ON APPROXIMATION OF HELIX BY 3rd, 5th AND 7th ORDER BEZIER CURVES IN E^3

by

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Approximation of helices has been studied by using in many ways. In this study, it has been examined how a circular helix can be written as Bezier curve and written the 3th degree, 5th degree, and the 7th degree Maclaurin series expansions of helices for the polynomial forms. Hence, they can be written cubic, 5th order, and 7th order Bezier curves, based on the control points with matrix form we have already given in E^3. Further we have given the control points of the Bezier curve based on the coefficients of the Maclaurin series expansion of the circular helix.

Key words: *Helix curve, Maclaurin series, 7th order Bezier curve*

Introduction and preliminaries

A Bezier curve is frequently used in computer graphics and related fields, in vector graphics, used in animation as a tool to control motion, see in [1, 2]. In animation applications such as Adobe Flash and Synfig, Bezier curves are used to outline for example movement. Users outline the wanted path in Bezier curves, and the application creates the needed frames for the object to move along the path. For 3-D animation Bezier curves are often used to define 3-D paths as well as 2-D curves for keyframe interpolation. In [3], a dual unit spherical Bezier-like curve corresponds to a ruled surface by using Study's transference principle and closed ruled surfaces are determined via control points and also, integral invariants of these surfaces are investigated. Researchers have written many publications on Bezier curves, but some of these studies inspired this article. For example: in [4], Bezier curves with curvature and torsion continuity has been examined. In [4, 5], Bezier curves and surfaces has been given. In [6], Bezier curves are designed for computer-aided geometric. Recently equivalence conditions of control points and application to planar Bezier curves have been examined. In [7], Frenet apparatus of the cubic Bezier curves has been examined in $\mathbf{E}^3$. In [8], a cubic trigonometric Bezier-like curve similar to the cubic Bezier curve, with a shape parameter, is presented. In here, first 5th order Bezier curve and its first, second and third derivatives have been examined based on the control points of 5th order Bezier Curve in $\mathbf{E}^3$. We have already examine in cubic Bezier curves and involutes in [7, 9]. The Bertrand and the Mannheim mate of a cubic Bezier curve by using matrix representation have been researhed in $\mathbf{E}^3$ [10, 11], respectively. In [12], it has been examined the 5th order Bezier curve and its derivatives. In [13], it has been researched the answer of the question: how to find a nth order Bezier curve if we know the first, second and third derivatives? Also in [14] it has been given the way how we

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can determine the wanted 5th order Bezier curve, if we know its the first, the second, and the third derivatives, which it has the wanted control points. And finally, in [15, 16], approximation of circular arcs and helices have been studied.

Generaly n^{th} order Beziers curve can be defined by $n+1$ control points $P_0, P_1, \ldots, P_n$ with the parametrization:

$$\mathbf{B}(t) = \sum_{i=0}^{n} \binom{n}{i} t^i (1-t)^{n-i} \left[P_i\right]$$

For more detail see [17, 18]. Approximation of helices has been studied by using in many ways.

In this study, it has been examined how a circular helix can be written as Bezier curve by using Maclaurin series. It is well known that Taylor series:

$$f(x) = \sum_{n=0}^{\infty} f^{(n)}(a) \frac{(x-a)^n}{n!}$$

of a function is an infinite sum of the functions derivatives at a single point a. Also, a Maclaurin series

$$f(x) = \sum_{n=0}^{\infty} f^{(n)}(0) \frac{x^n}{n!}$$

is a taylor series where $a = 0$.

Helix as a cubic Bezier curve

In this section, it has been examined a more complex helix $[a\cos(bt-c), a\sin(bt-c), dt]$ as a cubic Bezier curve.

Theorem 1. The matrix representation of the helix curve of $[a\cos(bt-c), a\sin(bt-c), dt]$ as a cubic Bezier curve is:

$$[a\cos(bt-c), a\sin(bt-c), dt] = [t^3 \quad t^2 \quad t \quad 1][B^3][P_0 \quad P_1 \quad P_2 \quad P_3]^T$$

with the control points:

$$\begin{bmatrix} P_0 \\ P_1 \\ P_2 \\ P_3 \end{bmatrix} = \begin{bmatrix} a\cos c & -a\sin c & 0 \\ a\cos c + \frac{1}{3}ab\sin c & \frac{1}{3}ab\cos c - a\sin c & \frac{d}{3} \\ -\frac{ab^2\cos c}{6} + \frac{2ab\sin c}{3} + a\cos c & \frac{1}{6}ab^2\sin c + \frac{2ab\cos c}{3} - a\sin c & \frac{2d}{3} \\ \frac{-ab^3\sin c}{6} - \frac{ab^2\cos c}{2} + ab\sin c + a\cos c & \frac{-ab^3\cos c}{6} + \frac{ab^2\sin c}{2} + ab\cos c - a\sin c & d \end{bmatrix}$$

