THE HYPERBOLIC-TYPE *k*-FIBONACCI SEQUENCES AND THEIR APPLICATIONS

by

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In this study, we define hyperbolic-type k-Fibonacci numbers and then give the relationships between the k-step Fibonacci numbers and the hyperbolic-type k-Fibonacci numbers. In addition, we study the hyperbolic-type k-Fibonacci sequence modulo m and then we give periods of the Hperbolic-type k-Fibonacci sequences for any k and m which are related the periods of the k-step Fibonacci sequences modulo m. Furthermore, we extend the hyperbolic-type k-Fibonacci sequences to groups. Finally, we obtain the periods of the hyperbolic-type 2-Fibonacci sequences in the dihedral group D_{2m}, ($m \geq 2$) with respect to the generating pairs (*x*, *y*) *and* (*y*, *x*).

Key words: *hyperbolic number, hyperbolic-type sequence, group, matrix, period*

Introduction

The set of hyperbolic numbers $\mathbb{H}$ can be described in the form:

$$\mathbb{H} = \{z = x + hy \mid h \notin \mathbb{R}, h^2 = 1,\ x, y \in \mathbb{R}\}$$

Addition, substruction and multiplication of two hyperbolic numbers z_1 and z_2 are defined by:

$$z_1 \pm z_2 = (x_1 + hy_1) \pm (x_2 + hy_2) = (x_1 \pm x_2) + h(y_1 \pm y_2)$$

$$z_1 \times z_2 = (x_1 + hy_1)(x_2 + hy_2) = (x_1x_2) + (y_1y_2) + h(x_1y_2 + y_1x_2)$$

On the other hand, the division of two hyperbolic numbers are given by:

$$\frac{z_1}{z_2} = \frac{x_1 + hy_1}{x_2 + hy_2}$$

$$\frac{(x_1 + hy_1)(x_2 - hy_2)}{(x_2 + hy_2)(x_2 - hy_2)} = \frac{x_1x_2 + y_1y_2}{x_2^2 - y_2^2} + h\frac{(x_1y_2 + y_1x_2)}{x_2^2 - y_2^2}$$

If $x_2^2 - y_2^2 \neq 0$, then the division z_1/z_2 is possible. The hyperbolic conjugation of $z = x + hy$ is defined by $\overline{z} = x - hy$.

The *k*-step Fibonacci sequence $\{f_n^{(k)}\}$ is defined by the following recurrence relarion:

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$$f_{n+k}^{(k)} = f_{n+k-1}^{(k)} + f_{n+k-2}^{(k)} + \cdots + f_{n}^{(k)}, \quad n \geq 0, \quad (k = 2,3,\cdots) \tag{1}$$

where $f_0^{(k)} = f_1^{(k)} = \cdots = f_{k-2}^{(k)} = 0$ and $f_{k-1}^{(k)} = 1.$

The hyperbolic Fibonacci sequences $\{Hf_n\}$ is defined by a two-order recurrence equation:

$$Hf_n = f_n + hf_{n+1}$$

For $n \geq 0,$ where $h^2 = 1, h \notin \mathbb{R}$ and f_n is the n^{th} Fibonacci number.

In [1], Kalman derived a number of closed-form formulas for the generalized sequence by the companion matrix method:

$$A_k = \begin{bmatrix} 0 & 1 & 0 & \ldots & 0 & 0 \\ 0 & 0 & 1 & \ldots & 0 & 0 \\ 0 & 0 & 0 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ldots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & 0 & 1 \\ c_0 & c_1 & c_2 & \ldots & c_{k-2} & c_{k-1} \end{bmatrix}$$

He showed that:

$$A_k^n \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_{k-1} \end{bmatrix} = \begin{bmatrix} a_n \\ a_{n+1} \\ \vdots \\ a_{n+k-1} \end{bmatrix}$$

Number theoretic properties such as these obtained from homogeneous linear reccurence relations relevant to this article have been studied recently by many authors, see for example, [2-15]. In section *The hyperbolic-type k-Fibonacci numbers*, we define the hyperbolic-type k-Fibonacci numbers and then give the relationships between the k-step Fibonacci numbers and the hyperbolic-type k-Fibonacci numbers.

