NUMERICAL COMPARISONS FOR SOLVING FRACTIONAL ORDER INTEGRO-DIFFERENTIAL EQUATIONS WITH NON-LOCAL BOUNDARY CONDITIONS

by

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In this paper, univariate Pade approximation is applied to fractional power series solutions of fractional integro-differential equations with non-local boundary conditions. As it is seen from comparisons, univariate Pade approximation gives reliable solutions and numerical results.

Key words: *integro-differential equations, univariate Pade approximation, non-local boundary conditions*

Introduction

In the last few decades, fractional calculus became more popular due to its wide range of applications in various fields. Some recent applications of fractional calculus are investigated in the fields of science and engineering such as viscoelasticity [1-3], signal processing [4], physics [5, 6], bioengineering [7, 8], hydrology [9] and biology [10, 11]. For the details, the development of fractional calculus can be found in Podlubny [12]. It can be said that fractional calculus deals with the concept of non-integer order integrals and derivatives. The Riemann-Liouville and Caputo fractional derivatives are considered as the classical fractional derivatives.

It is known that fractional integro-differential equations (FIDE) are a combination of fractional derivative and integral terms. It can be said that many research works have been done for analytical and numerical methods to solve the FIDE. The adaptive Huber scheme for weakly singular FIDE was presented by the authors in [13]. The Tau approximation method was applied to solve the space fractional diffusion equation by Saadatmandi and Dehghan [14]. The collocation method with convergence were applied to solve the generalized FIDE by Sharma *et al.* [15]. Legendre collocation method was presented by Saadatmandi and Dehghan [16] to solve the FIDE. The approximation of fractional integrals and Caputo derivatives with applications in solving Abel's integral equation were presented by Kumar *et al.* [17], The Volterra integro-differential equations were presented for investigating the fractal heat-transfer by Yang *et al.* [18].

The idea of univariate and multivariate Pade approximations are based on expanding a function as a ratio of two power series and determining both the numerator and denominator coefficients using the coefficients of Taylor series expansion of a function $f(x)$ [19]. Many authors applied univariate and multivariate Pade approximation on different type of differential

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equations of integer and fractional order [20-26]. More details can be found about univariate and multivariate Pade approximations in [19, 27].

In this paper univariate Pade approximation was applied on the frractional power series solution of FIDE of the form [28]:

$$\mathrm{D}^q y(x) = f(x) + \int_a^x k_1(x,t) y(t)\mathrm{d}t + \int_a^b k_2(x,t) y(t)\mathrm{d}t \tag{1}$$

$m-1 < q \le m,\ \ a < x < b$ and $m \in \mathrm{N}$ with the non-local boundary conditions:

$$\sum_{j=1}^{m}\left[\gamma_{ij} y^{(j-1)}(a) + \eta_{ij} y^{(j-1)}(b)\right] + \lambda_i \int_a^b H_i(t) y(t)\mathrm{d}t = \mathrm{d}_i, \qquad i = 1,2,...,m \tag{2}$$

where D^q denotes a differential operator with fractional order q, $f(x)$ and $k_i(x,t)$, (i = 1, 2) are holomorphic functions, $H_i(t)$ – a continuous function, $\gamma_{ij}, \eta_{ij}, \lambda_i$ and d_i (i = 1, 2, …, m) are constants and $y(x)$ is the function of class C, a class of functions that are piecewise continuous on $J' = (0,\infty)$ and integrable on any finite subinterval $J = [0,\infty)$.

The fractional differential transform method (FDTM)

Let us expand the analytic function $f(x)$ as the fractional power series:

$$f(x) = \sum_{k=0}^{\infty} F(k)(x - x_0)^{\frac{k}{\alpha}} \tag{3}$$

where α is the order of the fraction and $F(k)$ is the fractional differential transform of $f(x)$ In order to avoid fractional initial and boundary conditions, it is defined the fractional derivative in the Caputo sense. The relation between the Riemann-Liouville and Caputo operators is given by:

$$\mathrm{D}_{*x_0}^q f(x) = \mathrm{D}_{x_0}^q \left[f(x) - \sum_{k=0}^{m-1} \frac{1}{k!}(x - x_0)^k f^{(k)}(x_0) \right] \tag{4}$$

$$\mathrm{D}_{x_0}^q f(x) = \frac{1}{\Gamma(m-q)} \frac{\mathrm{d}^m}{\mathrm{d}x^m} \int_{x_0}^{x} (x-t)^{m-q-1} f(t)\mathrm{d}t, \quad x > x_0 \tag{5}$$