Proof. Let's examine the $[a\cos(bt-c), a\sin(bt-c), dt]$ as a cubic Bezier curve. First we need to write $[a\cos(bt-c), a\sin(bt-c), dt]$ in Maclaurin series expansion. Also 3rd degree Maclaurin series expansion of helix can be written as in parametric form and a 3rd degree polynomial function as in the following way:

$$\alpha(t)=[a\cos(bt-c),a\sin(bt-c),dt]=\begin{pmatrix} -\dfrac{ab^3\sin c}{3!}t^3-\dfrac{ab^2\cos c}{2!}t^2+(ab\sin c)t+a\cos c, \\ \dfrac{-ab^3\cos c}{3!}t^3+\dfrac{ab^2\sin c}{2!}t^2+abt(\cos c)-a\sin c, \\ dt \end{pmatrix}$$

Also this can be written in matrix form with the matrix representation of cubic Bezier curve as in:

$$\alpha(t)=[a\cos(bt-c),a\sin(bt-c),dt]=\begin{bmatrix} t^3 \\ t^2 \\ t \\ 1 \end{bmatrix}^T \begin{bmatrix} \dfrac{-ab^3\sin c}{3!} & \dfrac{-ab^3\cos c}{3!} & 0 \\ \dfrac{-ab^2\cos c}{2!} & \dfrac{ab^2\sin c}{2!} & 0 \\ ab\sin c & ab\cos c & d \\ a\cos c & -a\sin c & 0 \end{bmatrix}=\begin{bmatrix} t^3 \\ t^2 \\ t \\ 1 \end{bmatrix}^T \begin{bmatrix} B^3 \end{bmatrix}\begin{bmatrix} P_0 \\ P_1 \\ P_2 \\ P_3 \end{bmatrix}$$

Solving the previous equation we get the control points $P_0, P_1, P_2,$ and P_3 as in the result of the matrix product and so, we get the proof:

$$[P_0 \quad P_1 \quad P_2 \quad P_3]^T=[B^3]^{-1}\begin{bmatrix} \dfrac{-ab^3\sin c}{3!} & \dfrac{-ab^2\cos c}{2!} & ab\sin c & a\cos c \\ \dfrac{-ab^3\cos c}{3!} & \dfrac{ab^2\sin c}{2!} & ab\cos c & -a\sin c \\ 0 & 0 & d & 0 \end{bmatrix}^T$$

Corollary 2. The co-ordinates are:

$$\begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{bmatrix}=[B^3]^{-1}\begin{bmatrix} \dfrac{-ab^3\sin c}{3!} \\ \dfrac{-ab^2\cos c}{2!} \\ ab\sin c \\ a\cos c \end{bmatrix}, \quad \begin{bmatrix} y_0 \\ y_1 \\ y_2 \\ y_3 \end{bmatrix}=[B^3]^{-1}\begin{bmatrix} \dfrac{-ab^3\cos c}{3!} \\ \dfrac{ab^2\sin c}{2!} \\ ab\cos c \\ -a\sin c \end{bmatrix}, \quad \begin{bmatrix} z_0 \\ z_1 \\ z_2 \\ z_3 \end{bmatrix}=[B^3]^{-1}\begin{bmatrix} 0 \\ 0 \\ d \\ 0 \end{bmatrix}$$

Corollary 2.2. There is the following relations among the coefficients a and b and apscissas, ordinates and applicates of the control points:

$$a=\frac{1}{b^3\sin c}\left(6x_0-18x_1+18x_2-6x_3\right)$$

If

$$6x_0\cos c - 18x_1\cos c + 18x_2\cos c - 6x_3\cos c - b^3x_0\sin c = 0$$
$$-3x_0\cos c + 9x_1\cos c - 9x_2\cos c + 3x_3\cos c - 3bx_0\sin c + 6bx_1\sin c - 3bx_2\sin c = 0$$
$$6x_0 - 18x_1 + 18x_2 - 6x_3 + 3b^2x_0 - 3b^2x_1 = 0,\ b^3(\sin c) \neq 0$$

$$a = \frac{1}{b^3\cos c}(6y_0 - 18y_1 + 18y_2 - 6y_3)$$

If:

$$-6y_0\sin c + 18y_1\sin c - 18y_2\sin c + 6y_3\sin c - b^3y_0\cos c = 0$$
$$3y_0\sin c - 9y_1\sin c + 9y_2\sin c - 3y_3\sin c - 3by_0\cos c + 6by_1\cos c - 3by_2\cos c = 0$$
$$6y_0 - 18y_1 + 18y_2 - 6y_3 + 3b^2y_0 - 3b^2y_1 = 0, b^3\left(\cos c\right) \neq 0 \text{ and } d = 3z_1$$

if $z_0 = 0,\ -3z_1 + 3z_2 - z_3 = 0$, $6z_1 - 3z_2 = 0.$

Helix as a 5th order Bezier curve

In this section it will be examined the helix $[a\cos(bt-c), a\sin(bt-c), dt]$ as the 5th order Bezier curve. We have already known that the matrix representation of the polynomial curve $\alpha(t) = (t, a_5t^5 + a_4t^4 + a_3t^3 + a_2t^2 + at_1 + a_0)$ as 5th order Bezier curve in $\mathbf{E}^2$ is:

$$\alpha(t) = [t^5 \quad t^4 \quad t^3 \quad t^2 \quad t \quad 1][B^5][P_0 \quad P_1 \quad P_2 \quad P_3 \quad P_4 \quad P_5]^T$$

where the coefficient matrix and the inverse of the 5^{th} order Bezier curve are:

$$[B^5] = \begin{bmatrix} -1 & 5 & -10 & 10 & -5 & 1 \\ 5 & -20 & 30 & -20 & 5 & 0 \\ -10 & 30 & -30 & 10 & 0 & 0 \\ 10 & -20 & 10 & 0 & 0 & 0 \\ -5 & 5 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad [B^5]^{-1} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & \frac{1}{5} & 1 \\ 0 & 0 & 0 & \frac{1}{10} & \frac{2}{5} & 1 \\ 0 & 0 & \frac{1}{10} & \frac{3}{10} & \frac{3}{5} & 1 \\ 0 & \frac{1}{5} & \frac{2}{5} & \frac{3}{5} & \frac{4}{5} & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

Theorem 2. The matrix representation of the helix curve of $[a\cos(bt-c), a\sin(bt-c), dt]$ as a 5th order Bezier curve is:

$$\alpha(t) = [a\cos(bt-c), a\sin(bt-c), dt] =$$
$$= [t^5 \quad t^4 \quad t^3 \quad t^2 \quad t \quad 1][B^5][P_0 \quad P_1 \quad P_2 \quad P_3 \quad P_4 \quad P_5]^T$$

where the control points P_0, P_1, P_2, P_3, P_4, and P_5 are:

$$\begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} a\cos c \\ a\cos c + \frac{1}{5}ab\sin c \\ -\frac{ab^2\cos c}{20} + \frac{2ab\sin c}{5} + a\cos c \\ -\frac{ab^3\sin c}{60} - \frac{3ab^2\cos c}{20} + \frac{3ab\sin c}{5} + a\cos c \\ \frac{ab^4\cos c}{120} - \frac{ab^3\sin c}{15} - \frac{3ab^2\cos c}{10} + \frac{4ab\sin c}{5} + a\cos c \\ -\frac{ab^5\sin c}{120} + \frac{ab^4\cos c}{24} - \frac{ab^3\sin c}{6} - \frac{ab^2\cos c}{2} + ab\sin c + a\cos c \end{bmatrix}$$

$$\begin{bmatrix} y_0 \\ y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \end{bmatrix} = \begin{bmatrix} -a\sin c \\ \frac{1}{5}ab\cos c - a\sin c \\ \frac{ab^2\sin c}{20} + \frac{2ab\cos c}{5} - a\sin c \\ -\frac{ab^3\cos c}{60} + \frac{3ab^2\sin c}{20} + \frac{3ab\cos c}{5} - a\sin c \\ -\frac{ab^4\sin c}{120} - \frac{ab^3\cos c}{15} + \frac{3ab^2\sin c}{10} + \frac{4ab\cos c}{5} - a\sin c \\ \frac{ab^5\cos c}{120} - \frac{ab^4\sin c}{24} - \frac{ab^3\cos c}{6} + \frac{ab^2\sin c}{2} + ab\cos c - a\sin c \end{bmatrix}$$

and

$$[z_0 \quad z_1 \quad z_2 \quad z_3 \quad z_4 \quad z_5]^T = \begin{bmatrix} 0 & \frac{d}{5} & \frac{2d}{5} & \frac{3d}{5} & \frac{4d}{5} & d \end{bmatrix}^T$$

