A}$ denote the class of functions f of the form:

$$f(z)=z+\sum_{k=2}^{\infty}a_k z^k \tag{1}$$

which are analytic in the open unit disk $\mathcal{U}=\{z:|z|<1\}$. We also denote by $\mathcal{S}$ the class of all functions in $\mathcal{A}$ which are univalent in $\mathcal{U}$.

Al-Oboudi [1] introduced the following differential operator for $f(z)\in\mathcal{A}$ which is called the Al-Oboudi differential operator:

$$D^0 f(z)=f(z)$$

$$D^1 f(z)=(1-\delta)f(z)+\delta z f'(z)=D_\delta f(z), \quad (\delta\ge 0)$$

$$D^n f(z)=D_\delta[D^{n-1}f(z)] \quad (n\in\mathbb{N}=1,2,3,...)$$

We note that:

$$D^n f(z)=z+\sum_{k=2}^{\infty}[1+(k-1)\delta]^n a_k z^k \quad (n\in\mathbb{N}_0=\mathbb{N}\cup\{0\}) \tag{2}$$

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when $\delta = 1$ in (2), we get Salagean's differential operator [2].

It is well known that every $f \in \mathcal{S}$ has an inverse function f^{-1} satisfying:

$$f^{-1}[f(z)] = z \quad (z \in \mathcal{U})$$

and

$$f[f^{-1}(w)] = w \quad \left[|w| < r_0(f); r_0(f) \geq \frac{1}{4}\right]$$

where

$$f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2 a_3 + a_4)w^4 + \ldots$$

A function $f \in \mathcal{A}$ is said to be bi-univalent in $\mathcal{U}$ if both $f(z)$ and $f^{-1}(z)$ are univalent in $\mathcal{U}$. Let Σ denote the class of bi-univalent functions in $\mathcal{U}$ given by (1). Lewin [3] introduced the bi-univalent function class and showed that $|a_2| < 1.51$. Subsequently, Brannan and Clunie [4] conjectured that $|a_2| \leq \sqrt{2}$. Netanyahu [5], otherwise, showed that $\max\limits_{f \in \Sigma} |a_2| = 4/3$. The coefficient estimate problem for each of the following Taylor Maclaurin coefficients: $|a_n|$ $(n \in \mathbb{N} \setminus \{1, 2\}; \mathbb{N} = \{1, 2, 3, \ldots\})$ is still an open problem. Recently, several researchers such as [1, 3-25] obtained the coefficients $|a_2|$, $|a_3|$ of bi-univalent functions for the various subclasses of the function class Σ. Motivating with their work, we introduce a new subclass of the function class Σ and find estimates on the coefficients $|a_2|$ and $|a_3|$ for functions in these new subclass of the function class Σ employing the techniques used earlier by Srivastava *et al.* [20] and Frasin and Aouf [13].

Let φ be an analytic and univalent function with positive real part in $\mathcal{U}$, $\varphi(0) = 1$, $\varphi'(0) > 0$ and φ maps the unit disk $\mathcal{U}$ onto a region starlike with respect to 1 and symmetric with respect to the real axis. The Taylor's series expansion of such function is:

$$\varphi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + \ldots, \tag{3}$$

where all coefficients are real and $B_1 > 0$. Throughout this paper we assume that the function φ satisfies the above conditions unless otherwise stated.

Definition 1. A function $f \in \Sigma$ given by (1) is said to be in the class $\Sigma_\delta^n(\lambda, \gamma, \varphi)$ if the following conditions are satisfied:

$$1 + \frac{1}{\lambda}\{[D^n f(z)]' + \gamma z [D^n f(z)]'' - 1\} \prec \varphi(z) \quad (0 \leq \gamma \leq 1,\ \delta \geq 0, \lambda \in \mathbb{C} / \{0\},\ n \in \mathbb{N},\ z \in \mathcal{U})$$

and

$$1 + \frac{1}{\lambda}\{[D^n g(w)]' + \gamma w [D^n g(w)]'' - 1\} \prec \varphi(w) \big(0 \leq \gamma \leq 1,\ \delta \geq 0, \lambda \in \mathbb{C}/\{0\}, n \in \mathbb{N}, w \in \mathcal{U}\big)$$

where the function g is given by:

$$g(w) = f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2 a_3 + a_4)w^4 + \ldots$$

and D^n is the Al-Oboudi differential operator.

In this paper, we obtain the estimates on the coefficients $|a_2|$ and $|a_3|$ for $\Sigma_\delta^n(\lambda, \gamma, \varphi)$ as well as its special classes.

Firstly, in order to derive our main results, we need the following lemma.