Replacing $f(t)$ by:

$$f(t) - \sum_{k=0}^{m-1} \frac{1}{k!}(x - x_0)^k f^{(k)}(x_0) \tag{6}$$

in (5) and using (4), it is obtained the fractional derivative in the Caputo sense as [28]:

$$\mathrm{D}_{*x_0}^q f(x) = \frac{1}{\Gamma(m-q)} \frac{\mathrm{d}^m}{\mathrm{d}x^m} \int \frac{f(t) - \sum_{k=0}^{m-1} \frac{1}{k!}(x - x_0)^k f^{(k)}(x_0)}{(x-t)^{1+q-m}} \mathrm{d}t \tag{7}$$

Since the initial conditions are implemented by the integer-order derivative, the transformations of the initial conditions for are $k = 0,1,...,(\alpha q - 1)$ defined by:

$$F(k)=\begin{cases}0, & \dfrac{k}{\alpha}\notin\mathbb{Z}^+\\ \dfrac{1}{\left(\dfrac{k}{\alpha}\right)!}\left(\dfrac{\mathrm{d}^{k/\alpha}}{\mathrm{d}x^{k/\alpha}}f(x)\right)_{x=x_0}, & \dfrac{k}{\alpha}\in\mathbb{Z}^+\end{cases} \tag{8}$$

where q is the order of the corresponding fractional equation [28]. More details about definitions and theory differential Transform Method and fractional Differential Transform Method can be found in [29-31].

Univariate Pade approximation

Consider a formal power series:

$$f(x)=c_0+c_1x+c_2x^2+\dots \tag{9}$$

with $(c_0\neq 0)$ [27]. The Pade approximation problem of order (m, n) or $[m, n]$ for f consists in finding polynomials:

$$p(x)=\sum_{i=0}^{m}a_ix^i,\quad q(x)=\sum_{i=0}^{n}b_ix^i \tag{10}$$

such that in the power series $(fq-p)$ [25]. To find the coefficients we get following linear systems of equations:

$$\begin{cases}c_{m+1}b_0+c_mb_1+\ldots+c_{m-n+1}b_n=a_m\\ \vdots\\ c_{m+n}b_0+c_{m-n+1}b_1+\ldots+c_mb_n=0\end{cases} \tag{11}$$

$$\begin{cases}c_0b_0=0\\ c_1b_0+c_0b_1=a_1\\ \vdots\\ c_mb_0+c_{m-1}b_1+\cdots+c_{m-n}b_n=a_m\end{cases} \tag{12}$$

with $c_i=0$ for $i<0$ [27].

In general a solution for the coefficients a_i is known after substitution of a solution for the b_i in the left hand side of (11). So the crucial point is to solve the homogeneous system of n eq. (12) in the $n+1$ unknowns b_i. This system has at least one nontrivial solution because one of the unknowns can be chosen freely [27].

In short, by solving eqs. (11) and (12) the coefficients a_i and b_i are found. Then the Pade eqs. (10) are found. After finding these polynomials we get The Pade approximation of order (m, n) or $[m, n]$ for f.

Applications and results

In this section univariate Pade series solutions of FIDE with non-local boundary conditions shall be illustrated by two examples. The full FDTM solutions of examples can be seen in [28].

Example 1.

Consider the following linear FIDE with the given non-local condition:

$$\mathrm{D}^{\frac{1}{2}} y(x) = -x^2 \mathrm{e}^x y(x) - \frac{1}{2}x^2 + \frac{1}{\Gamma\left(\frac{3}{2}\right)} x^{\frac{1}{2}} + \mathrm{e}^x \int_0^x t y(t)\mathrm{d}t + \int_0^1 x^2 y(t)\mathrm{d}t \tag{13}$$

$$y(0) + y(1) - 3\int_0^1 t y(t)\mathrm{d}t = 0 \tag{14}$$

where the order of fraction is $\alpha = 2$. The exact solution is for eq. (13) is given as $y(x) = x$ in [28]. Nazari and Shahmorad obtained following solution (15) by applying FDTM on eqs. (13) and (14).