A sequence is periodic if, after a certain point, it consists only of repetitions of a fixed subsequence. The number of elements in the shortest repeating subsequence is called the period of the sequence. For example, the sequence $a,b,c,d,b,c,d,b,c,d,\cdots$ is periodic after the initial element a and has period 3. A sequence is simply periodic with period k if the first k elements in the sequence form a repeating subsequence. For example, the sequence $a,b,c,d,a,b,c,d,a,b,c,d,\cdots$ is simply periodic with period 4. Lu and Wang [16] contributed to study of the Wall number for the k-step Fibonacci sequence. Lu and Wang [16] proved that the k-step Fibonacci sequence modulo m is simply periodic.

The notaion $h_k(m)$ denotes the period of the sequence $\{f_n^{(k)}(m)\}$. It is clear that:

$$f_{h_k(m)}^{(k)} \equiv 0(\text{mod}m), f_{h_k(m)+1}^{(k)} \equiv 0(\text{mod}m), \cdots, f_{h_k(m)+k-2}^{(k)} \equiv 0(\text{mod}m)$$

and

$$f_{h_k(m)+k-1}^{(k)} \equiv 1(\text{mod}m)$$

In section *The period of the hyperbolic-type k-Fibonacci sequence modulo m,* we extend the concept to the hyperbolic-type k-Fibonacci sequence and then give the relationships between $h_k(m)$ and the hyperbolic-type *k*-Fibonacci sequence modulo *m*. Furthermore, in this section, we produce the cyclic groups from the multiplicative orders of the generating matrix of the hyperbolic-type *k*-Fibonacci sequences such that the elements of the generating matrix when read modulo *m*. Then we derive the relationships among the orders of the cyclic groups obtained and the periods of the hyperbolic-type *k*-Fibonacci sequences modulo *m*.

The study of recurrence sequences in algebraic structures began with the earlier work of Wall [17] where the ordinary Fibonacci sequences in cyclic groups were investigated. The theory extended to some special linear recurrence sequences by several authors, see for example, [18-30]. In section *The hyperbolic-type k-Fibonacci sequence in groups*, we give the definition of the hyperbolic-type *k*-Fibonacci sequences in groups generated by two or more elements and then we investigate these sequences in the finite groups for any *k*. Finally, we obtain the periods of the hyperbolic-type 2-Fibonacci sequences in the dihedral group $D_{2m}, (m \geq 2)$ as applications of the results produced in the same section.

The hyperbolic-type *k*-Fibonacci numbers

We now define the hyperbolic-type *k*-Fibonacci numbers for any given $k(k = 2,3,\ldots,)$ by the following homogeneous linear recurrence relation:

$$HF_{n+k}^{(h,k)} = hHF_{n+k-1}^{(h,k)} + h^2 HF_{n+k-2}^{(h,k)} + \cdots + h^{k-1} HF_{n+1}^{(h,k)} + h^k HF_n^{(h,k)} \tag{2}$$

for $n \geq 0,$, where $HF_0^{(h,k)} = \cdots = HF_{k-2}^{(h,k)} = 0, HF_{k-1}^{(h,k)} = 1$ and $h^2 = 1$. Since $F_0^{(k)} = HF_0^{(h,k)}$, $F_1^{(k)} = HF_1^{(h,k)}$, $F_{k-2}^{(k)} = HF_{k-2}^{(h,k)}$ and $F_{k-1}^{(k)} = HF_{k-1}^{(h,k)}$ and from eqs. (1) and (2), we derive the following relations:

If *k* is odd:

$$HF_n^{(h,k)} = \begin{cases} hF_n^{(k)}, n \equiv 0(\text{mod}2) \\ F_n^{(k)}, n \equiv 1(\text{mod}2) \end{cases}$$

