ON TUBULAR SURFACES WITH MODIFIED ORTHOGONAL FRAME IN GALILEAN SPACE $\mathbb{G}_3$

by

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In this study, we give the curves and tubular surfaces in Galilean space $\mathbb{G}_3$ with modified orthogonal frame. Firstly, we consider any curve in $\mathbb{G}_3$ and we obtain derivative formulas of modified orthogonal frame of any curve in $\mathbb{G}_3$. Therefore, we get tubular surface with modified orthogonal frame in Galilean space $\mathbb{G}_3$. Consequently, we give some examples and figures.

Key words: *Galilean space $\mathbb{G}_3$, modified orthogonal frame, tubular surface*

Introduction

The foundations of Euclidean geometry were laid by Euclid in 300 BC, and Euclidean geometry was thought for a long time as the only existing geometric system. However, later Gauss (1816), Lobachevsky (1829), and Bolyai (1832), independently of each other, proposed a non-Euclidean geometry, hyperbolic geometry. Then, the elliptic (spherical) geometry, which is also a non-Euclidean geometry, was expressed by Riemann. Although it was thought that non-Euclidean geometry had no purpose at first, the superiority of non-Euclidean geometries was proven when the similarity between Einstein's general theory of relativity and geometry considered at the beginning of the 20th century. In addition, the importance of non-Euclidean geometries began to be better understood with the understanding that Hilbert's infinite-dimensional metric geometry could explain the mathematical structure of atomic theory. Considering the geometries on the plane in the literature, it is possible to come across many planar geometries in addition to the Euclidean and non-Euclidean geometries mentioned above. According to the definitions and classifications given by Yaglom [1, 2], these geometries were nine. The relations of these geometries with each other were shown by Klein in 1871 and these geometries were named as Cayley-Klein geometries based on Cayley's previous studies [3, 4]. These geometries were classified by the concepts of *measure of the distance between two points on a line* and *measure of the angle between lines passing through a point*. In this book given by Yaglom [1], there are physical foundations of Galilean geometry and basic information about Galilean geometry. Afterwards, some basic concepts about sur-

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faces in Galilean space were introduced by Roschel, Divjak and Sipus *et al.* given by [5, 6]. In addition, Pavković *et al.* mentioned some concepts about curves in Galilean space in [7].

Studies on the theory of surfaces date back to the 18th century. In 1785, Monge investigated the formation of surfaces called envelopes in $\mathbb{E}_3$. Towards the end of the 19th century, mathematicians such as Lie, Beltrami, Codazzi and Darboux contributed to the theory of surfaces with their important results in $\mathbb{E}_3$, [8]. The canal surface is the surface that is defined by one parameter sphere families with a center curve (u) and radius (r(u)). If the radius function [r(u)] is constant, the canal surface is typically called a tube surface. Tubular surfaces are used to show 3-D shapes such as pipes, ropes, poles. In addition, tubular surfaces are used for modeling solids and surfaces for computer aided geometric design and fabrication. The canal surface (tubular surface) was first defined by Monge and later this surface was discussed by many geometers from different perspectives. In the study [9], some geometric properties of the tubular surface were investigated by Do Carmo. The study on the canal and tubular surfaces defined with the help of the Bishop frame, which is created as an alternative to the Frenet framework in $\mathbb{E}_3$, is given in [10]. In the study [11], the Weingarten surface of the tubular surface was examined with respect to the 2nd type Bishop frame. In addition, studies on the canal and tubular surface are also included in spaces other than Euclidean space. For example, tubular surfaces in Minkowski space [12, 13] and tubular surfaces in Galilean space were investigated in studies [14-16]. Then, a lot of papers were studied on "Modified Orthogonal Frame" [17-25].

Preliminaries

The Galilean space $\mathbb{G}_3$ contains an ordered triad $\{w, f, I\}$. This triad is *w*: absolute (ideal) plane, *f*: absolute line in the absolute plane, and *I*: the fixed elliptical involution of the absolute line.