$$\begin{gathered} y_1(x) = -0.96658\times10^{-6} + x + 0.24306\times10^{-5}x^{5/2} - 0.831\times10^{-7}x^{7/2} - 0.36933\times10^{-7}x^{9/2} - \\ -0.16178\times10^{-6}x^5 - 0.11192\times10^{-7}x^{11/2} - 0.10295\times10^{-6}x^6 - 0.25827\times10^{-8}x^{13/2} + \\ +0.157\times10^{-6}x^7 + 0.37576\times10^{-7}x^{15/2} - 0.10389\times10^{-7}x^8 + 0.64758\times10^{-7}x^{17/2} + \\ +0.89992\times10^{-8}x^9 + 0.13509\times10^{-7}x^{19/2} + 0.73501\times10^{-7}x^{10} \end{gathered} \tag{15}$$

By applying eqs. (11) and (12) to put eq. (15) into Pade series, following Pade equations respectively $r_{10,8}(x)$, $r_{10,7}(x)$, and $r_{10,6}(x)$ were obtained for different values of *m* and *n*:

$$\begin{gathered} r_{10,8}(x) = [-9.665800002\times10^{-7} + 1.419202103\times10^{(-6)}\sqrt{x} + 0.9999992671x - \\ -1.468270770x^{3/2} + 0.7584647740x^2 - 1.009162736x^{5/2} + 0.6765228633x^3 - \\ -0.5361405544x^{7/2} - 0.05847060361x^4 + 0.05241522878x^{9/2} - \\ -0.1165250370x^5]/(1.0 - 1.468271746\sqrt{x} + 0.7584654280x - 1.009165685x^{3/2} + \\ +0.6765263755x^2 - 0.5361422641x^{5/2} - 0.05846838538x^3 + \\ +0.05241368437x^{7/2} - 0.1165237101x^4) \end{gathered} \tag{16}$$

$$\begin{gathered} r_{10,7}(x) = [-9.665800003\times10^{-7} - 2.592446073\times10^{(-8)}\sqrt{x} + \\ +0.9999995223x + 0.02682101480x^{3/2} + 0.4942901382x^2 - \\ -0.2095188449x^{5/2} + 0.5497334320x^3 + 0.3382399268x^{7/2} + \\ +0.03269493562x^4 + 0.08376079934x^{9/2} + \\ +6.767640621\times10^{-7}x^5]/(0.9999999999 + 0.02682081227\sqrt{x} + \\ +0.4942906696x - 0.2095209486x^{3/2} + 0.5497333984x^2 + \\ +0.3382388894x^{5/2} + 0.03269544711x^3 + 0.08375954116x^{7/2}) \end{gathered} \tag{17}$$

$$r_{10,6}(x) = (-9.665799997\times10^{-7} + 2.804104916\times10^{-6}\sqrt{x} + 0.9999989582x - \\ -2.901059816x^{3/2} + 1.077744481x^{2} + 1.584698085x^{5/2} + 1.054703859x^{3} - \\ -3.650616682x^{7/2} - 0.1360488854x^{4} + 2.43708636910^{-6}x^{9/2} - \\ -9.059518204\times10^{-6}x^{5})/(0.9999999999 - 2.901058284\sqrt{x} + 1.077745500x + \\ +1.584692126x^{3/2} + 1.054710779x^{2} - 3.650619218x^{5/2} - 0.1360529783x^{3}) \quad (18)$$

Table 1. Numerical values for exact solution $y(x) = x$, $y_1(x)$ (FDTM Solution) and Pade approximations of $y_1(x)$

x	$r_{10,8}(x)$	$r_{10,7}(x)$	$r_{10,6}(x)$	$y_1(x)$	$y(x) = y$
10.0	10.00008692	10.00023159	10.00069340	818.7945465	10.0
10.1	10.10008795	10.10023566	10.10071050	902.7897906	10.1
10.2	10.20008899	10.20023976	10.20072790	994.5386236	10.2
10.3	10.30009003	10.30024389	10.30074553	1094.671666	10.3
10.4	10.40009106	10.40024804	10.40076340	1203.863979	10.4
10.5	10.50009210	10.50025221	10.50078152	1322.837711	10.5
10.6	10.60009315	10.60025641	10.60079990	1452.364894	10.6
10.7	10.70009418	10.70026064	10.70081851	1593.270362	10.7
10.8	10.80009522	10.80026488	10.80083740	1746.434824	10.8
10.9	10.90009626	10.90026915	10.90085652	1912.798074	10.9
11.0	11.00009730	11.00027344	11.00087591	2093.362350	11.0