Proof. Lets examine the $(a\cos(bt-c), a\sin(bt-c), dt)$ as a 5th order Bezier curve. First we need to write $[a\cos(bt-c), a\sin(bt-c), dt]$ in Maclaurin series expansion. For helix 5th degree Maclaurin series expansion can be written as in parametric form and a 5th degree polynomial function:

$$\alpha(t) = [a\cos(bt-c), a\sin(bt-c), dt] =$$

$$= \begin{pmatrix} \frac{-ab^5\sin(c)t^5}{5!} + \frac{ab^4\cos c}{4!}t^4 - \frac{ab^3\sin c}{3!}t^3 - \frac{ab^2\cos c}{2!}t^2 + (ab\sin c)t + a\cos c \\ \frac{ab^5\cos c}{5!}t^5 + \frac{-ab^4\sin c}{4!}t^4 + \frac{-ab^3\cos c}{3!}t^3 + \frac{ab^2\sin c}{2!}t^2 + abt(\cos c) - a\sin c \\ dt \end{pmatrix}$$

Also this can be written in matrix form with the matrix representation of 5th order Bezier curve as in:

$$\alpha(t) = [a\cos(bt-c), a\sin(bt-c), dt]$$

$$= [t^5 \quad t^4 \quad t^3 \quad t^2 \quad t \quad 1]\begin{bmatrix} \frac{-ab^5\sin c}{5!} & \frac{ab^4\cos c}{4!} & \frac{-ab^3\sin c}{3!} & \frac{-ab^2\cos c}{3!} & ab\sin c & a\cos c \\ \frac{ab^5\cos c}{5!} & \frac{-ab^4\sin c}{4!} & \frac{-ab^3\cos c}{3!} & \frac{ab^2\sin c}{2!} & ab\cos c & -a\sin c \\ 0 & 0 & 0 & 0 & d & 0 \end{bmatrix}$$

$$= [t^5 \quad t^4 \quad t^3 \quad t^2 \quad t \quad 1][B^5][P_0 \quad P_1 \quad P_2 \quad P_3 \quad P_4 \quad P_5]^T$$

Solving the equation we get the control points $P_0, P_1, P_2, P_3, P_4,$ and P_5 as in the result of the matrix product

$$[P_0 \quad P_1 \quad P_2 \quad P_3 \quad P_4 \quad P_5]^T = [B^5]^{-1}\begin{bmatrix} \frac{-ab^5\sin c}{\frac{ab^5\cos c}{4b^4\cos c}} & \frac{-ab^3\sin c}{4!} & \frac{-ab^2\cos c}{3!}c & ab\sin c & a\cos c \\ \frac{5!}{4!} & \frac{-ab^3\cos c}{3!} & \frac{ab^2\sin c}{2!} & ab\cos c & -a\sin c \\ 0 & 0 & 0 & d & 0 \end{bmatrix}^T$$

The result give us the proof.

$$\begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = [B^5]^{-1}\begin{bmatrix} \frac{-ab^5\sin c}{5!} \\ \frac{ab^4\cos c}{4!} \\ \frac{-ab^3\sin c}{3!} \\ \frac{-ab^2\cos c}{2!} \\ ab\sin c \\ a\cos c \end{bmatrix}, \quad \begin{bmatrix} y_0 \\ y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \end{bmatrix} = [B^5]^{-1}\begin{bmatrix} \frac{ab^5\cos c}{5!} \\ \frac{-ab^4\sin c}{4!} \\ \frac{-ab^3\cos c}{3!} \\ \frac{ab^2\sin c}{2!} \\ ab\cos c \\ -a\sin c \end{bmatrix}, \quad \begin{bmatrix} z_0 \\ z_1 \\ z_2 \\ z_3 \\ z_4 \\ z_5 \end{bmatrix} = [B^5]^{-1}\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ d \\ 0 \end{bmatrix}$$

Helix as a 7th order Bezier curve

In this section we it will be focused on how we can write a general helix curve as a 7th order Bezier curve.