Definition 1. The following definitions apply to the lines and planes in $\mathbb{G}_3$. A line is referred to as a non-isotropic line if the intersection of the line with an absolute line is an empty set. Or, if the intersection of this line with the absolute line is not an empty set, this line is isotropic line. On the other hand, if a plane does not contain an absolute ideal line, it is called the isotropic plane, otherwise the Euclidean plane [1].

Definition 2. Consider that the vector $\vec{\omega} = (\omega_1, \omega_2, \omega_3)$ is any vector in $\mathbb{G}_3$. Therefore, if $\omega_1 \neq 0$ then, the vector $\vec{\omega}$ is non-isotropic vector, if $\omega_1 = 0$ then, the vector $\vec{\omega}$ is isotropic vector [1].

Definition 3. Let the vectors $\vec{\omega} = (\omega_1, \omega_2, \omega_3)$ and $\vec{\mu} = (\mu_1, \mu_2, \mu_3)$ be two any vectors in Galilean 3-space. Therefore, the scalar product of these vectors in Galilean 3-space is written by:

$$\langle \vec{\omega}, \vec{\mu} \rangle_G = \begin{cases} \omega_1 \mu_1, & \text{if } \omega_1 \neq 0 \text{ or } \mu_1 \neq 0 \\ \omega_2 \mu_2 + \omega_3 \mu_3, & \text{if } \omega_1 = 0 \text{ and } \mu_1 = 0 \end{cases}$$

The orthogonality in Galilean Space of these vectors in $\mathbb{G}_3$ is defined by $\langle \vec{\omega}, \vec{\mu} \rangle_G = 0$.

Definition 4. Suppose that the vector $\vec{\omega} = (\omega_1, \omega_2, \omega_3)$ is in $\mathbb{G}_3$. In that case, the norm of this vector in $\mathbb{G}_3$ is:

$$\|\vec{\omega}\|_G = \sqrt{\langle \vec{\omega}, \vec{\omega} \rangle_G} = \begin{cases} |\omega_1|, & \text{if } \omega_1 \neq 0 \\ \sqrt{\omega_2^2 + \omega_3^2}, & \text{if } \omega_1 = 0 \end{cases}$$

If $\|\vec{\omega}\|_G = 1$ for the norm of the vector $\vec{\omega}$ in $\mathbb{G}_3$ then, vector in $\mathbb{G}_3$ is unit vector [1].

Definition 5. In the Galilean space $\mathbb{G}_3$, the distance between two points $\vec{\Omega} = (\omega_1, \omega_2, \omega_3)$ and $\vec{M} = (\mu_1, \mu_2, \mu_3)$ in $\mathbb{G}_3$ is [1, 16]:

$$d(\Omega, M) = \begin{cases} |\mu_1 - \omega_1|, & \text{if } \omega_1 \neq \mu_1 \\ \sqrt{(\mu_2 - \omega_2)^2 + (\mu_3 - \omega_3)^2}, & \text{if } \omega_1 = \mu_1 \end{cases}$$

Definition 6. Let two arbitrary vectors in $\mathbb{G}_3$ be $\vec{\omega} = (\omega_1, \omega_2, \omega_3)$ and $\vec{\mu} = (\mu_1, \mu_2, \mu_3)$. In that case, the vector product of these vectors in $\mathbb{G}_3$ is [1, 7]:

$$\vec{\omega} \wedge_G \vec{\mu} = \begin{cases} \begin{vmatrix} 0 & e_2 & e_3 \\ \omega_1 & \omega_2 & \omega_3 \\ \mu_1 & \mu_2 & \mu_3 \end{vmatrix}, & \text{if } \omega_1 \neq 0 \text{ or } \mu_1 \neq 0 \\ \\ \begin{vmatrix} e_1 & e_2 & e_3 \\ \omega_1 & \omega_2 & \omega_3 \\ \mu_1 & \mu_2 & \mu_3 \end{vmatrix}, & \text{if } \omega_1 = 0 \text{ or } \mu_1 = 0 \end{cases}$$