If *k* is even:

$$HF_n^{(h,k)} = \begin{cases} F_n^{(k)}, n \equiv 0(\text{mod}2) \\ hF_n^{(k)}, n \equiv 1(\text{mod}2) \end{cases}$$

From eq. (2), we have:

$$\begin{bmatrix} HF_{n+k+1}^{(h,k)} \\ HF_{n+k}^{(h,k)} \\ \vdots \\ HF_{n+2}^{(h,k)} \\ HF_{n+1}^{(h,k)} \end{bmatrix} = \begin{bmatrix} h & h^2 & \ldots & h^{k-1} & h^k \\ 1 & 0 & \ldots & 0 & 0 \\ 0 & 1 & \ldots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & 1 & 0 \end{bmatrix} \begin{bmatrix} HF_{n+k}^{(h,k)} \\ HF_{n+k-1}^{(h,k)} \\ \vdots \\ HF_{n+1}^{(h,k)} \\ HF_n^{(h,k)} \end{bmatrix}$$

for the sequence of the hyperbolic-type *k*-Fibonacci numbers. Letting:

$$M_k = \begin{bmatrix} h & h^2 & \cdots & h^{k-1} & h^k \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix}$$

The companion matrix M_k is said to be the hyperbolic-type k-Fibonacci matrix.

By mathematical induction on n, we find:

$$(M_k)^n = \begin{bmatrix} HF_{n+k+1}^{(h,k)} & & h^{k-1}HF_{n+k-2}^{(h,k)} + h^k HF_{n+k-3}^{(h,k)} & h^k HF_{n+k-2}^{(h,k)} \\ HF_{n+k}^{(h,k)} & & \cdots h^{k-1}HF_{n+k-3}^{(h,k)} + h^k HF_{n+k-4}^{(h,k)} & h^k HF_{n+k-3}^{(h,k)} \\ \vdots & C_1\, C_2 \cdots & \vdots & \vdots \\ HF_{n+2}^{(h,k)} & \vdots & h^{k-1}HF_{n}^{(h,k)} + h^k HF_{n-1}^{(h,k)} & h^k HF_{n}^{(h,k)} \\ HF_{n+1}^{(h,k)} & \cdots & h^{k-1}HF_{n-1}^{(h,k)} + h^k HF_{n-2}^{(h,k)} & h^k HF_{n-1}^{(h,k)} \\ & \cdots & & \end{bmatrix} \tag{3}$$

for $n \ge 1$, , where the column matrices C_1 and C_2 are:

$$C_1 = \begin{bmatrix} h^2 HF_{n+k-2}^{(h,k)} + h^3 HF_{n+k-3}^{(h,k)} + \cdots + h^{k-1}HF_{n+1}^{(h,k)} + h^k HF_{n}^{(h,k)} \\ h^2 HF_{n+k-3}^{(h,k)} + h^3 HF_{n+k-4}^{(h,k)} + \cdots + h^{k-1}HF_{n}^{(h,k)} + h^k HF_{n-1}^{(h,k)} \\ \vdots \\ h^2 HF_{n}^{(h,k)} + h^3 HF_{n-1}^{(h,k)} + \cdots + h^{k-1}HF_{n-k+3}^{(h,k)} + h^k HF_{n-k+2}^{(h,k)} \\ h^2 HF_{n-1}^{(h,k)} + h^3 HF_{n-2}^{(h,k)} + \cdots + h^{k-1}HF_{n-k+2}^{(h,k)} + h^k HF_{n-k+1}^{(h,k)} \end{bmatrix}$$