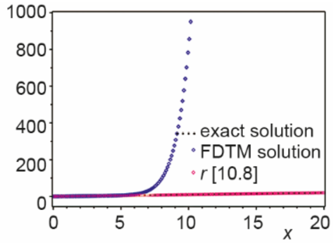

Figure 1. $y(x) = x$, $y_1(x)$, $r_{10,8}(x)$

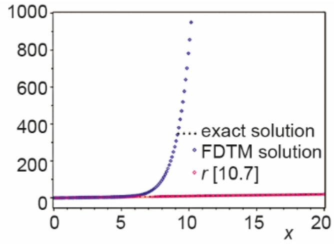

Figure 2. $y(x) = x$, $y_1(x)$, $r_{10,7}(x)$

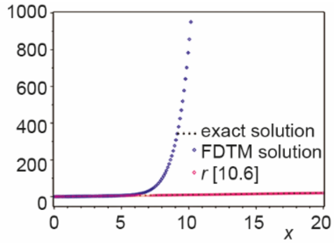

Figure 3. $y(x) = x$, $y_1(x)$, $r_{10,6}(x)$

Example 2.

Consider the following fractional integro-differential equation:

$$D^{5/4}y(x) = (\cos x - \sin x)y(x) + f(x)\int_0^1 \sin t y(t)\mathrm{d}t \quad (19)$$

with the non-local conditions:

$$y(0) + y(1) + \left(\frac{e+1}{e+2}\right)y'(0) + \frac{1}{2}y'(1) - \int_0^1 t y(t)\mathrm{d}t = 0, \quad 2y(0) + 2y(1) + \left(\frac{e}{e+2}\right)y'(0) - y'(1) = 0 \quad (20)$$

Function $f(x)$ and exact solution of eq. (19) are respectively given as:

$$f(x)=\frac{8}{3}\frac{x^{3/4}}{\Gamma(3/4)}-2\cos x-2x\sin x+2$$

and $y(x)=x^2$ in [28] by Nazari and Shahmorad.

Nazari and Shahmorad obtained following solution (21) by applying FDTM on (19) and (20):

$$\begin{gathered}y_2(x)=0.29447\times10^{-6}-0.12796\times10^{-6}x+0.25990\times10^{-6}x^{5/4}+\\+0.99999x^2-0.16571\times10^{-6}x^{9/4}+0.88607\times10^{-7}x^{5/2}-0.23831\times10^{-5}x^{13/4}-\\-0.93280\times10^{-7}x^{7/2}+0.17754\times10^{-7}x^{15/4}+0.1716\times10^{-5}x^{17/4}-0.35889\times10^{-6}x^{9/2}-\\-0.26854\times10^{-7}x^{19/4}+0.24539\times10^{-8}x^{5}+0.3218\times10^{-6}x^{21/4}+0.51203\times10^{-6}x^{11/2}-\\-0.33534\times10^{-7}x^{23/4}-0.48810\times10^{-8}x^{6}-0.17322\times10^{-6}x^{25/4}-0.67336\times10^{-7}x^{13/2}-\\+0.8618\times10^{-7}x^{27/4}-0.11207\times10^{-8}x^{7}-0.12477\times10^{-7}x^{29/4}-0.52817\times10^{-7}x^{30/4}\end{gathered}\quad(21)$$

By applying eqs. (11) and (12) to put eq. (21) into Pade series, following Pade equations respectively $r_{10,5}(x)$,$r_{10,4}(x)$, and $r_{10,3}(x)$, were obtained for different values of m and n.

$$\begin{gathered}r_{10,5}(x)=(2.944700001\times10^{-7}-1.152628002\times10^{-8}x^{1/4}+2.641816012\times10^{-9}\sqrt{x}+\\+1.459099016\times10^{-15}x^{3/4}-1.279600003\times10^{-7}x+2.649093709\times10^{-7}x^{5/4}-\\-1.132110896\times10^{-8}x^{3/2}+2.331673157\times10^{-9}x^{7/4}+0.9999900002x^2-\\-0.03914223370x^{9/4}+0.008971432047x^{5/2})/(1.0-0.03914245941x^{1/4}+\\+0.008971426668\sqrt{x}+4.955000564\times10^{-9}x^{3/4}-7.949383311\times10^{-10}x+\\+2.383123833\times10^{-6}x^{5/4})\end{gathered}\quad(22)$$