Theorem 3. The coefficients matrix and the inverse matrix of any 7th order Bezier curve are:

$$[B^7]=\begin{bmatrix} -1 & 7 & -21 & 35 & -35 & 21 & -7 & 1 \\ 7 & -42 & 105 & -140 & 105 & -42 & 7 & 0 \\ -21 & 105 & -210 & 210 & -105 & 21 & 0 & 0 \\ 35 & -140 & 210 & -140 & 35 & 0 & 0 & 0 \\ -35 & 105 & -105 & 35 & 0 & 0 & 0 & 0 \\ 21 & -42 & 21 & 0 & 0 & 0 & 0 & 0 \\ -7 & 7 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

and

$$[B^7]^{-1}=\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{7} & 1 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{21} & \frac{2}{7} & 1 \\ 0 & 0 & 0 & 0 & \frac{1}{35} & \frac{1}{7} & \frac{3}{7} & 1 \\ 0 & 0 & 0 & \frac{1}{35} & \frac{4}{35} & \frac{2}{7} & \frac{4}{7} & 1 \\ 0 & 0 & \frac{1}{21} & \frac{1}{7} & \frac{2}{7} & \frac{10}{21} & \frac{5}{7} & 1 \\ 0 & \frac{1}{7} & \frac{2}{7} & \frac{3}{7} & \frac{4}{7} & \frac{5}{7} & \frac{6}{7} & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

Theorem 4. The matrix representation of the helix curve of $\alpha(t)=[a\cos(bt-c), a\sin(bt-c), dt]$ as a 7th order Bezier curve is:

$$\alpha(t)=[a\cos(bt-c), a\sin(bt-c), dt]=$$
$$=[t^7 \quad t^6 \quad t^5 \quad t^4 \quad t^3 \quad t^2 \quad t \quad 1][B^7][P_0 \quad P_1 \quad P_2 \quad P_3 \quad P_4 \quad P_5 \quad P_6 \quad P_7]^T$$

where the control points $P_0, P_1, P_2, P_3, P_4, P_5, P_6,$ and P_7 have the following co-ordinates:

$$\begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \end{bmatrix} = \begin{bmatrix} a\cos c \\ a\cos c + \frac{1}{7}ab\sin c \\ \frac{-ab^2\cos c}{42} + \frac{2ab\sin c}{7} + a\cos c \\ \frac{-ab^3\sin c}{210} - \frac{ab^2\cos c}{14} + \frac{3ab\sin c}{7} + a\cos c \\ \frac{ab^4\cos c}{840} - \frac{2ab^3\sin c}{14} - \frac{ab^2\cos c}{7} + \frac{4ab\sin c}{7} + a\cos c \\ \frac{-ab^6\cos c}{5040} - \frac{ab^5\sin c}{420} + \frac{ab^4\cos c}{56} - \frac{2ab^3\sin c}{21} - \frac{5ab^2\cos c}{7} + \frac{6ab\sin c}{7} + a\cos c \\ \frac{-ab^6\cos c}{720} - \frac{ab^3\sin c}{120} + \frac{ab^4\cos c}{24} - \frac{ab^3\sin c}{6} - \frac{ab^2\cos c}{2} + ab\sin c + a\cos c \end{bmatrix}$$

$$\begin{bmatrix} y_0 \\ y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \\ y_6 \\ y_7 \end{bmatrix} = \begin{bmatrix} -a\sin c \\ \frac{ab\cos c}{7} - a\sin c \\ \frac{ab^2\sin c}{42} + \frac{2ab\cos c}{7} - a\sin c \\ \frac{-ab^2\cos c}{210} + \frac{1ab^2\sin c}{7} + \frac{3ab\cos c}{7} - a\sin c \\ \frac{-ab^4\sin c}{84ab^3\cos c} + \frac{ab^2\sin c}{7} + \frac{4ab\cos c}{7} - a\sin c \\ \frac{-ab^7\cos c}{5040} + \frac{ab^6\sin c}{720} + \frac{ab^5\cos c}{120} - \frac{ab^4\sin c}{24} - \frac{ab^3\cos c}{6} + \frac{ab^2\sin c}{2} + ab\cos c - a\sin c \end{bmatrix}$$

and

$$[z_0 \quad z_1 \quad z_2 \quad z_3 \quad z_4 \quad z_5 \quad z_6 \quad z_7] = [0 \quad \frac{d}{7} \quad \frac{2d}{7} \quad \frac{3d}{7} \quad \frac{4d}{7} \quad \frac{5d}{7} \quad \frac{6d}{7} \quad d]$$