Let the curve (α) of the class C^∞ be any non-isotropic curve $\mathbb{G}_3$ and this curve is given by $\alpha(s) = (x(s), y(s), z(s))$ where s is Galilean invariant parameter. In that case, if the Galilean parameter s is considered as the arc-length parameter of the curve, α, the curve, α, can be written by $\alpha(s) = [s, y(s), z(s)]$ [7, 16]. Now, the Frenet vectors and curvatures of the curve, α, is considered as $\{\mathbf{t}(s), \mathbf{n}(s), \mathbf{b}(s)\}$ and $\{\kappa(s), \tau(s)\}$. Therefore, the Frenet vectors of the Galilean curve, α, can be written as:

$$\mathbf{t}(s) = [1, y'(s), z'(s)], \quad \mathbf{n}(s) = \frac{1}{\kappa(s)}[0, y''(s), z''(s)], \quad \mathbf{b}(s) = \frac{1}{\kappa(s)}[0, -z''(s), y''(s)]$$

where the Frenet curvatures κ, τ are $\kappa(s) = \sqrt{y''^2(s) + z''^2(s)}$ and:

$$\tau(s) = \frac{y''(s)z'''(s) - z''(s)y'''(s)}{\kappa^2(s)}$$

In addition to that, the Frenet formulae of the Galilean curve, α, with the Galilean invariant arc-length parameter s are obtained by [7]:

$$\nabla_{\mathbf{t}}\mathbf{t} = \kappa\mathbf{n}, \nabla_{\mathbf{t}}\mathbf{n} = \tau\mathbf{b}, \nabla_{\mathbf{t}}\mathbf{b} = -\tau\mathbf{n}$$

Theorem 1. Consider that α is any curve in Galilean space $\mathbb{G}_3$ and $\{\kappa, \tau\}$ are the Frenet curvatures of the Galilean curve, α. Thus, the curve α is general helix therefore, the equation is hold [17].

$$\kappa/\tau = \text{constant}$$

Suppose that M is any surface in Galilean space $\mathbb{G}_3$. Therefore, the Galilean surface M can be parametrized by:

$$\vartheta(\rho^1,\rho^2)=[x(\rho^1,\rho^2),y(\rho^1,\rho^2),z(\rho^1,\rho^2)]$$

where $x(\rho^1,\rho^2),y(\rho^1,\rho^2),z(\rho^1,\rho^2)\in C^3$ functions and $\rho^1,\rho^2\in\mathbb{R}$.

Therefore, if we consider that the unit normal vector field of the Galilean surface M is the isotropic vector field $\mathbf{n}$ and we get isotropic normal vector of tubular surface as:

$$\mathbf{n}=\frac{\vartheta_{,1}\wedge\vartheta_{,2}}{\left\|\vartheta_{,1}\wedge\vartheta_{,2}\right\|_1}$$

where $\vartheta_{,1}=[\partial\vartheta(\rho^1,\rho^2)]/[\partial\rho^1]$, $\vartheta_{,2}=[\partial\vartheta(\rho^1,\rho^2)]/[\partial\rho^2]$, and $\|\ \|_1$ is the isotropic norm [16]. Moreover, the first fundamental form of the Galilean surface M can be calculated by $I=(g_{ij}+\varepsilon h_{ij})d\rho^i d\rho^j$, where $g_{ij}=\left\langle\vartheta_{,i},\vartheta_{,j}\right\rangle,h_{ij}=\left\langle\vartheta_{,i},\vartheta_{,j}\right\rangle_1$, $\langle,\rangle_1$ is the isotropic scalar product and ε is:

$$\varepsilon=\begin{cases}0, & d\rho^1:d\rho^2 \text{ non-isotropic}\\ \\ 1, & d\rho^1:d\rho^2 \text{ isotropic}\end{cases}$$

In addition to that, the second fundamental form of the Galilean surface M can be defined by:

$$II=L_{11}d\rho^{12}+2L_{12}d\rho^1 d\rho^2+L_{22}d\rho^{22}$$

where the coefficients L_{ij} is:

$$L_{ij}=<\frac{\vartheta_{,ij}x_{,1}-x_{,ij}\vartheta_{,1}}{x_{,1}},\quad \mathbf{n}>_1$$