Lemma 1. [26] Let $p(z)=1+c_1z+c_2z^2+...\in P,$ where P is the family of all functions p, analytic in $\mathcal{U}$, for which $\operatorname{Re}p(z)>0 \quad (z\in\mathcal{U}).$ Then:

$$|c_n|\leq 2; n=1,2,3,....$$

Coefficients bounds for the class $\Sigma_\delta^n(\lambda,\gamma,\varphi)$

Theorem 1. Let $f(z)\in\Sigma_\delta^n(\lambda,\gamma,\varphi)$ be of the form (1). Then:

$$|a_2|\leq\frac{|\lambda|\sqrt{B_1^3}}{\sqrt{\left|3\lambda B_1^2(1+2\delta)^n(1+2\gamma)+4(1+\delta)^{2n}(1+\gamma)^2(B_1-B_2)\right|}} \tag{4}$$

and

$$|a_3|\leq B_1|\lambda|\left[\frac{B_1|\lambda|}{4(1+\delta)^{2n}(1+\gamma)^2}+\frac{1}{3(1+2\delta)^n(1+2\gamma)}\right] \tag{5}$$

Proof. Since $f\in\Sigma_\delta^n(\lambda,\gamma,\varphi),$ there exist two analytic functions $u,v:\mathcal{U}\to\mathcal{U},$ with $u(0)=v(0)=0,$ such that:

$$1+\frac{1}{\lambda}\{[D^nf(z)]'+\gamma z[D^nf(z)]''-1\}=\varphi[u(z)] \quad (z\in\mathcal{U}) \tag{6}$$

and

$$1+\frac{1}{\lambda}\{[D^ng(w)]'+\gamma w[D^ng(w)]''-1\}=\varphi[v(w)] \quad (w\in\mathcal{U}) \tag{7}$$

Define the function p and q as:

$$p(z)=\frac{1+u(z)}{1-u(z)}=1+c_1z+c_2z^2+c_3z^3+...$$

and

$$q(w)=\frac{1+v(w)}{1-v(w)}=1+b_1w+b_2w^2+b_3w^3+...$$

or equivalently:

$$u(z)=\frac{p(z)-1}{p(z)+1}=\frac{c_1}{2}z+\frac{1}{2}\left(c_2-\frac{c_1^2}{2}\right)z^2+\frac{1}{2}\left[c_3+\frac{c_1}{2}\left(\frac{c_1^2}{2}-c_2\right)-\frac{c_1c_2}{2}\right]z^3... \tag{8}$$

and

$$v(w)=\frac{q(w)-1}{q(w)+1}=\frac{b_1}{2}w+\frac{1}{2}\left(b_2-\frac{b_1^2}{2}\right)w^2+\frac{1}{2}\left[b_3+\frac{b_1}{2}\left(\frac{b_1^2}{2}-b_2\right)-\frac{b_1b_2}{2}\right]w^3.... \tag{9}$$

If we use (8) and (9) in (6) and (7) along with (3), we have:

$$1+\frac{1}{\lambda}\{[D^nf(z)]'+\gamma z[D^nf(z)]''-1\}=1+\frac{1}{2}B_1c_1z+\left[\frac{1}{2}B_1\left(c_2-\frac{c_1^2}{2}\right)+\frac{1}{4}B_2c_1^2\right]z^2+... \tag{10}$$

and

$$1+\frac{1}{\lambda}\{[D^n g(w)]' + \gamma w[D^n g(w)]'' - 1\} = 1+\frac{1}{2}B_1 b_1 w + \left[\frac{1}{2}B_1\left(b_2 - \frac{b_1^2}{2}\right) + \frac{1}{4}B_2 b_1^2\right]w^2 + \dots \quad (11)$$

It follows from (10) and (11) that:

$$\frac{2(1+\delta)^n(1+\gamma)a_2}{\lambda} = \frac{1}{2}B_1 c_1 \quad (12)$$

$$\frac{3(1+2\delta)^n(1+2\gamma)a_3}{\lambda} = \frac{1}{2}B_1\left(c_2 - \frac{c_1^2}{2}\right) + \frac{1}{4}B_2 c_1^2 \quad (13)$$

and

$$-\frac{2(1+\delta)^n(1+\gamma)a_2}{\lambda} = \frac{1}{2}B_1 b_1 \quad (14)$$

$$\frac{3(1+2\delta)^n(1+2\gamma)(2a_2^2 - a_3)}{\lambda} = \frac{1}{2}B_1\left(b_2 - \frac{b_1^2}{2}\right) + \frac{1}{4}B_2 b_1^2 \quad (15)$$

From (12) and (14) we obtain:

$$c_1 = -b_1 \quad (16)$$

By adding (13) to (15) and combining this with (12) and (14), we get:

$$a_2^2 = \frac{\lambda^2 B_1^3(b_2 + c_2)}{4[3\lambda B_1^2(1+2\delta)^n(1+2\gamma) + 4(1+\delta)^{2n}(1+\gamma)^2(B_1 - B_2)]} \quad (17)$$

Subtracting (13) from (15), if we use (12) and applying (16), we have:

$$a_3 = \frac{\lambda^2 B_1^2 b_1^2}{16(1+\delta)^{2n}(1+\gamma)^2} + \frac{\lambda B_1(c_2 - b_2)}{12(1+2\delta)^n(1+2\gamma)} \quad (18)$$

Finally, in view of *Lemma 1*, we get results (4) to (5) asserted by the *Theorem 1*.