and

$$C_2 = \begin{bmatrix} h^3 HF_{n+k-2}^{(h,k)} + h^4 HF_{n+k-3}^{(h,k)} + \cdots + h^{k-1}HF_{n+2}^{(h,k)} + h^k HF_{n+1}^{(h,k)} \\ h^3 HF_{n+k-3}^{(h,k)} + h^4 HF_{n+k-4}^{(h,k)} + \cdots + h^{k-1}HF_{n+1}^{(h,k)} + h^k HF_{n}^{(h,k)} \\ \vdots \\ h^3 HF_{n}^{(h,k)} + h^4 HF_{n-1}^{(h,k)} + \cdots + h^{k-1}HF_{n-k+4}^{(h,k)} + h^k HF_{n-k+3}^{(h,k)} \\ h^3 HF_{n-1}^{(h,k)} + h^4 HF_{n-2}^{(h,k)} + \cdots + h^{k-1}HF_{n-k+3}^{(h,k)} + h^k HF_{n-k+2}^{(h,k)} \end{bmatrix}$$

Also, we derive the following relationships between the hyperbolic-type k-Fibonacci numbers and the k-step Fibonacci numbers for $n \ge k-1$:

$$(M_k)^n = \begin{bmatrix} h^{k+u}F_{n+k+1}^{(k)} & h^{k+u+1}[F_{n+k-2}^{(k)}+F_{n+k-3}^{(k)}+\cdots+F_n^{(k)}] & h^{k+u+2}[F_{n+k-2}^{(k)}+F_{n+k-3}^{(k)}+\cdots+F_{n+1}^{(k)}] & \cdots & h^{2k+u-2}[F_{n+k-2}^{(k)}+F_{n+k-3}^{(k)}] & h^{2k+u-1}F_{n+k-2}^{(k)} \\ h^{k+u-1}F_{n+k+1}^{(k)} & h^{k+u}[F_{n+k-3}^{(k)}+F_{n+k-4}^{(k)}+\cdots+F_{n-1}^{(k)}] & h^{k+u+1}[F_{n+k-3}^{(k)}+F_{n+k-4}^{(k)}+\cdots+F_n^{(k)}] & \cdots & h^{2k+u-3}[F_{n+k-3}^{(k)}+F_{n+k-4}^{(k)}] & h^{2k+u-2}F_{n+k-3}^{(k)} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ h^{u+2}F_{n+k+1}^{(k)} & h^{u+3}[F_n^{(k)}+F_{n-1}^{(k)}+\cdots+F_{n-k+2}^{(k)}] & h^{u+4}[F_n^{(k)}+F_{n-1}^{(k)}+\cdots+F_{n-k+3}^{(k)}] & \cdots & h^{k+u}[F_n^{(k)}+F_{n-1}^{(k)}] & h^{k+u+1}F_n^{(k)} \\ h^{u+1}F_{n+k+1}^{(k)} & h^{u+2}[F_{n-1}^{(k)}+F_{n-2}^{(k)}+\cdots+F_{n-k+1}^{(k)}] & h^{u+3}[F_{n-1}^{(k)}+F_{n-2}^{(k)}+\cdots+F_{n-k+2}^{(k)}] & \cdots & h^{k+u-1}[F_{n-1}^{(k)}+F_{n-2}^{(k)}] & h^{k+u}F_{n-1}^{(k)} \end{bmatrix} \quad (4)$$

where

$$u = \begin{cases} 0, k-n \equiv 0(\text{mod}2) \\ 1, k-n \equiv 1(\text{mod}2) \end{cases}$$

The period of the hyperbolic-type *k*-Fibonacci sequence modulo *m*

If we reduce the hyperbolic-type *k*-Fibonacci sequence $\{HF_n^{(h,k)}\}$ modulo *m*, taking least non-negative residues, then we can get the repeating sequence, denoted by:

$$\{HF_n^{(h,k)}(m)\} = \{HF_0^{(h,k)}(m), HF_1^{(h,k)}(m), \cdots, HF_i^{(h,k)}(m), \cdots\}$$

where $HF_i^{(h,k)}(m)$ is used to mean n^{th} hyperbolic-type *k*-Fibonacci number modulo *m*. We note here that the sequence $\{HF_n^{(h,k)}(m)\}$ has the same recurrence as in the $\{HF_n^{(h,k)}\}$.