Corollary 1. Consider that the surface M is any Galilean surface in $\mathbb{G}_3$. As a result, the mean curvature H and the Gauss curvature K of M are provided by [16]:

$$K=\frac{\det L_{ij}}{\left\|\vartheta_{,1}\wedge\vartheta_{,2}\right\|_1^2},\quad 2H=g^{ij}L_{ij}$$

where

$$g^1=\frac{x_{,2}}{\left\|\vartheta_{,1}\wedge\vartheta_{,2}\right\|_1},\quad g^2=\frac{x_{,1}}{\left\|\vartheta_{,1}\wedge\vartheta_{,2}\right\|_1},\quad g^{ij}=g^i g^j \tag{1}$$

Main theorems and proofs

On curves with modified orthogonal frame in $\mathbb{G}_3$

In this section, we expressed a new orthogonal frame which is called *Modified Orthogonal Frame* for the curves in Galilean space $\mathbb{G}_3$. The modified orthogonal frame in

Euclidean space was first defined by Sasai [18]. Sasai expressed that if the curvature κ of any curve in Euclidean space is hold $\kappa = 0$ then, the Frenet frame of this curve can not be constructed. Therefore, Sasai modified the Frenet frame and called modified orthogonal frame. Now, we defined the modified orthogonal frame for the Galilean curves in $\mathbb{G}_3$.

Let (δ) in $\mathbb{G}_3$ be any Galilean curve with arc-length parameter $s \in I$ and $\{\mathcal{T}, \mathcal{N}, \mathcal{B}\}$ be the modified orthogonal frame in Galilean curve, δ. Therefore, we can define:

$$\mathcal{T} = \frac{d\delta}{ds}, \quad \mathcal{N} = \frac{d\mathcal{T}}{ds}, \quad \mathcal{B} = \mathcal{T} \times \mathcal{N}$$

In that case, the relation between the modified orthogonal frame and the Frenet frame in Galilean space $\mathbb{G}_3$ is:

$$\mathcal{T} = \mathbf{t}, \quad \mathcal{N} = \kappa\mathbf{n}, \quad \mathcal{B} = \kappa\mathbf{b} \tag{2}$$

where $\mathcal{N}(s_0) = \mathcal{B}(s_0) = 0$ when $\kappa(s_0) = 0$ and the squares of the length of $\mathcal{N}$ and $\mathcal{B}$ vary analytically in parameter *s*.

Now, we take the derivatives eq. (3.1). Therefore, we get:

$$\begin{bmatrix} \nabla_{\mathcal{T}}\mathcal{T} \\ \nabla_{\mathcal{T}}\mathcal{N} \\ \nabla_{\mathcal{T}}\mathcal{B} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & \dfrac{\kappa'}{\kappa} & \tau \\ 0 & -\tau & \dfrac{\kappa'}{\kappa} \end{bmatrix} \begin{bmatrix} \mathcal{T} \\ \mathcal{N} \\ \mathcal{B} \end{bmatrix} \tag{3}$$

where $\kappa^2(s) = 0$ and $\tau(s) = [\det(\delta', \delta'', \delta''')]/[\kappa^2]$ which we know that if there is any point with $\kappa^2 = 0$. Then, the point is a removable singularity of τ. In addition to that, the modified orthogonal frame satisfies:

$$\langle \mathcal{T}, \mathcal{T} \rangle = 1, \quad \langle \mathcal{N}, \mathcal{N} \rangle = \langle B, B \rangle = \kappa^2, \quad \langle \mathcal{T}, \mathcal{N} \rangle = \langle \mathcal{T}, B \rangle = \langle \mathcal{N}, B \rangle = 0$$

If we take the curvature κ as $\kappa = 1$ then, the modified orthogonal frame coincides with the Frenet frame.

Now, we give some theorems for general helices with the modified orthogonal frame $\{\mathcal{T}, \mathcal{N}, \mathcal{B}\}$ of any Galilean curve in $\mathbb{G}_3$.