Proof. First we need to write $[a\cos(bt-c), a\sin(bt-c), dt]$ in Maclaurin series expansion. For sine function 7th degree Maclaurin series expansion can be written as in parametric form and a 7th degree polynomial function:

$$\alpha(t) = [a\cos(bt-c), a\sin(bt-c), dt]$$

$$= \begin{pmatrix} \frac{ab^7t^7\sin c}{7!} + \frac{-ab^6t^6\cos c}{7!} - \frac{ab^5t^5\sin c}{6b^6} + \frac{ab^4\cos c}{4!}t^4 - \frac{ab^3\sin c}{3!}t^3 - \frac{ab^2\cos c}{2!}t^2 + abt\sin c + a\cos c \\ \frac{-ab^7t^7\cos c}{7!} + \frac{ab^6t^6\sin c}{6!} + \frac{ab^5t^5t^5\cos c}{5!} + \frac{-ab^4t^4\sin c}{4!} - \frac{ab^3t^3\cos c}{3!} + \frac{ab^2t^2\sin c}{2!} + abt\cos c - a\sin c \end{pmatrix}$$

Also this can be written in matrix form with the matrix representation of 7th order Bezier curve as in:

$$\alpha(t) = [a\cos(bt-c), a\sin(bt-c), dt]$$

$$= [t^7\, t^6\, t^5\, t^4\, t^3\, t^2\, t 1][B^7][P_0\, P_1\, P_2\, P_3\, P_4\, P_5\, P_6\, P_7]^T$$

$$= \begin{bmatrix} t^7 \\ t^6 \\ t^5 \\ t^4 \\ t^3 \\ t^2 \\ t \\ 1 \end{bmatrix}^T \begin{bmatrix} \frac{ab^7\sin(c)}{7!} & -\frac{ab^7\cos c}{7!} & 0 \\ \frac{-ab^6\cos c}{6!} & \frac{ab^6\sin c}{\sin c} & 0 \\ \frac{-ab^5\sin c}{ab^{6!}} & \frac{ab^5\cos c}{5!} & 0 \\ \frac{ab^4\cos c}{4!} & \frac{-ab^5\sin c}{4!} & 0 \\ \frac{-ab^3\sin c}{3!} & \frac{-ab^3\cos c}{3!} & 0 \\ \frac{-ab^2\cos c}{2!} & \frac{ab^2\sin c}{2!} & 0 \\ ab\sin c & ab\cos c & d \\ a\cos c & -a\sin c & 0 \end{bmatrix}$$

Solving the previous equation we get the control points $P_0, P_1, P_2, P_3, P_4, P_5, P_6,$ and P_7, as in the result of the matrix product:

$$\begin{bmatrix} P_0 \\ P_1 \\ P_2 \\ P_3 \\ P_4 \\ P_5 \\ P_6 \\ P_7 \end{bmatrix} = [B^7]^{-1} \begin{bmatrix} \frac{ab^7\sin(c)}{7!} & -\frac{ab^7\cos c}{7!} & 0 \\ \frac{-ab^6\cos c}{6!} & \frac{ab^6\sin c}{6!} & 0 \\ \frac{-ab^5\sin c}{\frac{65}{5!}} & \frac{ab^5\cos c}{5!} & 0 \\ \frac{ab^4\cos c}{4!} & \frac{-ab^4\sin c}{4!} & 0 \\ \frac{-ab^3\sin c}{3!} & \frac{-ab^3\cos c}{3!} & 0 \\ \frac{-ab^2\cos c}{2!} & \frac{ab^2\sin c}{2!} & 0 \\ ab\sin c & ab\cos c & d \\ a\cos c & -a\sin c & 0 \end{bmatrix}$$

Hence it is easy to say that:

$$\begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \end{bmatrix} = [B^7]^{-1} \begin{bmatrix} \dfrac{ab^7 \sin c}{7!} \\ \dfrac{-ab^6 \cos c}{6!} \\ \dfrac{-ab^5 \sin c}{5!} \\ \dfrac{ab^4 \cos c}{4!} \\ \dfrac{-ab^3 \sin c}{3!} \\ \dfrac{-ab^2 \cos c}{2!} \\ ab \sin c \\ a \cos c \end{bmatrix}, \quad \begin{bmatrix} y_0 \\ y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \\ y_6 \\ y_7 \end{bmatrix} = [B^7]^{-1} \begin{bmatrix} -\dfrac{ab^7 \cos c}{7!} \\ \dfrac{ab^6 \sin c}{6!} \\ \dfrac{ab^5 \cos c}{5!} \\ \dfrac{-ab^4 \sin c}{4!} \\ \dfrac{-ab^3 \cos c}{3!} \\ \dfrac{ab^2 \sin c}{2!} \\ ab \cos c \\ -a \sin c \end{bmatrix}$$

and $[z_0 \quad z_1 \quad z_2 \quad z_3 \quad z_4 \quad z_5 \quad z_6 \quad z_7]^T = [B^7]^{-1}[0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad d \quad 0]^T$. So, this complete the proof.