Corollaries and consequences

i) If we set:

$$\lambda = e^{i\theta}\cos\theta \left(-\frac{\pi}{2} < \theta < \frac{\pi}{2}\right)$$

and

$$\varphi(z) = \frac{1+(1-2\tau)z}{1-z} = 1 + 2(1-\tau)z + 2(1-\tau)z^2 + \dots \quad (0 \le \tau < 1)$$

which gives $B_1 = B_2 = 2(1-\tau)$, in *Theorem 1*, we can have the following corollary.

*Corollary 1***.** Let:

$$f(z)\in\Sigma_{\delta}^{n}\left[e^{i\theta}cos\theta,\gamma,\frac{1+(1-2\tau)z}{1-z}\right]$$

be of the form (1). Then:

$$|a_2|\le\sqrt{\frac{2(1-\tau)}{3(1+2\delta)^n(1+2\gamma)}\cos\theta} \tag{19}$$

and

$$|a_3|\le 2(1-\tau)\left[\frac{(1-\tau)\cos\theta}{2(1+\delta)^{2n}(1+\gamma)^2}+\frac{1}{3(1+2\delta)^n(1+2\gamma)}\right]\cos\theta \tag{20}$$

Remark 1. For $\gamma=0$, *Corollary 1* simplifies to the following form.
Corollary 2. Let:

$$f(z)\in\sum_{\delta}^{n}\left[e^{i\theta}cos\theta,0,\frac{1+(1-2\tau)z}{1-z}\right]$$

be of the form (1). Then:

$$|a_2|\le\sqrt{\frac{2(1-\tau)}{3(1+2\delta)^n}\cos\theta} \tag{21}$$

and

$$|a_3|\le 2(1-\tau)\left[\frac{(1-\tau)\cos\theta}{2(1+\delta)^{2n}}+\frac{1}{3(1+2\delta)^n}\right]\cos\theta \tag{22}$$

ii) If we set $\lambda=1$ and:

$$\varphi(z)=\left(\frac{1+z}{1-z}\right)^{\alpha}=1+2\alpha z+2\alpha^2 z^2+\ldots\ \ (0<\alpha\le 1)$$

which gives $B_1=2\alpha$, $B_2=2\alpha^2$, in *Theorem 1*, we can obtain the following corollary.
Corollary 3. Let:

$$f(z)\in\sum_{\delta}^{n}\left[1,\gamma,\left(\frac{1+z}{1-z}\right)^{\alpha}\right]$$

be of the form (1). Then:

$$|a_2|\le\alpha\sqrt{\frac{2}{3(1+2\delta)^n(1+2\gamma)\alpha+2(1+\delta)^{2n}(1+\gamma)^2(1-\alpha)}} \tag{23}$$

and

$$|a_3|\le\left[\frac{\alpha^2}{(1+\delta)^{2n}(1+\gamma)^2}+\frac{2\alpha}{3(1+2\delta)^n(1+2\gamma)}\right] \tag{24}$$

Remark 2. In its special case when $\gamma = 0$ in *Corollary 3.*, we can get the following corollary.

Corollary 4. Let:

$$f(z) \in \sum_{\delta}^{n}\left[1, 0, \left(\frac{1+z}{1-z}\right)^{\alpha}\right]$$

be of the form (1). Then:

$$|a_2| \le \alpha\sqrt{\frac{2}{3\alpha(1+2\delta)^n + 2(1-\alpha)(1+\delta)^{2n}}} \tag{25}$$

and

$$|a_3| \le \left[\frac{\alpha^2}{(1+\delta)^{2n}} + \frac{2\alpha}{3(1+2\delta)^n}\right] \tag{26}$$

Remark 3.

i. If we take $n = 0$ in *Theorem 1*, we obtain the corresponding result given earlier by Deniz [12] (also Srivastava and Bansal [22]).

ii. Putting $\lambda = 1$, $\gamma = 0$, $n = 0$ in *Theorem 1*, we have the corresponding result given earlier by Ali *et al*. [6].

iii. For $\beta = 0$, $n = 0$ in *Corollary 2* and $\gamma = 0$, $n = 0$ in *Corollary 3*, we get the corresponding result given earlier by Srivastava *et al.* [21].

iv. Putting $\delta = 1$ in *Theorem 1*, we obtain the corresponding result given earlier by Caglar and Deniz [10].

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