Theorem 2. Suppose that $\delta = \delta(s)$ is any curve with arc-length parameter *s* in Galilean space $\mathbb{G}_3$ and $\{\mathcal{T}, \mathcal{N}, \mathcal{B}\}$ is the modified orthogonal frame of this Galilean curve (δ). The curve (δ) is a general helix with respect to the modified orthogonal frame in $\mathbb{G}_3$ therefore, the necessary and sufficient condition is that:

$$\nabla_{\mathcal{T}}\nabla_{\mathcal{T}}\nabla_{\mathcal{T}}T - \mho\nabla_{\mathcal{T}}T = \frac{3\tau'}{\tau}\nabla_{\mathcal{T}}\mathcal{N}$$

where $\mho = \dfrac{\kappa''}{\kappa} - \tau^2 - \dfrac{3\kappa'^2}{\kappa^2}$.

Proof. Consider that the Galilean curve, δ, has the modified orthogonal frame $\{\mathcal{T}, \mathcal{N}, \mathcal{B}\}$ in $\mathbb{G}_3$. Therefore, we can write that:

$$\nabla_{\mathcal{T}}(\nabla_{\mathcal{T}}\mathcal{T}) = \frac{\kappa'}{\kappa}\mathcal{N} + \tau B$$

and

$$\nabla_{\mathcal{T}}\nabla_{\mathcal{T}}\nabla_{\mathcal{T}}\mathcal{T}=\left(\frac{\kappa''}{\kappa}-\tau^2\right)\mathcal{N}+\left(2\frac{\kappa'\tau}{\kappa}+\tau'\right)B \tag{4}$$

from eq. (3.2). On the other hand, we know that the Galilean curve, δ, is a general helix and the ratio:

$$\frac{\kappa}{\tau}=\text{constant} \tag{5}$$

is hold. Therefore, with derivatives of eq. (5) we get:

$$\kappa'\tau=\kappa\tau' \tag{6}$$

In that case, using eqs. (3), (4) and (6) we have:

$$\nabla_{\mathcal{T}}\nabla_{\mathcal{T}}\nabla_{\mathcal{T}}\mathcal{T}=(\frac{\kappa''}{\kappa}-\tau^2)\mathcal{N}+(3\tau')B \tag{7}$$

and consequently, we get:

$$\nabla_{\mathcal{T}}\nabla_{\mathcal{T}}\nabla_{\mathcal{T}}\mathcal{T}=\left(\frac{\kappa''}{\kappa}-\tau^2-3\frac{\kappa'^2}{\kappa^2}\right)\mathcal{N}+\left(3\frac{\tau'}{\tau}\right)\nabla_{\mathcal{T}}\mathcal{N}$$

Then, we use $\nabla_{\mathcal{T}}\mathcal{T}=\mathcal{N}$ and taking the derivative of the last equation twice, we get:

$$\nabla_{\mathcal{T}}\nabla_{\mathcal{T}}\nabla_{\mathcal{T}}\mathcal{T}=\nabla_{\mathcal{T}}\nabla_{\mathcal{T}}\mathcal{N}$$

Thus, from (3) and (7), we have:

$$\left(\frac{\kappa''}{\kappa}-\tau^2\right)\mathcal{N}+3\tau'B=\left(\frac{\kappa''}{\kappa}-\tau^2\right)\mathcal{N}+\left(2\frac{\kappa'\tau}{\kappa}+\tau'\right)B$$

Consequently, we obtain:

$$\frac{\kappa'}{\kappa}=\frac{\tau'}{\tau}$$

and

$$\left(\frac{\tau}{\kappa}\right)'=0$$

Therefore we see that τ/κ = constant and the Galilean curve, δ, is general helix.

Tubular surfaces with modified orthogonal frame in Galilean space $\mathbb{G}_3$

This section provides the tubular surfaces in a modified orthogonal frame in Galilean space $\mathbb{G}_3$. Then, using a modified orthogonal frame in $\mathbb{G}_3$, we obtain the mean curvature H and the Gaussian curvature K of these tubular surfaces.

A canal surface is expressed as the envelope of the one parameter sphere families with center curve $\delta(\upsilon)$ and radius $r(\upsilon)$. If the radius function $r(\upsilon)$ is a constant function therefore, the canal surface is defined the tubular surface.