Now, it will be examined $\alpha(t) = (a\cos t, a\sin t, bt)$.

Theorem 5. The matrix representation of the helix of function $\alpha(t) = (a\cos t, a\sin t, bt)$ as any 7th order Bezier curve based on the coefficients is:

$$\alpha(t) = (a\cos t, a\sin t, bt) = [t^7 \quad t^6 \quad t^5 \quad t^4 \quad t^3 \quad t^2 \quad t \quad 1][B^7][P_0 \quad P_1 \quad P_2 \quad P_3 \quad P_4 \quad P_5 \quad P_6 \quad P_7]^T$$

with the control points $P_0, P_1, P_2, P_3, P_4, P_5, P_6,$ and P_7 are:

$$\begin{bmatrix} P_0 \\ P_1 \\ P_2 \\ P_3 \\ P_4 \\ P_5 \\ P_6 \\ P_7 \end{bmatrix} = \begin{bmatrix} a & 0 & 0 \\ a & \dfrac{1}{7}a & \dfrac{1}{7}b \\ \dfrac{41}{42}a & \dfrac{2}{7}a & \dfrac{2}{7}b \\ \dfrac{13}{14}a & \dfrac{89}{210}a & \dfrac{3}{7}b \\ \dfrac{103}{120}a & \dfrac{58}{105}a & \dfrac{4}{7}b \\ \dfrac{43}{56}a & \dfrac{1681}{2520}a & \dfrac{5}{7}b \\ \dfrac{330}{504}a & \dfrac{107}{140}a & \dfrac{6}{7}b \\ \dfrac{389}{720}a & \dfrac{4241}{5040}a & b \end{bmatrix}$$

Proof. For helix $\alpha(t)=(a\cos t, a\sin t, bt)$, the 7th degree Maclaurin series expansion is:

$$\alpha(t)=(a\cos t, a\sin t, bt)=$$
$$=\left(0-\frac{a}{6!}t^6+\frac{a}{4!}t^4-\frac{at^2}{2!}+a,-\frac{a}{7!}t^7+\frac{a}{5!}t^5-\frac{a}{3!}t^3+at,bt\right)$$

This 7th degree polynomial function can be written as in parametric form in $\mathbf{E}^3$. Also this can be written in matrix form with the matrix representation of 7th order Bezier curve as in:

$$\alpha(t)=(a\cos t, a\sin t, bt)=\begin{bmatrix}t^7\\t^6\\t^5\\t^4\\t^3\\t^2\\t\\1\end{bmatrix}^T\begin{bmatrix}0 & -\frac{a}{7!} & 0\\ -\frac{a}{6!} & 0 & 0\\ 0 & \frac{a}{5!} & 0\\ \frac{a}{4!} & 0 & 0\\ 0 & -\frac{a}{3!} & 0\\ -\frac{a}{2!} & 0 & 0\\ 0 & a & b\\ a & 0 & 0\end{bmatrix}=\begin{bmatrix}t^7\\t^6\\t^5\\t^4\\t^3\\t^2\\t\\1\end{bmatrix}^T[B^7]\begin{bmatrix}P_0\\P_1\\P_2\\P_3\\P_4\\P_5\\P_6\\P_7\end{bmatrix}$$

Solving the equation we get the control points $P_0, P_1, P_2, P_3, P_4, P_5, P_6,$ and P_7, as in the following, result of the matrix product:

$$[P_0 \quad P_1 \quad P_2 \quad P_3 \quad P_4 \quad P_5 \quad P_6 \quad P_7]^T=[B^7]^{-1}\begin{bmatrix}0 & -\frac{a}{6!} & 0 & \frac{a}{4!} & 0 & -\frac{a}{2!} & 0 & a\\ -\frac{a}{7!} & 0 & \frac{a}{5!} & 0 & -\frac{a}{3!} & 0 & a & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & b & 0\end{bmatrix}$$

This complete the proof.