Theorem 1. $\{HF_n^{(h,k)}(m)\}$ forms a simply periodic sequence for any $k \geq 2$.

Proof. Let us consider the set:

$$X = \{(x_1, x_2, \cdots, x_k) \mid 0 \leq x_v \leq m-1\}$$

It is clear that *X* is a finite set. Let the notation $|X|$ denote the cardinality of the set *X*, then there are $|X|$ distinct *k*-tuples of the hyperbolic numbers modulo *m*. Then, at least one of the *k*-tuples appears twice in the sequence $\{HF_n^{(h,k)}(m)\}$. Thus, the subsequence following this *k*-tuple repeats; therefore, the sequence $\{HF_n^{(h,k)}(m)\}$ is periodic. Since periodicity, for any $s \geq 0$, there exist $r \geq s+k$ such that:

$$HF_s^{(h,k)}(m) \equiv HF_r^{(h,k)}(m), \quad HF_{s+1}^{(h,k)}(m) \equiv HF_{r+1}^{(h,k)}(m), \cdots, HF_{s+k}^{(h,k)}(m) \equiv HF_{r+k}^{(h,k)}(m)$$

From the recurrence relation of the hyperbolic-type *k*-Fibonacci sequence $HF_n^{(h,k)}$, we can easily derive that:

$$HF_{s-1}^{(h,k)}(m) \equiv HF_{r-1}^{(h,k)}(m), \quad HF_{s-2}^{(h,k)}(m) \equiv HF_{r-2}^{(h,k)}(m), \cdots, HF_0^{(h,k)}(m) \equiv HF_{r-s}^{(h,k)}(m)$$

which implies that the sequence $\{HF_n^{(h,k)}(m)\}$ is simply periodic.

We denote the period of the sequence $\{HF_n^{(h,k)}(m)\}$ by $p_k^h(m)$.

The hyperbolic-type *k*-Fibonacci sequence in groups

Let G be a finite k-generator group and let:

$$X = \left\{ (x_1, x_2, \cdots, x_k) \in \underbrace{G \times G \times \cdots G}_{k} \,|\, \{x_1, x_2, \cdots, x_k\} = G \right\}$$

We call $(x_1, x_2, \cdots, x_k)$ a generating k-tuple for G.

Definition 1. Let G be a k-generator group. For generating k-tuple $(x_1, x_2, \cdots, x_k)$, we define the hyperbolic-type k-Fibonacci orbit by:

$$a_0 = x_1, a_1 = x_2, \cdots, a_{k-1} = x_k, a_{n+k} = (a_n)^{h^k} (a_{n+1})^{h^{k-1}} \cdots (a_{n+k-2})^{h^2} (a_{n+k-1})^h \; n \geq 0$$

The following conditions hold for every $x, y \in G$.

i. if $xy \neq yx$, then $x^h y^h \neq y^h x^h$.
ii. $(xy)^h = y^h x^h$,
iii. $xy^h = y^h x$ and $x^h y = yx^h$.

For generating k-tuple $(x_1, x_2, \cdots, x_k)$, we define the hyperbolic-type k-Fibonacci orbit is denoted by $HF^h_{(x_1, x_2, \cdots, x_k)}(G)$.

Theorem 2. Let G be a k-generator group. If G is finite, then the hyperbolic-type k-Fibonacci orbit of G is periodic.

Proof. Consider the set:

$$X = \Big\{ [(x_1)^{a_1(\text{mod}|x_1|) + hb_1(\text{mod}|x_1|)},$$

$$(x_2)^{a_2(\text{mod}|x_2|) + hb_2(\text{mod}|x_2|)}, \cdots,$$

$$(x_k)^{a_k(\text{mod}|x_k|) + hb_k(\text{mod}|x_k|)}] \,|$$

$$x_1, x_2, \cdots, x_k \in G \text{ and } a_n, b_n \in \mathbb{Z} \text{ such that} 1 \leq n \leq k \Big\}$$

Since G is finite, the X is a finite set. Then for any $t \geq 0$, there exists $v \geq t + k$ such that $a_{t+1} = a_{v+1}, a_{t+2} = a_{v+2}, \cdots, a_{t+k} = a_{v+k}$. Because of the repeating for all generating k-tuples, the sequence $HF^h_{(x_1, x_2, \cdots, x_k)}(G)$ is periodic.