Let $\delta=\delta(\upsilon)$ be a Galilean curve with arc-length parameter υ and $\{\mathcal{T},\mathcal{N},\mathcal{B}\}$ be the modified orthogonal frame of (δ). Therefore, the parametric equation of the Galilean tubular surface in $\mathbb{G}_3$ can be written:

$$\xi(\upsilon,\eta)=\delta(\upsilon)+r(\cos\eta\mathcal{N}+\sin\eta\mathcal{B}) \tag{8}$$

where η is Galilean angle between the vectors $\mathcal{N}$ and $r(\cos\eta\mathcal{N}+\sin\eta\mathcal{B})$, r is the radius of the Galilean tubular surface, and $\delta(\upsilon)$ is the center Galilean curve of this surface.

Now, we take the partial derivatives in according to υ and η of the Galilean tubular surface $\xi(\upsilon,\eta)$. Therefore, we obtain:

$$\xi_\upsilon=\mathcal{T}+r(\sigma\cos\eta-\tau\sin\eta)\mathcal{N}+r(\tau\cos\eta-\sigma\sin\eta)\mathcal{B} \tag{9}$$

where $\sigma=\kappa'/\kappa$ and:

$$\xi_\eta=-r\sin\eta\mathcal{N}+r\cos\eta\mathcal{B} \tag{10}$$

we know that $\mathcal{B}\wedge\mathcal{N}=0$ and $\mathcal{T}\wedge\mathcal{N}=\mathcal{B}$. Therefore, the cross product of these two vectors in eq. (9) and (10) is given by:

$$\xi_\upsilon\wedge\xi_\eta=-r\cos\eta\mathcal{N}-r\sin\eta\mathcal{B} \tag{11}$$

and we get:

$$\|\xi_\upsilon(\upsilon,\eta)\wedge\xi_\eta(\upsilon,\eta)\|_1=r \tag{12}$$

Thus, using eqs. (11) and (12), we get the tubular surface's isotropic normal vector as:

$$\mathbf{n}=-\cos\eta\mathcal{N}-\sin\eta\mathcal{B} \tag{13}$$

The surface ξ is second order partial differentials are computed:

$$\begin{aligned}\xi_{\upsilon\upsilon}&=1+[r\cos\eta(\sigma^2+\sigma'-\tau^2)-r\sin\eta(\tau'+2\sigma\tau)]\mathcal{N}+\\&+[r\cos\eta(\tau'+2\sigma\tau)+r\sin\eta(\sigma^2+\sigma'-\tau^2)]\mathcal{B}\end{aligned} \tag{14}$$

$$\xi_{\eta\upsilon}=-r(\sigma\sin\eta+\tau\cos\eta)\mathcal{N}+r(-\tau\sin\eta+\sigma\cos\eta)\mathcal{B} \tag{15}$$

$$\xi_{\eta\eta}=-r\cos\eta\mathcal{N}-r\sin\eta\mathcal{B} \tag{16}$$

Equations (13)-(16) and (8) are used as a guide to obtain the coefficients of the second fundamental form:

$$L_{11}=\kappa^2[-\cos\eta-r(\sigma^2+\sigma'-\tau^2)] \tag{17}$$

$$L_{12}=\kappa^2 r\tau \tag{18}$$

$$L_{22}=\kappa^2 r \tag{19}$$

Furthermore, the Gaussian curvature K is defined:

$$K=\frac{-\kappa^4[\cos\eta+r(\sigma^2+\sigma')]}{r} \tag{20}$$

Consequently, substituting (17)-(20) and (1) into (8), we obtain the tubular surface's mean curvature as:

$$H = \frac{1}{2r} \tag{21}$$

where $g^{11} = 0, \quad g^{12} = 0, \quad g^{22} = -1/2r$.

Corollary 2. Tubular surface are constant mean curvature surfaces in Galilean space $\mathbb{G}_3$.

Theorem 3. Tubular surfaces are Weingarten surfaces in Galilean space $\mathbb{G}_3$.