Theorem 6. There is the following relations among the coefficients *a* and *b* and apscissas, ordinates and applicates of the control points. Further the coefficient *a*, based on the apscissas, and the ordinates of the control points. Also the coefficient *b*, based on the only aplicates of the control points.

$$7x_1 - x_0 - 21x_2 + 35x_3 - 35x_4 + 21x_5 - 7x_6 + x_7 = 0$$
$$7x_0 - 42x_1 + 105x_2 - 140x_3 + 105x_4 - 42x_5 + 7x_6 = -\frac{1}{720}a$$
$$105x_1 - 21x_0 - 210x_2 + 210x_3 - 105x_4 + 21x_5 = 0$$
$$35x_0 - 140x_1 + 210x_2 - 140x_3 + 35x_4 = \frac{1}{24}a$$
$$105x_1 - 35x_0 - 105x_2 + 35x_3 = 0$$
$$21x_0 - 42x_1 + 21x_2 = -\frac{1}{2}a$$
$$7x_1 - 7x_0 = 0, x_0 = a$$
$$7y_1 - y_0 - 21y_2 + 35y_3 - 35y_4 + 21y_5 - 7y_6 + y_7 = -\frac{1}{5040}a$$
$$7y_0 - 42y_1 + 105y_2 - 140y_3 + 105y_4 - 42y_5 + 7y_6 = 0$$
$$105y_1 - 21y_0 - 210y_2 + 210y_3 - 105y_4 + 21y_5 = \frac{1}{120}a$$
$$35y_0 - 140y_1 + 210y_2 - 140y_3 + 35y_4 = 0$$
$$105y_1 - 35y_0 - 105y_2 + 35y_3 = \frac{1}{6}a$$
$$21y_0 - 42y_1 + 21y_2 = 0$$
$$7y_1 - 7y_0 = a, y_0 = 0$$
$$7z_1 - z_0 - 21z_2 + 35z_3 - 35z_4 + 21z_5 - 7z_6 + z_7 = 0$$
$$7z_0 - 42z_1 + 105z_2 - 140z_3 + 105z_4 - 42z_5 + 7z_6 = 0$$
$$105z_1 - 21z_0 - 210z_2 + 210z_3 - 105z_4 + 21z_5 = 0$$
$$35z_0 - 140z_1 + 210z_2 - 140z_3 + 35z_4 = 0$$
$$105z_1 - 35z_0 - 105z_2 + 35z_3 = 0$$
$$21z_0 - 42z_1 + 21z_2 = 0$$
$$7z_1 - 7z_0 = b, z_0 = 0$$

Proof. For the apscissas, applicates and ordinates we have the following equations:

$$[B^7]\begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \end{bmatrix} = \begin{bmatrix} 0 \\ -\dfrac{a}{6!} \\ 0 \\ \dfrac{a}{4!} \\ 0 \\ -\dfrac{a}{2!} \\ 0 \\ a \end{bmatrix}, \quad [B^7]\begin{bmatrix} y_0 \\ y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \\ y_6 \\ y_7 \end{bmatrix} = \begin{bmatrix} -\dfrac{a}{7!} \\ 0 \\ \dfrac{a}{5!} \\ 0 \\ -\dfrac{a}{3!} \\ 0 \\ a \\ 0 \end{bmatrix}$$

and

$$[B^7][z_0 \quad z_1 \quad z_2 \quad z_3 \quad z_4 \quad z_5 \quad z_6 \quad z_7]^T = [0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad b \quad 0]^T$$

The solutions of the above equilately give us the proof.

Here we will examine the helix $\alpha(t) = (\cos t, \sin t, t)$ as a 7th order Bezier curve.

Example 1. The numerical matrix representation of the helix $\alpha(t) = (\cos t, \sin t, t)$, as a 7th order Bezier curve is:

$$\alpha(t) = (\cos t, \sin t, t)$$

$$= \begin{bmatrix} t^7 \\ t^6 \\ t^5 \\ t^4 \\ t^3 \\ t^2 \\ t \\ 1 \end{bmatrix}^T [B^7] \begin{bmatrix} 1 & 0 & 0 \\ 1 & \dfrac{1}{7} & \dfrac{1}{7} \\ \dfrac{41}{42} & \dfrac{2}{7} & \dfrac{2}{7} \\ \dfrac{13}{14} & \dfrac{89}{210} & \dfrac{3}{7} \\ \dfrac{103}{120} & \dfrac{58}{105} & \dfrac{4}{7} \\ \dfrac{43}{56} & \dfrac{1681}{2520} & \dfrac{5}{7} \\ \dfrac{3329}{5040} & \dfrac{107}{140} & \dfrac{6}{7} \\ \dfrac{389}{720} & \dfrac{4241}{5040} & 1 \end{bmatrix}$$

Proof. For the helix $\alpha(t) = (a\cos t, a\sin t, bt)$, if we substitute $a = 1$ and $b = 1$ in *Theorem 5,* we have the proof.

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