We denote the length of the period of the orbit $HF^h_{(x_1, x_2, \cdots, x_k)}(G)$ by $L^h_{(x_1, x_2, \cdots, x_k)}(G)$. From the definition of the orbit $HF^h_{(x_1, x_2, \cdots, x_k)}(G)$ it is clear that the length of the period of this sequence in a finite group depend on the chosen generating set and the order in which the assignments of $x_1, x_2, \cdots, x_k$ are made.

It is well-known that the dihedral group D_{2m} of order $2m$ is defined by the presentation:

$$D_{2m} = \langle x, y \,|\, x^m = y^2 = (v)^2 = e \rangle$$

We shall now address the lengths of the periods of the orbits $HF^h_{(x,y)}(D_{2m})$ and $HF^h_{(y,x)}(D_{2m})$.

Theorem 3. For:

$$m \geq 2, L^h_{(x,y)}(D_{2m}) = L^h_{(y,x)}(D_{2m}) = lcm[6, p^h_2(m)]$$

Proof. Let us consider the hyperbolic-type 2-Fibonacci orbit of the dihedral group D_{2m} for generating pair (x, y).

$$a_0 = x, a_1 = y, a_2 = xy^h, a_3 = x^h, a_4 = x^2 y^h, a_5 = x^{3h} y, a_6 = x^5, \cdots$$

$$a_{s.r} = x^{u_{s.r}} y^{v_{s.r}}, a_{s.r+1} = x^{u_{s.r+1}} y^{v_{s.r+1}}, a_{s.r+2} = x^{u_{s.r+2}} y^{v_{s.r+2}}, \cdots$$

Since $|x| = m, |y| = 2$ and $p^h_2(2) = 6$:

$$a_{6.r} = x^{u_{6.r}}, a_{6.r+1} = x^{u_{6.r+1}} y, a_{6.r+2} = x^{u_{6.r+2}} y^h, \cdots$$

and

$$a_{p^h_2(m).r} = xy^{v_{p^h_2(m).r}}, a_{p^h_2(m).r+1} = y^{v_{p^h_2(m).r+1}}, a_{p^h_2(m).r+2} = x^h y^{v_{p^h_2(m).r+2}}, \cdots$$

where $u_0 = 1, u_1 = 0, u_{n+2} = hu_{n+1} + u_n$ and $v_0 = 0, v_1 = 1, v_{n+2} = hv_{n+1} + v_n, n \geq 0.$
Then we obtain:

$$a_{lcm[6, p^h_2(m)]} = x, a_{lcm[6, p^h_2(m)]+1} = y, a_{lcm[6, p^h_2(m)]+2} = x^h y, \cdots.$$

Thus it is verified that the length of the period of the sequence $HF^h_{(x,y)}(D_{2m})$ is $lcm[6, p^h_2(m)]$. There is a similar proof for length of the period of the sequence $HF^h_{(y,x)}(D_{2m})$.

Example 1. The sequence $L^h_{(x,y)}(D_6)$ is:

$$x, y, xy^h, x^h, x^2 y^h, y, x^2, x^{2h} y, xy^h, e, xy^h, x^h y,$$

$$x^2, y, x^2 y^h, x^{2h}, xy^h, y, x, x^h y, x^2 y^h, e, x^2 y^h, x^{2h} y,$$

$$x, y, \ldots,$$

which implies that $L^h_{(x,y)}(D_6) = 24.$ It is easy to see that $lcm[6, p^h_2(3)] = lcm[6,8] = 24.$

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