Proof. Differentiating K and H in according to υ and η gives:

$$K_\upsilon = -\kappa^4(2\sigma\sigma' + \sigma''), K_\eta = \frac{\kappa^4 \sin\eta}{r}$$

and

$$H_\upsilon = H\eta = 0$$

Therefore, the Jacobi equation is obtained by $\Phi(H,K) = K_\upsilon H_\eta - K_\eta H_\upsilon = 0$. Consequently, $\xi(\upsilon, \eta)$ is a Weingarten surface.

Theorem 4. Let $\xi(\upsilon, \eta)$ be a tubular surface by generated $\delta(s)$ with respect to modified orthogonal frame in $\mathbb{G}_3$. Therefore,

i. The η – parameter curves an $\xi(\upsilon, \eta)$ are asymptotic curves if:

$$r = -\frac{\cos\eta}{\sigma^2 + \sigma' - \tau^2}$$

ii. The υ – parameter curves an $\xi(\upsilon, \eta)$ cannot also be asymptotic curves.

Proof. i) It is well known that a curve $\delta(s)$ lying any surface is asymptotic curve if $k_n = \langle \mathbf{n}, \xi_{\eta\eta} \rangle = 0$ where **n** is the isotropic normal vector of tubular surface. Then we can calculate η – parameter curve:

$$\begin{aligned}\langle \mathbf{n}, \xi_{\eta\eta} \rangle &= \langle -\cos\eta\mathcal{N} - \sin\eta\mathcal{B}, \{1 + [r\cos\eta(\sigma^2 + \sigma' - \tau^2) - \\ &-r\sin\eta(\tau' + 2\sigma\tau)]\mathcal{N} + [r\cos\eta(\tau' + 2\sigma\tau) + r\sin\eta(\sigma^2 + \sigma' - \tau^2)]\mathcal{B}\}\rangle = 0\end{aligned}$$

so

$$-\cos\eta - r(\sigma^2 + \sigma' - \tau^2) = 0$$

we have:

$$r = -\frac{\cos\eta}{\sigma^2 + \sigma' - \tau^2}$$

ii) It is easy calculate the following equation:

$$\langle \mathbf{n}, \xi_{\upsilon\upsilon} \rangle = \langle -\cos\eta\mathcal{N} - \sin\eta\mathcal{B}, -r\cos\eta\mathcal{N} - r\sin\eta\mathcal{B} \rangle = r$$

Therefore, η – parameter curves can not be asymptotic curve an $\xi(\upsilon, \eta)$.

Theorem 5. For a tubular surface $\xi(\upsilon, \eta)$ in according to modified orthogonal frame

i. υ – parameter curve $\xi(\upsilon, \eta)$ is always geodesic curve.

ii. η – parameter curve of tubular surface $\xi(\upsilon, \eta)$ is always geodesic if the equation

$$\sin\eta - r(\tau' + 2\sigma\tau) = 0$$

is provided.

Proof. i) It is well known that a space curve (δ) is a geodesic curve of any tubular surface $\xi(\upsilon, \eta)$ surface if the vectors δ'' and $\mathbf{n}$ are linearly dependent such that $\mathbf{n}\times\delta'' = 0$. Then, we get υ – parameter curve:

$$\mathbf{n}\times\xi_{\eta\eta} = (-\cos\eta\mathcal{N} - \sin\eta\mathcal{B})\times(-r\cos\eta\mathcal{N} - r\sin\eta\mathcal{B}) = 0$$

ii) With a similar method, we get:

$$\mathbf{n}\times\xi_{\upsilon\upsilon} = (-\cos\eta\mathcal{N} - \sin\eta\mathcal{B})\times\{1 + [r\cos\eta(\sigma^2 + \sigma' - \tau^2) -$$
$$-r\sin\eta(\tau' + 2\sigma\tau)]\mathcal{N} + [r\cos\eta(\tau' + 2\sigma\tau) + r\sin\eta(\sigma^2 + \sigma' - \tau^2)]\mathcal{B}\} = 0$$

If the necessary calculations are made, we get the following equation.