$$\begin{gathered}r_{10,4}(x)=(2.944700001\times10^{-7}-9.025442875\times10^{-8}x^{1/4}+\\+23.89639020\sqrt{x}+3.959910427\times10^{-6}x^{3/4}-2.245367965\times10^{-6}x+\\+2.991194681\times10^{-7}x^{5/4}-10.38401913x^{3/2}+21.09101541x^{7/4}+\\+0.9999944155x^2-0.3064968429x^{9/4}+\\+8.114969687\times10^{7}x^{5/2})/(1.0-0.3064978733x^{1/4}+\\+8.115050838\times10^{7}\sqrt{x}+13.44758525x^{3/4}-7.190572774x)\end{gathered}\quad(23)$$

$$\begin{gathered}r_{10,3}(x)=(2.944700000\times10^{-7}-89.38050004x^{1/4}+0.00001481139537\sqrt{x}+\\+7.919819617\times10^{-6}x^{3/4}-1.279600000\times10^{-7}x+38.83970816x^{5/4}-\\-78.88747192x^{3/2}+9.631071199\times10^{-6}x^{7/4}+0.9999969900x^2-\\-3.035270357\times10^{8}x^{9/4}+100.5959518x^{5/2})/(0.9999999999-\\-3.035300711\times10^{8}x^{1/4}+50.29848664\sqrt{x}+26.89516629x^{3/4})\end{gathered}\quad(24)$$

Table 2. Numerical values for exact solution $y(x) = x^2$, $y_2(x)$ (FDTM Solution) and Pade approximations of $y_2(x)$

x	$r_{10,5}(x)$	$r_{10,4}(x)$	$r_{10,3}(x)$	$y_2(x)$	$y(x) = x^2$
10.0	99.99458229	99.99900226	99.99900213	98.26822922	100.0
10.1	102.0044168	102.0089821	102.0089821	100.1424244	102.01
10.2	104.0342480	104.0389620	104.0389619	102.0275267	104.04
10.3	106.0840755	106.0889415	106.0889415	103.9230298	106.09
10.4	108.1538996	108.1589209	108.1589209	105.8284017	108.16
10.5	110.2437200	110.2489001	110.2489001	107.7430886	110.25
10.6	112.3535366	112.3588791	112.3588791	109.6665115	112.36
10.7	114.4833498	114.4888579	114.4888579	111.5980670	114.49
10.8	116.6331592	116.6388365	116.6388365	113.5371241	116.64
10.9	118.8029649	118.8088150	118.8088150	115.4830251	118.81
11.0	120.9927666	120.9987932	120.9987931	117.4350854	121.0

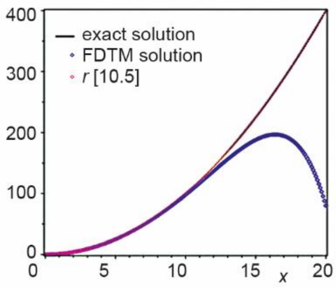

Figure 4. $y(x) = x^2$, $y_2(x)$, $r_{10,5}(x)$

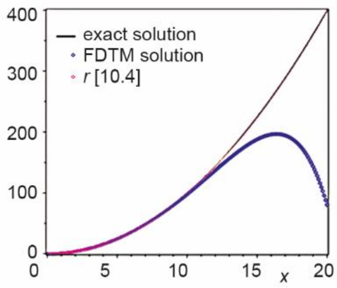

Figure 5. $y(x) = x^2$, $y_2(x)$, $r_{10,4}(x)$

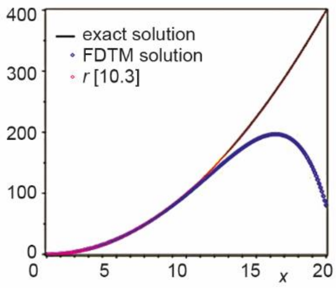

Figure 6. $y(x) = x^2$, $y_2(x)$, $r_{10,3}(x)$

Conclusion

As it is seen from the tables and figures in two examples, it can be said that obtained numerical results by using univariate Pade approximation are very powerful and efficient. Especially to show the efficiency of Pade approximation on figures, a very large interval has been chosen. The proposed method is very simple in application and is more accurate in comparison with other mentioned method.

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