$$\sin\eta - r(\tau' + 2\sigma\tau) = 0$$

Example 1. $\delta(\upsilon) = (\upsilon, -3/2\cos 2\upsilon, 3/2\sin 2\upsilon)$ be a circular helix in $\mathbb{G}_3$. Modified orthogonal frame apparatus of the curve δ can be calculated as:

$$\mathcal{T} = (1, 3\sin 2\upsilon, 3\cos 2\upsilon), \quad \mathcal{N} = (0, 6\cos 2\upsilon, -6\sin 2\upsilon), \quad \mathcal{B} = (0, 6\sin 2\upsilon, 6\cos 2\upsilon)$$

Therefore, Frenet curvatures of the δ is $\kappa = 6$, $\tau = 2$. The equation for the tubular surface around the curve δ is:

$$\xi(\upsilon,\eta) = \delta(\upsilon) + r(\cos\eta\mathcal{N} + \sin\eta\mathcal{B})$$

Moreover, for $r = \sqrt{2}$, we have tubular surface parametrized by:

$$\xi(\upsilon,\eta) = \Big(\upsilon, -\frac{3}{2}\cos 2\upsilon + 6\sqrt{2}\cos 2\upsilon\cos\eta +$$
$$+6\sqrt{2}\sin 2\upsilon\sin\eta + \frac{3}{2}\sin 2\upsilon -$$
$$-6\sqrt{2}\sin 2\upsilon\cos\eta + 6\sqrt{2}\cos 2\upsilon\sin\eta\Big)$$

Figure 1 shows the graphs of tubular surface and its special curves.

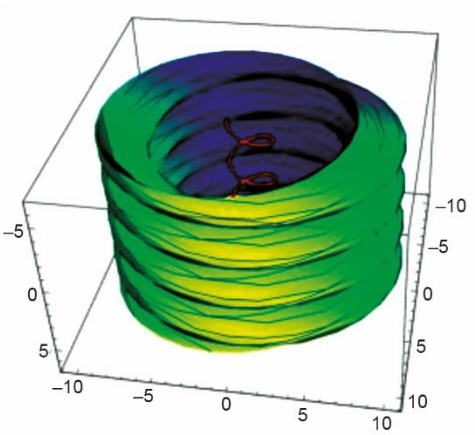

Figure 1. The circular helix δ (red) and its tubular surface

Example 2. $\delta(\zeta) = (\zeta, -\zeta\cos\zeta + 2\sin\zeta, -\zeta\sin\zeta - 2\cos\zeta)$ is a curve in $\mathbb{G}_3$. Modified orthogonal frame apparatus of the curve δ can be calculated as:

$$\mathcal{T} = (1, \zeta\sin\zeta + \cos\zeta, -\zeta\cos\zeta + \sin\zeta),$$
$$\mathcal{N} = (0, \zeta\cos\zeta, \zeta\sin\zeta),$$
$$\mathcal{B} = (0, -\zeta\sin\zeta, \zeta\cos\zeta)$$

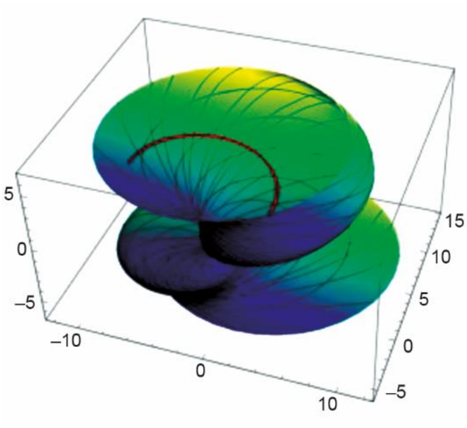

Figure 2. The curve δ (red) and its tubular surface

Then, the curvatures of the δ is $\kappa = \zeta$, $\tau = 1$. Therefore, for r = 2, we have tubular surface of δ parametrized by:

$$\xi(\zeta,\eta) = (\zeta, -\zeta\cos\zeta + 2\sin\zeta - 2\zeta\cos\zeta\cos\eta + \\ +2\zeta\sin\zeta\sin\eta - \zeta\sin\upsilon - 2\cos\zeta + \\ +2\zeta\sin\zeta\cos\eta + 2\zeta\cos\zeta\sin\eta)$$

Figure 2 shows the graphs of tubular surface and its special curves.

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